TJMM
5 (2013), No. 1, 59-64

LAPLACE TRANSFORM TECHNIQUE FOR PSEUDOPARABOLIC EQUATION WITH NONLOCAL CONDITIONS

AHCENE MERAD AND ABDELFATAH BOUZIANI

Abstract. In this paper we present a Laplace transform technique for solving the pseudoparabolic equation with nonlocal (integral) conditions. A Laplace transform technique is described for the solution of the considered equation. Finally, we obtain the solution using a numerical technique by inverting the Laplace transform.

1. Introduction

In this paper we are concerned with the following integrodifferential pseudoparabolic equation

$$\frac{\partial u}{\partial t}(x,t)-\frac{\partial^2 u}{\partial x^2}(x,t)-\frac{\partial^3 u}{\partial t\partial x^3}(x,t)= \\ =f(x,t)+\int_0^t a(t-s)\,u(x,s)\,ds, 0\le x\le 1, t>0, \tag{1}$$

Subject to the initial condition

$$u(x,0)=\varphi(x), 0\le x\le 1, \tag{2}$$

and the nonlocal (integral) conditions

$$u(x,0)=\int_0^1 p(x)\,u(x,t)\,dx+r(t), t>0, \tag{3}$$

$$u(x,1)=\int_0^1 q(x)\,u(x,t)\,dx+v(t), t>0, \tag{4}$$

where, x, t, are space and time coordinates, respectively, and f, φ, p, q, r, and u are suitably chosen functions. Ang [2] has considered a one dimensional heat equation with nonlocal integral conditions. The author has taken the Laplace transform of the problem and then used a numerical technique for the inverse Laplace transform to obtain the numerical solution. Ekolin [5] and Liu [9] have used the finite difference method to solve a similar type problem numerically. After, Shruti [13] studied Sobolev-type differential equation subject to nonlocal initial boundary conditions by Laplace transform method. Merad [10] used the Adomian decomposition method to solve a similar type equation.

The purpose of the present article is to give a method of solution to problem (1)–(4) using Laplace transform technique. In recent years, Laplace transform method has been used to approximate the solution of different classes of linear partial differential equations [2, 3, 8, 11]. Suying, Minzhen, Zichen and Wencheng [14] established a numerical method based on Laplace transform for solving the initial problem of nonlinear dynamic differential equations. The main difficulty in using Laplace transform method consists in finding its inverse. Numerical inversion methods are then used to overcome this difficulty. There

2010 *Mathematics Subject Classification.* 40A05, 40A25, 45G05.
Key words and phrases. pseudoparabolic equation, Laplace transform technique, nonlocal conditions.

are many numerical techniques available in literature to invert Laplace transform. In this paper we focus exclusively on the Stehfest inversion algorithm[12] in order to efficiently and accurately invert the Laplace transform (which cannot be done analytically).

2. Laplace transform method

Lapalce transform method is widely used in the engineering technology and mathematical science. There are many problems whose solution may be found in terms of Laplace transform. In fact, it is an efficient method for solving various differential equations. Laplace transform of the function $v(x,t)$ with respect to t can be expressed as

$$V(x;s) = \mathcal{L}\{v(x,t);t \longrightarrow s\} = \int_0^\infty v(x,t)\exp(-st)\,dt,$$

where s is positive real parameter. Taking the Laplace transforms on both sides of (1), we have

$$(s - A(s))\,U(x;s) - (1+s)\frac{d^2}{dx^2}U(x;s) = F(x;s) + \varphi(x) - \frac{\partial^2\varphi}{\partial x^2}, \tag{5}$$

where $U(x;s) = \mathcal{L}\{u(x,t);t \longrightarrow s\}$ and $F(x;s) = \mathcal{L}\{f(x,t);t \longrightarrow s\}$. Similarly, we have

$$U(0;s) = \int_0^1 p(x)\,U(x;s)\,dx + R(s),$$
$$U(1;s) = \int_0^1 q(x)\,U(x;s)\,dx + Q(s),$$

where

$$R(s) = \mathcal{L}\{r(t);t \longrightarrow s\}$$

and

$$V(s) = \mathcal{L}\{q(t);t \longrightarrow s\}.$$

Now, we have the following cases:

Case 1: If $s - A(s) > 0$.

Case 2: If $s - A(s) < 0$.

Case 3: If $s - A(s) = 0$.

We only consider the cases 2 and 3, as the case 1 can be similarly treated as in [2]. For $(s - A(s)) = 0$ we have

$$\frac{d^2}{dx^2}U(x,s) = -\frac{1}{1+s}\left[F(x;s) + \varphi(x) - \frac{\partial^2\varphi}{\partial x^2}\right], \tag{6}$$

The general solution for the case 3 is given by

$$U(x,s) = -\frac{1}{1+s}\int_0^x\int_0^y\left[F(z;s) + \varphi(z) - \frac{\partial^2\varphi}{\partial z^2}\right]dzdy + C_1(s)\,x + C_2(s), \tag{7}$$

Putting the boundary conditions in (4) we get

$$C_1(s)\int_0^1 xp(x)\,dx + C_2(s)\left[\int_0^1 p(x)\,dx - 1\right]$$
$$= \frac{1}{1+s}\int_0^1\int_0^x\int_0^y p(x)\left[F(z;s) + \varphi(z) - \frac{\partial^2\varphi}{\partial z^2}\right]dzdydx - R(s),$$

$$C_1(s)\left[\int_0^1 xq(x)\,dx - 1\right] + C_2(s)\left[\int_0^1 q(x)\,dx - 1\right]$$

$$= \frac{1}{1+s}\left[\begin{array}{c}\int_0^1\int_0^x\int_0^y q(x)\left(F(z;s)+\varphi(z)-\frac{\partial^2\varphi}{\partial z^2}\right)dzdy- \\ \int_0^1\int_0^y\left(F(z;s)+\varphi(z)-\frac{\partial^2\varphi}{\partial z^2}\right)dzdy\end{array}\right]-V(s),$$

So C_1, C_2 are given by

$$\begin{pmatrix} C_1(s) \\ C_2(s)\end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s)\end{pmatrix}^{-1}\times\begin{pmatrix} b_1(s) \\ b_2(s)\end{pmatrix},$$

where

$$a_{11}(s) = \int_0^1 xp(x)\,dx,\quad a_{12}(s) = \int_0^1 p(x)\,dx - 1,$$

$$a_{21}(s) = \int_0^1 xq(x)\,dx - 1,\quad a_{22}(s) = \int_0^1 q(x)\,dx - 1,$$

$$b_1(s) = \frac{1}{1+s}\int_0^1\int_0^x\int_0^y p(x)\left[F(z;s)+\varphi(z)-\frac{\partial^2\varphi}{\partial z^2}\right]dzdydx - R(s),$$

$$b_2(s) = \frac{1}{1+s}\left[\begin{array}{c}\int_0^1\int_0^x\int_0^y q(x)\left(F(z;s)+\varphi(z)-\frac{\partial^2\varphi}{\partial z^2}\right)dzdy- \\ \int_0^1\int_0^y\left(F(z;s)+\varphi(z)-\frac{\partial^2\varphi}{\partial z^2}\right)dzdy\end{array}\right]-V(s).$$

For the case 2, that is $(s - A(s)) < 0$, using the method of variation of parameter we have the general solution as,

$$U(x;s) = \sqrt{\frac{1+s}{A(s)-s}}\int_0^x\left(F(\tau;s)+\varphi(\tau)-\frac{\partial^2\varphi}{\partial\tau^2}\right)\sin\left(\sqrt{\frac{A(s)-s}{1+s}}\right)(x-\tau)\,d\tau$$

$$+ d_1(s)\cos\sqrt{\left(\frac{A(s)-s}{1+s}\right)}x + d_2(s)\sin\sqrt{\left(\frac{A(s)-s}{1+s}\right)}x \tag{8}$$

From the boundary conditions we get

$$d_1(s)\left(1-\int_0^1 p(x)\cos\sqrt{\left(\frac{A(s)-s}{1+s}\right)}xdx\right) - d_2(s)\int_0^1 p(x)\sin\sqrt{\left(\frac{A(s)-s}{1+s}\right)}xdx =$$

$$\sqrt{\frac{1+s}{A(s)-s}}\int_0^1\int_0^x p(x)\left(F(\tau;s)+\varphi(\tau)-\frac{\partial^2\varphi}{\partial\tau^2}\right)\sin\left(\sqrt{\frac{A(s)-s}{1+s}}\right)(x-\tau)\,d\tau dx + R(s),$$

$$d_1(s)\left(\cos\sqrt{\left(\frac{A(s)-s}{1+s}\right)} - \int_0^1 q(x)\cos\sqrt{\left(\frac{A(s)-s}{1+s}\right)}xdx\right)$$

$$+ d_2(s)\left(\sin\sqrt{\left(\frac{A(s)-s}{1+s}\right)} - \int_0^1 q(x)\sin\sqrt{\left(\frac{A(s)-s}{1+s}\right)}xdx\right)$$

$$= \sqrt{\frac{1+s}{A(s)-s}}\int_0^1\int_0^x q(x)\left(F(\tau;s)+\varphi(\tau)-\frac{\partial^2\varphi}{\partial\tau^2}\right)\sin\left(\sqrt{\left(\frac{A(s)-s}{1+s}\right)}(x-\tau)\right)d\tau dx$$

$$- \sqrt{\frac{1+s}{A(s)-s}}\int_0^1\left(F(\tau;s)+\varphi(\tau)-\frac{\partial^2\varphi}{\partial\tau^2}\right)\sin\left(\sqrt{\left(\frac{A(s)-s}{1+s}\right)}(1-\tau)\right)d\tau + V(s).$$

Thus d_1, d_2 are given by

$$\begin{pmatrix} d_1(s) \\ d_2(s)\end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s)\end{pmatrix}^{-1}\times\begin{pmatrix} b_1(s) \\ b_2(s)\end{pmatrix},$$

and

$$a_{11}(s) = 1 - \int_0^1 p(x) \cos \sqrt{\left(\frac{A(s)-s}{1+s}\right)} x dx,$$

$$a_{12}(s) = -\int_0^1 p(x) \sin \sqrt{\left(\frac{A(s)-s}{1+s}\right)} x dx,$$

$$a_{21}(s) = \cos \sqrt{\left(\frac{A(s)-s}{1+s}\right)} - \int_0^1 q(x) \cos \sqrt{\left(\frac{A(s)-s}{1+s}\right)} x dx,$$

$$a_{22}(s) = \sin \sqrt{\left(\frac{A(s)-s}{1+s}\right)} - \int_0^1 q(x) \sin \sqrt{\left(\frac{A(s)-s}{1+s}\right)} x dx,$$

$$b_1(s) =$$

$$\sqrt{\frac{1+s}{A(s)-s}} \int_0^1 \int_0^x p(x) \left(F(\tau; s) + \varphi(\tau) - \frac{\partial^2 \varphi}{\partial \tau^2}\right) \sin\left(\sqrt{\frac{A(s)-s}{1+s}}\right)(x-\tau)\, d\tau dx + R(s),$$

$$b_2(s) =$$

$$\sqrt{\frac{1+s}{A(s)-s}} \int_0^1 \int_0^x q(x) \left(F(\tau; s) + \varphi(\tau) - \frac{\partial^2 \varphi}{\partial \tau^2}\right) \sin\left(\sqrt{\left(\frac{A(s)-s}{1+s}\right)}(x-\tau)\right) d\tau dx -$$

$$\sqrt{\frac{1+s}{A(s)-s}} \int_0^1 \left(F(\tau; s) + \varphi(\tau) - \frac{\partial^2 \varphi}{\partial \tau^2}\right) \sin\left(\sqrt{\left(\frac{A(s)-s}{1+s}\right)}(1-\tau)\right) d\tau + V(s)$$

If it is not possible to calculate them integrals directly, then we calculate it numerically. To do this, we proceed in the same way as given in [2]. If the Laplace inversion is possible directly for (4) and (5), we shall get our solution. In the other case we use a suitable approximate method and then we use the numerical inversion of the Laplace transform. Considering $\frac{A(s)-s}{1+s} = k(s)$ and using Gauss's formula given in Abramowitz and Stegun [1] we have the following approximations of the integrals:

$$\int_0^1 \begin{pmatrix} p(x) \\ q(x) \end{pmatrix} \cos \sqrt{k(s)} x dx$$
$$\simeq \frac{1}{2} \sum_{i=1}^{N} w_i \begin{pmatrix} p\left(\frac{1}{2}[x_i+1]\right) \\ q\left(\frac{1}{2}[x_i+1]\right) \end{pmatrix} \cos\left(\sqrt{k(s)} \frac{1}{2}[x_i+1]\right),$$

where $\widetilde{\tau} = \frac{x}{2}[x_i+1]$.

$$\int_0^1 \left[\left(F(\tau; s) + \varphi(\tau) - \frac{\partial^2 \varphi(\tau)}{\partial \tau^2}\right) \int_\tau^1 \begin{pmatrix} p(x) \\ q(x) \end{pmatrix} \sin\left(\sqrt{k(s)}\right)(x-\tau)\, dx\right] d\tau$$

$$\simeq \frac{1}{2} \sum_{i=1}^{N} w_i \left[F\left(\frac{x}{2}[x_i+1]; s\right) + \varphi\left(\frac{x}{2}[x_i+1]\right) - \frac{\partial^2 \varphi\left(\frac{x}{2}[x_i+1]\right)}{\partial \widetilde{\tau}^2}\right] \left(\frac{1-\frac{1}{2}[x_i+1]}{2}\right) \times$$

$$\sum_{i=1}^{N} w_j \begin{pmatrix} p\left(\frac{1-\frac{1}{2}[x_i+1]}{2} x_j + \frac{1-\frac{1}{2}[x_i+1]}{2}\right) \\ q\left(\frac{1-\frac{1}{2}[x_i+1]}{2} x_j + \frac{1-\frac{1}{2}[x_i+1]}{2}\right) \end{pmatrix} \times$$

$$\sin\left(\sqrt{k(s)} \left[\left(\frac{1-\frac{1}{2}[x_i+1]}{2}\right) x_j + \frac{1+\frac{1}{2}[x_i+1]}{2} - \frac{1}{2}(x_i+1)\right]\right),$$

where, $\widetilde{\tau} = \frac{x}{2}\left[x_i + 1\right]$. x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n\left(x\right), \qquad \omega_i \ = 2/\left(1 - x_i^2\right)\left[P_n^{'}\left(x\right)\right]^2.$$

Their tabulated values can be found in Abramowitz-Stegun [1] for different values of N.

2.1. **Numerical inversion of Laplace transform.** Now, we have a solution in Laplace transform domain as given in (4) and (5). So we expect to obtain a solution of original problem by means of inverting the Laplace transform. Simple transforms can often be inverted using readily available table. More complex functions can be analytically inverted through the complex inversion formula

$$g\left(t\right) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \exp\left(st\right) G\left(s\right) ds,$$

where, c is a positive real number such that all the poles of function $G\left(s\right)$ lie at the left of the line $\mathrm{Re}\left(s\right) = c$.

Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Pooladi-Darvish [7]. In this work we use the Stehfest's algorithm [12] that is easy to implement. This numerical technique was first introduced by Graver [6] and its algorithm then offered by [12]. Stehfest's algorithm approximates the time domain solution as

$$v\left(x, t\right) \approx \frac{\ln 2}{t}\sum_{n=1}^{2m} \beta_n V\left(x; \frac{n \ln 2}{t}\right),$$

where, m is the positive integer,

$$\beta_n = \left(-1\right)^{n+m}\sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)} \frac{k^m\left(2k\right)!}{\left(m-k\right)!k!\left(k-1\right)!\left(n-k\right)!\left(2k-n\right)!},$$

and $[q]$ denotes the integer part of the real number q.

References

[1] Abramowitz, M., Stegun, I.A., *Hand book of Mathematical Functions*, Dover, New York, 1972.

[2] Ang, W.T., *A Method of Solution for the One-Dimensional Heat Equation Subject to Nonlocal Conditions*, Southeast Asian Bulletin of Mathematics **26** (2001), 185-191.

[3] Babolian, E., Shamloo, A.S., *Numerical solution of Volterra integral and integro-differential equations of convolution type by using operational matrices of piecewise constant orthogonal function*, J. Comp. Appl. Math. **214** (2008), 494-508.

[4] Bahuguna, A., Abbas, S., Shukla, R.K., *Laplace transform method for one -dimensional heat and wave equations with nonlocal conditions*, Inter. Jour. Appl. Math. Stat. **16** (2010).

[5] Ekolin, G., *Finite difference methods for a nonlocal boundary value problem for the heat equation*, BIT **31** (1991), 245-261.

[6] Graver, D.P., *Observing stochastic processes and approximate transform inversion*, Oper. Res. **14** (1966), 444-459.

[7] Hassanzadeh, H., Pooladi-Darvish, M., *Comparison of different numerical Laplace inversion methods for engineering applications*, Appl. Math. Comp. **189** (2007), 1966-1981.

[8] Kong, W., Xionghua, W., *The Laplace transform and polynomial Trefftz method for a class of time dependent PDEs,* Appl. Math. Mod. in press.

[9] Liu, Y., *Numerical solution of the heat equation with non local boundary conditions*, J. Comp. Appl. Math. **110** (1997), 115-127.

[10] Merad, A.,*Adomian decomposition method of solution of parabolic equation to nonlocal conditions*, Int. J. Contemp. Math. Sciences **30** (6) (2011), 1491-1496.

[11] Maleknejad, K., Shamloo, A.S., *Numerical solution of singular Volterra integral equations system of convolution type by using operational matrices*, Appl. Math. Comp. **195** (2008), 500-505.
[12] Stehfest, H., *Numerical Inversion of the Laplace Transform*, Comm. ACM **13**, (1970), 47-49.
[13] Shruti, A.D., *Numerical Solution for Nonlocal Sobolev-type Differential Equations*, Electronic Journal of Differential Equations **19** (2010), 75-83.
[14] Suying, Z., Minzhen, Z., Zichen, D., Wencheng, L., *Solution of nonlinear dynamic differential equations based on numerical Laplace transform inversion*, Appl. Math. Comp. **189** (2007), 79-86.

Department of Mathematics
Larbi Ben M'hidi University
Oum El Bouaghi 04000, Algeria
E-mail address: merad_ahcene@yahoo.fr

Department of Mathematics
Larbi Ben M'hidi University
Oum El Bouaghi 04000, Algeria
E-mail address: aefbouziani@yahoo.fr