TJMM
5 (2013), No. 1, 65-68

COMPOSITION OPERATORS ON CESARO-ORLICZ SEQUENCE SPACES

KULDIP RAJ, SEEMA JAMWAL AND SUNIL K. SHARMA

Abstract. In this paper we characterize the boundedness, invertibility and Fredholmness of composition operators on the Cesaro-Orlicz sequence spaces Ces_φ.

1. Introduction and Preliminaries

An Orlicz function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous, non-decreasing and convex such that $\varphi(0) = 0$, $\varphi(x) > 0$ for $x > 0$ and $\varphi(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. An orlicz function is said to satisfy the δ_2–condition if their exists $K > 0$ such that $\varphi(Lx) \leq KL\varphi(x)$, for all $x \geq 0$, and for $L > 1$.

By w we shall denote the space of all real or complex sequences. The Cesaro-Orlicz sequence space (Ces_φ) is defined as

$$\mathrm{Ces}_\varphi = \left\{ x = (x_k) \in w : \sum_{m=1}^{\infty} \varphi\left(\frac{1}{m}\sum_{k=1}^{m}|\lambda x_k|\right) < \infty \right\}.$$

The space Ces_φ is a Banach space under the norm

$$||x|| = \inf\left\{ \lambda > 0 : \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m}|x_k|}{\lambda}\right) \leq 1 \right\} \text{(see [17])}.$$

The Cesaro-Orlicz sequence spaces Ces_φ appeared for the first time in 1988, when Lim and Lee [10] found their dual spaces. Recently, Cui et al. [5] obtained important properties of the spaces Ces_φ. In 2007 Maligranda et al. [14] showed that Ces_φ is not B-convex, if $\varphi \in \delta_2$ and $\mathrm{Ces}_\varphi \neq \{0\}$. The extreme points and strong $U-$ points of Ces_φ have been characterized by Foralewski et al. in [6]. Although the spaces Ces_φ have been studied by several mathematicians, some essential and basic properties remain still unknown but recently some of its properties have been discussed by Damian in [9].

In the case when $\varphi(x) = |x|^p$, $(p > 1)$ the Cesaro-Orlicz sequence space Ces_φ becomes the Cesaro sequence space Ces_p. For more details about Cesaro sequence spaces Ces_p see [2, 3, 4, 7, 11, 12, 13, 15, 20].

Let X and Y be two non empty sets and let $F(X,\mathbb{C})$ and $F(Y,\mathbb{C})$ be two topological vector spaces of complex valued functions on X and Y respectively. Suppose $T : Y \to X$ is a mapping such that $f \circ T \in F(Y,\mathbb{C})$ whenever $f \in F(X,\mathbb{C})$. Then we can define a composition transformation $C_T : F(X,\mathbb{C}) \to F(Y,\mathbb{C})$ by $C_T f = f \circ T$ for every $f \in F(X,\mathbb{C})$. If C_T is continuous, we call it a composition operator induced by T. By $B(\mathrm{Ces}_\varphi)$ we shall denote the set of all bounded linear operators from the space Ces_φ into itself. We denote $f_o = \frac{d\mu T^{-1}}{d\mu}$, the Radon-Nikodym derivative of the measure μT^{-1} with respect

2010 *Mathematics Subject Classification.* 47B38, 46A06.

Key words and phrases. composition operator, invertible operator, isometry, Cesaro sequence space, Cesaro-Orlicz sequence space, Fredholm operator.

to the measure μ. Throughout the paper we shall consider φ satisfies δ_2-condition. Also there should be no confusion to write $\text{Ces}_\varphi(\mathbb{N})$ instead of Ces_φ.

In the beginning composition operators were known as substitution operators. These operators found applications in ergodic theory, mathematical physics and other branches of mathematical sciences. Banach [1] himself used these operators in studying isometries of function spaces. The systematic study of composition operators was started with the paper of Nordgen [16] in 1968. After that the work was extended in several directions by several mathematicians. For more details about composition operators see [8, 18, 19, 21, 22] and references their in.

The main purpose of this paper is to study some properties of composition operators such as invertibility, Fredholmness, isometry, compactness and closed range etc. on Cesaro-Orlicz sequence spaces. It is shown that no composition operator is compact on the Cesaro-Orlicz sequence spaces.

2. Composition Operators

Theorem 1. *Let $T : \mathbb{N} \to \mathbb{N}$ be a mapping. Then $C_T \in B(Ces_\varphi)$ if and only if there exists $M > 0$ such that $\mu(T^{-1}(\{n\})) \leq M$, where μ is a counting measure.*

Proof. If $C_T \in B(\text{Ces}_\varphi)$, then

$$\|C_T \chi_{\{n\}}\| \leq M\|\chi_{\{n\}}\| \Rightarrow \frac{1}{\varphi^{-1}\left[\frac{1}{\mu T^{-1}(\{n\})}\right]} \leq M \frac{1}{\varphi^{-1}(\{1\})}$$

therefore

$$\frac{\varphi^{-1}(\{1\})}{M} \leq \varphi^{-1}\left[\frac{1}{\mu T^{-1}(\{n\})}\right].$$

This implies that $\{\mu T^{-1}(\{n\}) : n \in \mathbb{N}\}$ is a bounded set. Conversly, if the condition is true, then

$$\begin{aligned}
\sum_{m=1}^{\infty} \varphi\left[\frac{\left(\frac{1}{m}\sum_{n=1}^{m} |C_T f|(n)\right)}{M\|f\|}\right] &= \sum_{m=1}^{\infty} f_o(n)\varphi\left[\frac{\frac{1}{m}\sum_{n=1}^{m} |f(n)|}{M\|f\|}\right] \\
&\leq \sum_{m=1}^{\infty} \varphi\left[\frac{\left(\frac{1}{m}\sum_{n=1}^{m} |f(n)|\right)}{\|f\|}\right] \\
&\leq 1.
\end{aligned}$$

Hence $\|C_T f\| \leq M\|f\|$. □

Theorem 2. *Let $C_T \in B(Ces_\varphi)$. Then the following are equivalent:*

(i) T is invertible,
(ii) C_T is invertible,
(iii) C_T is isometry.

Proof. (i) $\Rightarrow$ (ii) Suppose T is invertible. Then there exist a mapping S such that $(ToS)(n) = (SoT)(n)$ for every $n \in \mathbb{N}$. Since S is one-to-one so $C_S \in B(\text{Ces}_\varphi)$ and $C_S o C_T = C_T o C_S = I$, the identity operator. Hence, C_T is invertible.

(ii) $\Rightarrow$ (i) Suppose C_T is invertible. If T is surjective, then $C_T e_n = 0$ for $n \in \mathbb{N}\backslash T(\mathbb{N})$ so that C_T has a non-trivial kernel. Now if $T(n_1) = T(n_2)$ for two distinct positive integers n_1 and n_2, then $\chi_{\{n_1\}}$, the characteristic function of $\{n_1\}$ is not in the range of C_T. Hence if C_T is invertible, then T is invertible.

(i) $\Rightarrow$ (iii) If T is invertible, then for $f \in \mathrm{Ces}_\varphi$ we have

$$\begin{aligned}
\|C_T f\| &= \inf\left\{\lambda > 0 : \sum_{m=1}^{\infty} \varphi\left[\frac{\frac{1}{m}\sum_{n=1}^{m} |f \circ T(n)|}{\lambda}\right] \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \sum_{m=1}^{\infty} \varphi\left[\frac{\frac{1}{m}\sum_{k=1}^{m} \sum_{n\in T^{-1}(\{k\})}(f(k))}{\lambda}\right] \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \sum_{m=1}^{\infty} \varphi\left[\frac{\frac{1}{m}\sum_{k=1}^{m} f(k)}{\lambda}\right] \le 1\right\} \\
&= \|f\|.
\end{aligned}$$

Hence C_T is isometry.

(iii) $\Rightarrow$ (i) For every $n \in \mathbb{N}$, $\|C_T e_n\| = \|e_n\|$ implies that $\varphi^{-1}\left[\frac{1}{\mu T^{-1}(\{n\})}\right] = \phi^{-1}(\{1\})$ for every $n \in \mathbb{N}$. Thus T is invertible. □

Theorem 3. *Let $C_T \in B(Ces_\varphi)$. Then C_T is not compact.*

Proof. Let C_T be a bounded operator on Ces_φ. For each $n \in T(\mathbb{N})$, define $\delta_n : \mathbb{N} \to \mathbb{R}$ as $\delta_n = \varphi^{-1}(\{1\})\chi_{\{n\}}$. Then $\|\delta_n\| = 1$ and $\delta_n \to 0$ weekly. But

$$\|C_T\delta_n\| = \frac{\varphi^{-1}(\{1\})}{\varphi^{-1}\left[\frac{1}{\mu(T^{-1}(\{n\}))}\right]} \not\to 0$$

strongly because $1 \le \mu(T^{-1}(\{n\})) \le M$ for every $n \in T(\mathbb{N})$. Hence C_T is not compact. □

Theorem 4. *Let $C_T \in B(Ces_\varphi)$. Then C_T has closed range.*

Proof. Let $A = \{f \in \mathrm{Ces}_\varphi : \forall n \in \mathbb{N}, f(m) = c_n, \forall m \in T^{-1}(\{n\}), c_1, c_2, \dots c_n \in \mathbb{R}\}$. By theorem 1, there exists $p > 0$ such that $\mu(T^{-1}(\{n\})) \le p$ for all $n \in \mathbb{N}$. It is very easy to see that A is a closed subspace of Ces_φ. Moreover, $\mathrm{ran}C_T \subset A$. It remains to prove that $A \subset \mathrm{ran}C_T$. Let $f \in A$, so we can regard $f \circ T^{-1}$ as an function on $T(\mathbb{N})$. Define $g : \mathbb{N} \to \mathbb{C}$ by $g(n) = 0$ for all $n \in \mathbb{N} \setminus T(\mathbb{N})$ and $g(n) = (f \circ T^{-1})(n)$ for every $n \in T(\mathbb{N})$. Then $(g \circ T)(n) = f(n)$ for all $n \in \mathbb{N}$. Hence $f \in \mathrm{ran}C_T$. Therefore $\mathrm{ran}C_T$ is closed. □

Theorem 5. *Let $C_T \in B(Ces_\varphi)$. Then C_T is Fredholm if and only if*

(i) $\mathbb{N} \setminus T(\mathbb{N})$ is a finite set;
(ii) the set $E = \{n \in \mathbb{N} : \mu(T^{-1}(\{n\})) \ge 2\}$ is a finite set.

Proof. Assume that C_T is Fredholm. If $\mathbb{N}\setminus T(\mathbb{N})$ is an infinite set, then $\mathrm{Ces}_\varphi(\mathbb{N}\setminus T(\mathbb{N})) \subset \ker C_T$ so that $\ker C_T$ is infinite dimensional, which is a contradiction. Hence $\mathbb{N} \setminus T(\mathbb{N})$ must be a finite set. Further, if $E = \{n \in \mathbb{N} : \mu(T^{-1}(\{n\})) \ge 2\}$ is an infinite set, choose $n_k, m_k \in E$ such that $T(n_k) = T(m_k) = p_k$. Take $f_k \in \mathrm{Ces}_\varphi$ to be such that $f_k(n_k) = 1$ and $f_k(m_k) = -1$ and equal to zero elsewhere. Clearly $f_k \notin \mathrm{ran}C_T$. Moreover $\{f_k : k \in \mathbb{N}\}$ is a linearly independent set of vectors contained in $\mathrm{Ces}_\varphi \setminus \mathrm{ran}C_T$. This contradicts the fact that $\mathrm{ran}C_T$ is finite co-dimensional. Thus E must be finite.

Conversely, if E is a finite set, then $\mathrm{ran}C_T$ is of finite co-dimensional and if $\mathbb{N} \setminus T(\mathbb{N})$ is a finite set, then $\mathrm{Ces}_\varphi(\mathbb{N}\setminus T(\mathbb{N}))$ is finite dimensional. By the theorem 4 C_T has closed range. Hence C_T is Fredholm. This completes the proof of the theorem. □

Theorem 6. *Let $C_T \in B(Ces_\varphi)$. Then $C_T^* : Ces_{\varphi^*} \to Ces_{\varphi^*}$ defined by*

$$(C_T^* f)(n) = \sum_{m\in T^{-1}(\{n\})} f(m)$$

is a bounded operator, where $Ces_{\varphi^*} = Ces_\psi$ *and* C_T^* *is the adjoint of* C_T*.*

Proof. It is easy to prove so we omit it. □

Theorem 7. *Let* $C_T \in B(Ces_\varphi)$*. Then following are equivalent:*
(i) T *is injective;*
(ii) C_T *is a co-isometry.*

Proof. It is easy to prove so we omit it. □

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School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India
E-mail address: kuldipraj68@gmail.com

School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India
E-mail address: sunilksharma42@yahoo.co.in