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GENERALIZED GROWTH OF HARMONIC FUNCTIONS IN HYPER SPHERES

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ABSTRACT. The paper deals with growth of functions which are harmonic in several variables. The type and lower type with respect to proximate order have been characterized in terms of derivatives of functions harmonic in the neighborhood of origin in $\mathbb{R}^n$. Our results generalize some results of Fryant and Shankar [5].

1. INTRODUCTION

A function $H(x), x = (x_1, x_2, \ldots, x_n)$ which has continuous partial derivatives of second order and satisfies the following Laplace's differential equation

$$\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \cdots + \frac{\partial^2 H}{\partial x_n^2} = 0,$$

is said to be harmonic in $n-$dimensional Euclidean space $\mathbb{R}^n$. If H satisfies Laplace's differential equation throughout a neighborhood of the origin in $\mathbb{R}^n$, it has the spherical harmonic expansion

$$H(x) = \sum_{k=0}^{\infty} H_k(x). \tag{1}$$

where $H_k(x)$ is a harmonic homogeneous polynomial of degree k in $x_1, \ldots, x_n$ [2, p.47]. Such homogenous harmonic polynomials are called spherical harmonics.

Let H_k denote the set of all spherical harmonics degree k in $\mathbb{R}^n$. Then H_k is a vector space of dimension

$$d_k = (n + 2k - 2)\frac{(n + k - 3)!}{k!(n - 2)!}.$$

(See [12, p.145]) $\{Q_k^j\}_{j=1}^{d_k}$ be an orthonormal basis for H_k with respect to the inner product

$$< f, g >= \frac{1}{A_n} \int_{|x|=1} f(x)g(x)d\sigma_1$$

where A_n denotes the area of the unit sphere $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} = 1$, and $d\sigma_1$ is the element of surface area on this sphere. Then the series (1) can be written as

$$H(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} C_{k_j} r^k Q_k^j(x/r), \tag{2}$$

where

$$C_{k_j} = \frac{1}{A_n} \int_{|x|=1} H(x)Q_k^j(x)d\sigma_1,$$

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and $|x| = r$.

If the series (2) converges uniformly on the sphere $|x| = \delta$, then we have

$$C_{k_j} = \frac{1}{\delta^{2k+n-1} A_n} \int_{|x|=\delta} H(x) Q_k^j(x) d\sigma,$$

where $d\sigma = \delta^{n-1} d\sigma_1$ is the element of surface area on the sphere $|x| = \delta$.

Sometimes it is useful to study the growth of harmonic functions H which are expressed in terms of derivatives of H at the origin. While such results are essentially equivalent to those expressed in terms of the spherical harmonic coefficients C_{k_j}. Results of the one kind can not easily be obtained directly from the other, and thus each requires a separate study. This fact leads us to deal herewith polynomials of several variables, for which the simple relationship between coefficients and derivatives at the origin that obtains in classical function theory (i.e., if $f(z) = \sum a_n z^n$, then $a_n = f^n(0)/n!$) is not available. The derivatives of a harmonic function H at the origin are equal to complicated linear combinations of the spherical harmonic coefficients.

For each $n-$tuple $a = (a_1, a_2, \ldots, a_n)$ of non-negative integers, the norms of the k^{th} gradients of H at origin i.e., the derivatives at origin is denoted by $|\nabla_k H(0)|$ and defined as

$$|\nabla_k H(0)| = \left(\frac{k!}{2^k} \sum_{|a|=k} \frac{[D^a H(0)]^2}{a!} \right)^{1/2} \tag{3}$$

where $|a| = a_1 + a_2 + \ldots a_n, a! = a_1!, a_2!, \ldots, a_n!$,

$$D^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \partial x_2^{a_2} \ldots \partial x_n^{a_n}}.$$

It has been shown [7] that the series (1) converges absolutely and uniformly on compact subsets of the open ball $|x| < R$, where

$$R^{-1} = \sqrt{2} \limsup_{k\to\infty} (|\nabla_k H(0)|/k!)^{1/k}. \tag{4}$$

In view of definition (3) and the result (4), it follows immediately that the series (1) converges absolutely and uniformly on compact subsets of open ball $|x| < R$, where

$$R^{-1} = \limsup_{k\to\infty} (|\nabla_k H(0)|/k!)^{1/k} \tag{5}$$

and such convergence can not obtain within any larger ball centered at origin.

It should be noted that if $H_j(x)$ is a spherical harmonic (i.e., a harmonic homogeneous polynomial) of degree $j \neq k$, then $|\nabla_k H_j(0)| = 0$.

The norm $\|H_k\|_2$ of spherical harmonic H_k of degree k can be expressed in terms of $|\nabla_k H_k|$. (See [1] and [6]).

$$\frac{1}{A_n} \int_{|x|=1} [H_k(x)]^2 d\sigma_1 = \frac{\sqrt{n/2}}{k! \sqrt{(k+n/2)}} |\nabla_k H_k|^2. \tag{6}$$

If f is any function which is square integrable on the unit sphere $|x| = 1$, it is easy to see that

$$\|f\|_2 \leq \|f\|_\infty$$

where

$$\|f\|_\infty = \sup_{|x|=1} |f(x)|,$$

and

$$\|f\|_2 = \left[\frac{1}{A_n}\int_{|x|=1} f^2(x)d\sigma_1\right]^{1/2}.$$

Motivation for the study of the growth of non-entire harmonic functions came from the wealth of results for analytic functions of single complex variable that obtain as consequences of the growth properties of analytic functions. Since an analytic function is a linear combination of harmonic functions in $\mathbb{R}^3$, it is natural to expect that a variety of results for harmonic function in $\mathbb{R}^n$ can be derived from the growth properties of non-entire harmonic functions in $\mathbb{R}^n$. Such results can be found in [3, 4, 5, 8]. All these results characterized the order and type of non-entire harmonic functions in terms of its spherical harmonic coefficients and in terms of derivatives at the origin. It has been noticed that these results are inadequate for comparing the growth of those non-entire harmonic functions which are of the same positive finite order but their types are infinity. To refine this scale we shall use the concept of proximate order introduced by Valiron [13]. Moreover, we restrict attention here to the fundamental objective of characterizing the type and lower type of harmonic function $H(x)$ as $|x| \to R$, with respect to a proximate order, in terms of the k^{th} gradients i.e., the derivatives of H at origin $|\nabla_k H(0)|$. These results are analogs of the classical results for analytic functions of a single complex variable. Also, Kumar [5, 11] and Kumar and Kasana [10] studied the growth of harmonic functions in $\mathbb{R}^n, n \geq 3$.

2. Some Classical Results

In this section we shall prove some classical results for functions analytic in finite disk.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be analytic in a disk $|z| < R$. To study the growth of $f(z)$ its order ρ_R is defined as

$$\rho_R = \limsup_{r\to R^-} \frac{\log^+\log^+ M(r)}{\log(R/R-r)}(0 \leq \rho_R \leq \infty) \tag{7}$$

where $M(r) = \max_{|z|=r}|f(z)|$, $r < R$.

Now we consider a real-valued function $\rho_R(r)(0 < r < R)$ having the following properties:

(i) $\rho_R(r)$ is positive, continuous and piecewise differentiable in $0 \leq r_0 < r < R$.
(ii) $\lim_{r\to R^-} \rho_R(r) \to \rho_R$; $(0 < \rho_R < \infty)$
(iii) $\lim_{r\to R^-} -\rho_R'(r)(R-r)\log((R-r)/R) \to 0$

where $\rho_R'(r)$ denotes the derivative of $\rho_R(r)$.

A function $\rho_R(r)$ satisfying the conditions (i), (ii) and (iii) is said to be a proximate order. For a function f, analytic in $|z| < R$ having nonzero finite order ρ_R, let

$$\begin{matrix} T_R^* \\ t_R^* \end{matrix} = \lim_{r\to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r)}{(R/R-r)^{\rho_R(r)}}. \tag{8}$$

The numbers T_R^* and t_R^* are said to be generalized type and generalized lower type of f with respect to the proximate order $\rho_R(r)$. If T_R^* is different from zero and infinity, then the function $\rho_R(r)$ satisfying (i),(ii) and (iii) is called a proximate order of f.

It is easily seen that if $\rho_R(r)$ is a proximate order of an analytic function f then $\rho_R(r) + \frac{\log T_R^*}{\log(R/R-r)}$ is also a proximate order for the same function. This show that the proximate order of an analytic function is not unique.

The following theorem shows that there exists a proximate order for every function, analytic in $|z| < R$ and having nonzero finite order.

Theorem 1. *Let f be analytic in a disk $|z| < R$ and have order ρ, $0 < \rho < \infty$. Then for every T_R^*, $0 < T_R^* < \infty$, there exists a proximate order of f satisfying (i) to (iii).*

Proof. Let

$$h(r) = (R/R-r)^\rho \log M(r)/T_R^*.$$

Putting $x = \log(R/R-r)$ and $h_1(x) = \log h(1-e^{-x})$, then

$$\limsup_{x\to\infty} \frac{h_1(x)}{x} = 0. \tag{9}$$

First, we assume that $\limsup_{x\to\infty} h_1(x) = \infty$ and $y = h_2(x)$ be the boundary curve of the smallest convex domain containing the curve $y = h_1(x)$ and the positive ray of $x-$axis. By making suitable changes to the small neighborhoods of the vertices in this curve, we may assume without loss of generality that the function $h_2(x)$ is differentiable in $0 \le x < \infty$. The curve $y = h_2(x)$ has the following properties:

(a) The curve $y = h_2(x)$ is concave in the sense that a chord joining any two points of the curve lies below the curve.

(b) The function $h_2(x)/x$ is monotonic decreasing and non negative it gives that this function must tend to a limit as $x \to \infty$.

Using (9) with the fact that the curves $y = h_2(x)$ and $y = h_1(x)$ have infinitely many common points $\{x_n\}$ such that $x_n \to \infty$, we obtain

$$\lim_{x\to\infty} \frac{h_2(x)}{x} = 0; \tag{10}$$

$$h_1(x) \le h_2(x) \texttt{ for all } x \ge 0. \tag{11}$$

In the consequence of (10), we have

$$\lim_{x\to\infty} h_2'(x) = 0. \tag{12}$$

Now in view of (11), we have

$$\log M(r) \le T_R^*(R/R-r)^{\rho+\frac{h_2(\log(R/R-r))}{\log(R/R-r)}}. \tag{13}$$

Set

$$\rho_R(r) = \rho + \frac{h_2(\log(R/R-r))}{\log(R/R-r)}. \tag{14}$$

It is clear that $\rho_R(r)$ is positive and differentiable in $0 \le r_0 < r < R$. Using (10) in (14), it gives $\rho_R(r) \to \rho$ as $r \to R$. Further,

$$-(R-r)\rho_R'(r)\log((R-r)/R) = h_2'(\log(R/(R-r))) - \frac{h_2(\log(R/(R-r)))}{\log(R/(R-r))}.$$

Again using (10) and (12) in above equality, we get

$$-(R-r)\rho_R'(r)\log((R-r)/R) \to 0 \text{ as } r \to R.$$

In view of (13) and (10), we get

$$\log M(r) \le T_R^*(R/(R-r))^{\rho_R(r)} \tag{15}$$

for all r in $0 \le r_0 < r < R$ and that there exists a sequence $r_n \to R$ as $n \to \infty$ on which

$$\log M(r_n) = T_R^*(R/(R-r_n))^{\rho_R(r_n)}. \tag{16}$$

Thus, $\rho_R(r)$ defined by (14) is a proximate order of f. □

To study the properties of a proximate order of an analytic function, first we need the concept of slowly increasing function. A real valued function $L(r)$, $0 < r < R$, is said to be slowly increasing if, for every k satisfying $1 < k < \infty$,

$$\lim_{x\to R} \frac{L\left(r + \frac{1}{k}((R-r)/R)\right)}{L(r)} = 1. \tag{17}$$

Now we prove

Theorem 2. *Let $\rho_R(r)$ be a proximate order of a function f analytic in a disk $|z| < R$ and having order ρ. Then*

$$L(r) = (R/(R-r))^{\rho_R(r)-\rho} \tag{18}$$

is a slowly increasing function of r on $0 < r < R$ and

$$(R/(R-r))^{\rho_R(r)} \tag{19}$$

is a monotonically increasing function of r in $0 \le r_0 < r < R$ and tends to ∞ as $r \to R$.

Proof. For $k > 1$, in view of mean value theorem and (iii), there exists $k' > 1$ such that for any $\varepsilon > 0$,

$$\begin{aligned}
\log \frac{L\left(r + \frac{1}{k}((R-r)/R)\right)}{L(r)} &= \left[\rho_R(r) + \frac{1}{k}((R-r)/R) - \rho\right] \\
&- \left[\log\left(1 - \frac{1}{k}\right) + \log(R/(R-r))\right] + (\rho_R(r) - \rho)\log((R-r)/R) \\
= &- \left[\rho_R(r) + \frac{1}{k}((R-r)/R) - \rho\right] \log\left(1 - \frac{1}{k}\right) \\
&- \left[\rho_R(r) + \frac{1}{k}((R-r)/R) - \rho_R(r)\right] \log((R-r)/R) \\
< &- \left[\rho_R(r) + \frac{1}{k}((R-r)/R) - \rho\right] \log\left(1 - \frac{1}{k}\right) + \frac{\varepsilon \log((R-r)/R)}{k(1-k')\log\left(1 - \frac{1}{k'}\right)(R-r)/R}.
\end{aligned}$$

Proceeding to limits, we get

$$\lim_{r\to R} \log \frac{L\left(r + \frac{1}{k}((R-r)/R)\right)}{L(r)} = 0$$

which proves the result (18).

To prove that $(R/(R-r))^{\rho_R(r)}$ is monotonically increasing, we have

$$\frac{d}{dr}[(R/(R-r))^{\rho_R(r)}] = \frac{\rho_R(r)}{r^2}(R/(R-r))^{\rho_R(r)+1} + \rho_R'(r)(R/(R-r))^{\rho_R(r)} \log(R/(R-r))$$

in view of (iii), we get

$$> \frac{(\rho - \varepsilon)}{R^2}(R/(R-r))^{\rho_R(r)+1} > 0.$$

Hence the proof of (19) is completed. □

Since $(R/R-r)^{\rho_R(r)}$ is a monotonically increasing function of r for $0 < r_0 < r < R$, a single valued function $\rho(x)$ can be defined for $x > x_0$, such that

$$x = (R/R-r)^{\rho_R(r)+1} \Leftrightarrow (R/R-r) = \varphi(x). \tag{20}$$

Now we shall prove

Lemma 1. *For the function $\varphi(t)$ defined in* (20) *we have*

$$\lim_{t\to\infty}\frac{d(\log\varphi(t))}{d(\log t)}=\frac{1}{\rho+1}$$

and, for $0<\eta<\infty$,

$$\lim_{t\to\infty}\frac{\varphi(\eta t)}{\varphi(t)}=\eta^{1/\rho+1}. \tag{21}$$

Proof. By the definition of proximate order and (20), we have

$$\begin{aligned}\frac{d[\log\varphi(t)]}{d[\log t]} &= \frac{1}{\rho+1+\rho_R'(r)(R-r)\log((R-r)/R)}\\ &\Leftrightarrow\ \lim_{t\to\infty}\frac{d[\log\varphi(t)]}{d[\log t]}=\frac{1}{\rho+1}.\end{aligned}$$

Now for a given $\varepsilon>0$ and $t>t_0$, we get

$$\int_t^{\eta t}\left(\frac{1}{\rho+1}-\varepsilon\right)d[\log t]<\int_t^{\eta t}d[\log\varphi(t)]\left(\frac{1}{\rho+1}+\varepsilon\right)d[\log t]$$

$$\left(\frac{1}{\rho+1}-\varepsilon\right)\log\eta<\log\frac{\varphi(\eta t)}{\varphi(t)}<\left(\frac{1}{\rho+1}+\varepsilon\right)\log\eta$$

$$\eta^{(1/\rho+1-\varepsilon)}<\frac{\varphi(\eta t)}{\varphi(t)}<\eta^{(1/\rho+1+\varepsilon)}$$

or

$$\lim_{t\to\infty}\frac{\varphi(\eta t)}{\varphi(t)}=\eta^{1/\rho+1}.$$

□

Theorem 3. *Let $f(z)=\sum_{k=0}^{\infty}a_kz^k$, analytic in a disk $|z|<R$, have order ρ $(0<\rho<\infty)$ and proximate order $\rho_R(r)$. Then the type T_R^* of f with respect to the proximate order $\rho_R(r)$ is given by*

$$\frac{(\rho+1)^{\rho+1}}{\rho^\rho}T_R^*=\limsup_{k\to\infty}\left[\frac{\varphi(k)\log^+|a_k|}{k}\right]^{\rho+1}. \tag{22}$$

Proof. From (8) for every $\varepsilon>0$ and for all r sufficiently close to R,

$$\log M(r)<(T_R^*+\varepsilon)(R/(R-r))^{\rho_R(r)}.$$

In view of Cauchy's estimate, we get

$$\log^+|a_k|<(T_R^*+\varepsilon)(R/(R-r))^{\rho_R(r)}-k\log r. \tag{23}$$

To estimate the right hand side let

$$(R/(R-r))^{\rho_R(r)+1}=k/\rho(T_R^*+\varepsilon)$$

and

$$V(k)=1-\frac{1}{\varphi(k/\rho(T_R^*+\varepsilon))}.$$

Then $V(k)\to1$ as $k\to\infty$. Substituting

$$P(k)=\frac{1}{1+\rho V(k)}.$$

By (23), for all sufficiently large k, we get

$$\log^+|a_k|<\frac{(T_R^*+\varepsilon)^{\rho(k)}k^{1-P(k)}}{1-P(k)}-k\log V(k)$$

$\Rightarrow$

$$\frac{\varphi(k)\log^+|a_k|}{k} < \frac{(T_R^*+\varepsilon)^{P(k)}}{\rho^{1-P(k)}}\frac{\varphi(k)}{k^{P(k)}}\left(1-\frac{\rho k^{P(k)}\log V(k)}{(\rho T_R^*+\varepsilon)^{P(k)}}\right). \tag{24}$$

Since

$$\lim_{k\to\infty}\frac{\varphi(k)}{k^{P(k)}}\to 1,\ \lim_{k\to\infty}\frac{k^{P(k)}\log V(k)}{\rho(T_R^*+\varepsilon)^{P(k)}}\to -1$$

and $\rho_R(r)\to\rho$ as $r\to R$, (24) gives that

$$\frac{(\rho+1)^{\rho+1}}{\rho^\rho}T_R^*\geq\limsup_{k\to\infty}\left[\frac{\varphi(k)\log^+|a_k|}{k}\right]^{\rho+1}. \tag{25}$$

To prove reverse inequality in (25), let β be defined by the equation

$$\limsup_{k\to\infty}\left[\frac{\varphi(k)\log^+|a_k|}{k}\right]^{\rho+1}=\frac{(\rho+1)^{\rho+1}}{\rho^\rho}\beta.$$

Then, for all r sufficiently close to R and for every $\alpha>\beta$,

$$|a_k|r^k<\exp\left[\frac{k(1+\rho)\beta^{1/\rho+1}}{\rho^{\rho/(1+\rho)}\varphi(k)}+k\log r\right].$$

In view of (20) and Lemma 1, we get

$$|a_k|r^k<\exp\left[\frac{k(1+\rho)}{\rho\varphi(k/\alpha\rho)}-n((R-r)/R)\right].$$

Thus, the maximum term $\mu(r)$ of f for $|z|=r$ satisfies

$$\log\mu(r)<\max_{k\geq 0}\left[\frac{k(1+\rho)}{\rho\varphi(k/\alpha\rho)}-k((R-r)/R)\right].$$

To estimate the value of right hand side we assume that

$$k=[\alpha\rho(R/(R-r))^{\rho_R(r)+1}],$$

which gives that

$$\frac{\log\mu(r)}{(R/(R-r))}<\alpha.$$

Proceeding to limit as $r\to R$, we get

$$T_R^*\leq\alpha,$$

since $\alpha>\beta$, it gives that

$$T_R^*\leq\beta. \tag{26}$$

Combining (25) and (26), the proof is completed. □

Theorem 4. *Let $f(z)=\sum_{k=0}^\infty a_kz^k$, analytic in the disk $|z|<R$, having order ρ $(0<\rho<\infty)$ and proximate order $\rho_R(r)$ be such that $v(k)=|a_{k-1}/a_k|$ forms a nondecreasing function of k for $k>k_0$. Then the lower type t_R^* of f, with respect to the proximate order $\rho_R(r)$ is given by*

$$\frac{(\rho+1)^{\rho+1}}{\rho^\rho}t_R^*=\liminf_{k\to\infty}\left[\frac{\varphi(k)\log^+|a_k|}{k}\right]^{\rho+1}. \tag{27}$$

Proof. Since $v(k)$ forms a nondecreasing function of k, it can be seen that $v(k+1) > v(k)$ for infinitely many values of k and $v(k) \to 1$ as $k \to \infty$. When $v(k+1) > v(k)$, the maximum term $\mu(r)$ and the central index $\nu(r)$ of f, for $v(k) \le r < v(k+1)$, are given by

$$\mu(r) = |a_k| r^k \text{ and } \nu(r) = k.$$

First, let $0 < t_R^* < \infty$. Applying Lemma 1 of Kapoor [8] for finite disk we get for given $\varepsilon > 0$ and for all r sufficiently close to R.

$$\log \mu(r) > (t_R^* - \varepsilon)(R/(R-r))^{\rho_R(r)}.$$

If $a_{k_1} z^{k_1}$ and $a_{k_1} z^{k_2}$ are consecutive maximum terms of f and $k_1 \le k \le k_2 - 1$. Then

$$v(k_1+1) = v(k_1+2) = \dots = v(k+1) = \dots v(k_2)$$

and

$$|a_k| r^k = |a_{k_2}| r^{k_2} \text{ for } r/k = v(n+1).$$

Therefore,

$$\log^+ |a_k| + k \log R\varphi(k+1) > (t_R^* - \varepsilon) \left[\frac{R - \varphi(k+1)}{R\varphi(k+1)} \right]^{-\rho\varphi(k+1)},$$

since $-\log Rx \ge \frac{1}{x} - \frac{1}{R}$ for $x > 0$, so we get

$$\frac{\varphi(k) \log^+ |a_k|}{k} > \frac{(t_R^* - \varepsilon)\varphi(k)}{k} \left[\frac{R - v(k+1)}{Rv(k+1)} \right]^{-\rho\varphi(k+1)} - \frac{k}{t_R^* - \varepsilon} \frac{R - v(k+1)}{R}. \tag{28}$$

Let

$$u(x) = [(R-x)/R]^{-\rho(x)} + \frac{k}{t_R^* - \varepsilon}((R-x)/R).$$

The minimum value of the function $u(x)$ occurs at a point $x_1 = x_1(k)$ given by, for k sufficiently large,

$$((R-x)/R)^{-\rho(x)-1} = \frac{k}{(t_R^* - \varepsilon)(\rho + o(1))}.$$

In view of definition $\varphi(x)$, we get

$$((R-x)/R)^{-1} = \varphi \left[\frac{k}{(t_R^* - \varepsilon)(\rho + o(1))} \right].$$

Thus, for sufficiently large value of k, we get

$$\begin{aligned} \inf_{0<x<R} u(x) &= \left[k/(t_R^* - \varepsilon)(\rho + o(1))\varphi \left(\frac{k}{(t_R^* - \varepsilon)(\rho + 0o(1))} \right) \right. \\ &\quad \left. + \frac{k}{(t_R^* - \varepsilon)\varphi(k/(t_R^* - \varepsilon)(\rho + o(1)))} \right] \\ &= \frac{k}{(t_R^* - \varepsilon)} \varphi \left[\frac{k}{(t_R^* - \varepsilon)(\rho + o(1))} \frac{(1 + \rho + o(1))}{\rho + o(1)} \right]. \end{aligned}$$

So in view of (28) and Lemma 1, we obtain

$$\liminf_{k\to\infty} \left[\frac{\varphi(k) \log^+ |a_k|}{k} \right]^{\rho+1} \ge \frac{(\rho+1)^{\rho+1}}{\rho^\rho} t_R^*. \tag{29}$$

Inequality (29) obviously holds if $t_R^* = 0$.

To prove the equality in (29) we shall prove the strict inequality can not hold in (29). For if it holds, then there exists a number $\gamma > t_R^*$, such that

$$\liminf_{k\to\infty}\left[\frac{\varphi(k)\log^+|a_k|}{k}\right]^{\rho+1} = \frac{(\rho+1)^{\rho+1}}{\rho^\rho}\gamma.$$

Let γ_1 be such that $\gamma > \gamma_1 > t_R^*$. Then, for all k sufficiently large we get

$$\log^+|a_k| > \frac{k}{\varphi(k)}\frac{1+\rho}{\rho^{\rho/(1+\rho)}}\gamma_1^{1/(1+\rho)}.$$

For all r sufficiently close to R and sufficiently large k,

$$\begin{aligned}\log M(r) &> \frac{k}{\varphi(k)}\frac{(1+\rho)}{\rho^{\rho/(1+\rho)}}\gamma_1^{1/(1+\rho)} + k\log r\\ &= \frac{k}{\varphi(k)}\frac{(1+\rho)}{\rho^{\rho/(1+\rho)}}\gamma_1^{1/(1+\rho)} - k(1-r).\end{aligned}$$

Let

$$k = [\gamma_1\rho((R-r)/R)^{-\rho_R(r)-1}].$$

Then, in view of definition of $\varphi(k)$, we get

$$\log M(r) > \frac{k}{\rho\varphi(k/\gamma_1\rho)} = \gamma_1(R/(R-r))^{\rho_R(r)}$$

so that $t_R^* \ge \gamma_1$, which is a contradiction. Hence the proof is completed. □

3. Main Results

In this section, we shall prove our main results but first we shall prove two lemmas. Lemma 2 gives the reverse inequality in $\|H_k\|_2 \le \|H_k\|_\infty$ and Lemma 3 gives upper and lower bounds on the maximum modulus of a harmonic function H, in terms of analytic function of r. These bounds are sufficiently good to determine the radius of convergence, order and type with respect to a proximate order, by the direct application of classical theory of analytic functions. These lemmas were first obtained in [4]. But for the sake of completeness, their derivation have been given here.

Lemma 2. *Let $H_k \in \aleph_k$. Then*

$$\|H_k\|_\infty \le \sqrt{d_k}\|H_k\|_2,$$

where

$$d_k = (n+2k-2)\frac{(n+k-3)!}{k!(n-2)!}$$

is the dimension of $\aleph_k$.

Proof. Let

$$Z_k(x,y) = \sum Q_k^i(x)\sum Q_k^j(y),$$

where $\{Q_k^j\}_{i=1}^{d_k}$ is an orthonormal basis for $\aleph_k$. Then we have

$$H_k(x) = \frac{1}{A_n}\int_{|y|=1} Z_k(x,y)H_k(y)d\sigma_1(y).$$

In view of Schwartz inequality, we get

$$|H_k(x)| \le \|Z_k(x,.)\|_2\|H_k\|_2$$

for all x in the unit sphere $|x| = 1$. This leads to

$$\|H_k\|_\infty \le \|Z_k(x,.)\|_2\|H_k\|_2.$$

Since

$$\begin{aligned} \|Z_k(x,.)\|_2^2 &= \frac{1}{A_n}\int_{|y|=1} Z_k(x,y)Z_k(x,y)d\sigma_1(y) \\ &= \sum_{i=1}^{d_k}[Q_k^j(x)]^2 = d_k. \end{aligned}$$

Therefore $\|Z_k(x,.)\|_2 = \sqrt{d_k}$ which immediately gives the required result. □

Lemma 3. *Let $H(k) = \sum_{k=0}^{\infty} H_k(x)$ is uniformly convergent in a neighborhood of the origin in $\mathbb{R}^n$. Then for all $r < R$.*

$$M_2(r) \le M(r) \le N(r)$$

where

$$\begin{aligned} M(r) &= \max_{|x|=r}|H(x)|, \\ M_2(r) &= \left[\Gamma(n/2)\sum_{k=0}^{\infty}\frac{|\nabla_k H(0)|^2}{k!\Gamma(k+n/2)}r^{2k}\right]^{1/2}, \end{aligned}$$

and

$$N(r) = \sqrt{\Gamma(n/2)}\sum_{k=0}^{\infty}\sqrt{d_k}\frac{|\nabla_k H(0)|}{\sqrt{k!\Gamma(k+n/2)}}r^k.$$

Here the upper bound of $M(r)$ holds for all $r \ge 0$, and the lower bound $M(r)$ obtains for all r such that the spherical harmonic series H is uniformly convergent on the sphere $|x| = r$.

Proof. Since

$$\begin{aligned} M_2^2(r) &= \frac{1}{A_n r^{n-1}}\int_{|x|=r} H^2(x)d\sigma(x) \\ &\le M(r)\frac{1}{A_n r^{n-1}}\int_{|x|=r} d\sigma(x) = M(r) \end{aligned}$$

and the homogeneous, harmonic polynomials $H_k(x)$ are orthogonal over any sphere $|x| = r$ centered at the origin, we get

$$\begin{aligned} M_2^2(r) &= \left(\frac{1}{A_n r^{n-1}}\right)\sum_{k=0}^{\infty}\int_{|x|=1} H_k^2(x)d\sigma(x) \\ &= \left(\frac{1}{A_n r^{n-1}}\right)\sum_{k=0}^{\infty}\int_{|x|=1} r^{2k}H_k^2(y)r^{n-1}d\sigma_1 \\ &= \sum_{k=0}^{\infty}\|H_k\|_2^2 r^{2k} \\ &= \sum_{k=0}^{\infty}\frac{\Gamma n/2}{k!\Gamma(k+n/2)}|\nabla_k H_k|^2 r^{2k}. \end{aligned}$$

Since $|\nabla_k H(0)| = |\nabla_k H_k|$, it gives

$$M_2^2(r) = \Gamma n/2\sum_{k=0}^{\infty}\left(|\nabla_k H(0)|^2/k!(k+n/2)\right)r^{2k}.$$

To obtain the right half inequality, in view of Lemma 2, we have

$$\begin{aligned} |H_k(x)| &= \left|\sum_{k=0}^{\infty} H_k(x)\right| \le \sum_{k=0}^{\infty} |H_k(x)| \\ &\le \sum_{k=0}^{\infty} \|H_k\|_\infty \le \sum_{k=0}^{\infty} \sqrt{d_k}\|H_k\|^2 \\ &= \sqrt{\Gamma n/2}\sum_{k=0}^{\infty} \sqrt{d_k}\frac{|\nabla_k H_k|}{\sqrt{k!\Gamma(k+n/2)}}r^k \\ &= \sqrt{\Gamma(n/2)}\sum_{k=0}^{\infty} \sqrt{d_k}\frac{|\nabla_k H_k|}{\sqrt{k\Gamma(k+n/2)}}r^k. \end{aligned}$$

Hence the proof is completed. □

Now we shall prove our main results.

Theorem 5. *Let H be harmonic function in a neighborhood of the origin in $\mathbb{R}^n$, have order $\rho(0 < \rho < \infty)$ and proximate order $\rho_R(r)$. Then the type T_R^* of H with respect to the proximate order $\rho_R(r)$ in the ball $|x| < R$ is given by*

$$\frac{(\rho+1)^{\rho+1}}{\rho^\rho}T_R^* = \limsup_{k\to\infty}\left[\frac{\varphi(k)\log^+\left[\frac{|\nabla_k H(0)|}{k!}\right]}{k}\right]^{\rho+1}. \tag{30}$$

Proof. It has been proved [4, Thm. 1] that the order of $[M_2(Z)]^2, H(x)$ and $N(z)$ are all equal. Thus the order of $[M_2(z)]^2$ is equal to ρ, and hence the type $T_R^*(M_2^2)$ with respect to proximate order $\rho_R(r)$, according to (25) is given by

$$T_R^*(M_2^2)\frac{\rho^\rho}{(\rho+1)^{\rho+1}} = \limsup_{k\to\infty}\left[\frac{\varphi(2k)\log^+\left[\frac{\Gamma n/2|\nabla_k H(0)|^2}{k!\Gamma(k+n/2)}R^{2k}\right]}{2k}\right]^{\rho+1}.$$

Since $\Gamma(k+n/2)/k! \sim k^{n/2-1}$, this leads to

$$\begin{aligned} &= \frac{\rho^\rho}{(\rho+1)^{\rho+1}}\limsup_{k\to\infty}\left[\frac{\varphi(2k)\log^+\left[\frac{|\nabla_k H(0)|}{k!}R^k\right]}{2k}\right]^{\rho+1} \\ &= \frac{2\rho^\rho}{(\rho+1)^{\rho+1}}\limsup_{k\to\infty}\left[\frac{\varphi(k)\log^+\left[\frac{|\nabla_k H(0)|}{k!}R^k\right]}{k!}\right]^{\rho+1}. \end{aligned}$$

Since $[M_2(Z)]^2$ has a Taylor series expansion with real, nonnegative coefficients,

$$\max_{|z|=r}[M_2(Z)]^2 = [M_2(r)]^2.$$

So we have

$$\begin{aligned} \frac{2\rho^\rho}{(\rho+1)^{\rho+1}}\limsup_{k\to\infty}\left[\frac{\varphi(k)\log^+\left[\frac{|\nabla_k H(0)|}{k!}R^k\right]}{k}\right]^{\rho+1} &= \limsup_{r\to R}\frac{\log^+[M_2(r)]^2}{(R/(R-r))^{\rho_R(r)}} \\ &\le 2\limsup_{r\to R}\frac{\log^+ M_2(r)}{(R/(R-r))^{\rho_R(r)}}. \end{aligned}$$

Applying Lemma 3, we get

$$\leq 2 \limsup_{r\to R} \frac{\log^+ M(r,H)}{(R/(R-r))^{\rho_R(r)}}. \tag{31}$$

Since the order of $N(z)$ is also ρ, again applying the result (25) the type $T^*_R(H)$ with respect to a proximate order $\rho_R(r)$ is given by

$$T^*_R(N) = \frac{\rho^\rho}{(\rho+1)^{\rho+1}} \limsup_{k\to\infty} \left\{ \frac{\varphi(k) \log^+ \frac{\sqrt{\Gamma n/2}|\nabla_k H(0)|}{\sqrt{d_k}\sqrt{k!\Gamma(k+n/2)}} R^k}{k} \right\}^{\rho+1}.$$

$$= \frac{\rho^\rho}{(\rho+1)^{\rho+1}} \limsup_{k\to\infty} \left\{ \frac{\varphi(k) \log^+ \frac{|\nabla_k H(0)|}{k!} R^k}{k} \right\}^{\rho+1}. \tag{32}$$

Combining (31)) and (32), we get

$$\frac{\rho^\rho}{(\rho+1)^{\rho+1}} \limsup_{k\to\infty} \left\{ \frac{\varphi(k) \log^+ \frac{|\nabla_k H(0)|}{k!} R^k}{k} \right\}^{\rho+1} = T^*_R(M_2) \leq T^*_R(H)$$

$$\leq T^*_R(N) = \frac{\rho^\rho}{(\rho+1)^{\rho+1}} \limsup_{k\to\infty} \left\{ \frac{\varphi(k) \log^+ \frac{|\nabla_k H(0)|}{k!} R^k}{k} \right\}^{\rho+1}.$$

Hence the proof is completed. □

Theorem 6. *Suppose H is harmonic function in a neighborhood of the origin in $\mathbb{R}^n$, have order $\rho(0 < \rho < \infty)$ and proximate order $\rho_R(r)$ be such that $\Omega(k) = \left\{ \frac{|\nabla_{k-1} H(0)|}{|\nabla_k H(0)|} \right\}$ forms a nondecreasing function of k for $k > k_0$. Then the lower type t^*_R of H, with respect to proximate order $\rho_R(r)$ in the ball $|x| < R$ is given by*

$$\frac{(\rho+1)^{\rho+1}}{\rho^\rho} t^*_R = \liminf_{k\to\infty} \left[\frac{\varphi(k) \log^+ \frac{|\nabla_k H(0)|}{k!} R^k}{k} \right]^{\rho+1}.$$

Proof. Since $\left\{ \frac{|\nabla_{k-1} H(0)|}{(k-1)!} / \frac{|\nabla_k H(0)|}{k!} R \right\}$ forms a nondecreasing function of k for $k > k_0$ if $\{|\nabla_{k-1} H(0)|/|\nabla_k H(0)|\}$ forms a nondecreasing function of $k > k_0$. Now in view of result (27) the proof follows in a similar manner as Theorem 5. □

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