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EXTENSION OF THE BERTRAND THEOREM TO THE RELATIVISTIC CLASSICAL MECHANICS

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Abstract. A new approach of the relativistic motion of a particle in a central field is presented. It is based on a modified form of the Hamilton function of the particle. This form allows us to extend the Bertrand theorem from the non-relativistic classical mechanics to the relativistic classical mechanics.

1. Introduction

In the non-relativistic classical mechanics, it is assumed that there exist central forces independent of velocities, which derive from simple potentials $U(r)$. The Hamilton function of a particle with the mass m_0 moving in such a central field with the linear momentum $\mathbf{p}$ is

$$H = \frac{p^2}{2m_0} + U(r). \tag{1}$$

During the motion of the particle, the angular momentum $\mathbf{l}$ with respect to the field centre and the energy E are conserved. The conservation law of the angular momentum has two important consequences. Firstly, the trajectory of the particle lies at a plane perpendicular to $\mathbf{l}$. Secondly, the areal velocity of the particle is constant (the Kepler second law). If we choose a system of Cartesian coordinates $Oxyz$ with the origin O at the field centre and with the axis Oz directed along the angular momentum, the motion takes place in the plane xOy. The field symmetry suggests using the polar coordinates (r, φ) as generalized coordinates in the study of this motion. The Hamilton function of the particle is

$$H = \frac{p_r^2}{2m_0} + \frac{l^2}{2m_0 r^2} + U(r) = E, \tag{2}$$

where p_r is the radial component of the linear momentum.

According to the Bertrand theorem [1], in the non-relativistic classical mechanics, the motions in the Coulomb field, with

$$U(r) = -\frac{\alpha}{r} \tag{3}$$

and in the field of the isotropic three-dimensional oscillator, with

$$U(r) = \frac{1}{2} m_0 \omega_0^2 r^2 \tag{4}$$

are the only motions in central fields in which all finite trajectories are closed.

In general, the trajectories in these fields are conic sections [2, 3]. In the case of the Coulomb field, the conic section can be a circle, an ellipse, a parabola or an hyperbola. The field centre lies at one of the conic-section foci. For circular and elliptical trajectories, the ratio of the cubes of the radiuses or major semi-axes to the squares of the periods of

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revolution is constant (the Kepler third law). In the case of the isotropic three-dimensional oscillator, the conic section is a circle or an ellipse. The field centre lies at centre of the conic section. In both cases, all finite trajectories (circles and ellipses) are closed.

For these potentials, extra constant of the motion arise, in addition to those expected from geometric invariance of the potentials [4, 5]. In the case of the Coulomb field, because the orbit has only a dynamical axis of symmetry, the extra constant is a vector (the Runge-Lenz vector) [6, 7, 8, 9, 10]. In the case of the isotropic three-dimensional oscillator, taking into account that the orbit has two dynamical axes of symmetry, the extra constant is a symmetric second-rank tensor [4, 5].

From the symmetry point of view, these potentials admit a Lie group larger than the $SO(3)$. For the Coulomb problem, the corresponding group is $SO(4)$ for bound states and $SO(3,1)$ for scattering states [11]. For the isotropic oscillator, this group is $SU(3)$ [12].

The extension of the Bertrand theorem to spherical geometry led again to two potential for which closed orbits exist [13]. In the limit of vanishing curvature, these potentials reduce to the Coulomb and oscillator potentials. For a combined potential consisting of the Coulomb potential or of the isotropic-three-dimensional-oscillator potential and of a term of the form $\frac{a}{r^2}$, closed orbits still exist for suitable angular momentum [14]. Particles moving in a central field can have closed orbits for discrete values of angular momentum even though the force law is different from the Coulomb and oscillator type [15]. A generalized Bertrand theorem for completely degenerate two-dimensional systems is presented in [16].

In the relativistic classical mechanics, the Hamilton function of a particle with the rest mass m_0, moving in a central field in which it possesses the linear momentum $\mathbf{p}$ and the potential energy $U(r)$, has the form

$$H = \sqrt{m_0^2 c^4 + c^2 p^2} + U\left(r\right), \tag{5}$$

where c is the velocity of light in the vacuum.

As in the non-relativistic classical mechanics, during the motion of the particle, the angular momentum $\mathbf{l}$ with respect to the field centre and the energy E are conserved. It follows from the conservation law of the angular momentum that the trajectory of the particle lies at a plane perpendicular to $\mathbf{l}$. In a system of polar coordinates (r, φ), the Hamilton function of the particle has the form

$$H = \sqrt{m_0^2 c^4 + c^2 p_r^2 + \frac{c^2 l^2}{r^2}} + U\left(r\right) = E, \tag{6}$$

where p_r is the radial component of the linear momentum.

The trajectory of a finite motion is a rosette both in the case of the Coulomb field [17] and in the case of the isotropic three-dimensional oscillator [18].

Under existing conditions, two alternatives arise:

- the Bertrand theorem is not available in the relativistic classical mechanics;
- the Bertrand theorem is valid in the relativistic classical mechanics for some central fields which differ from the Coulomb field and the isotropic-three-dimensional-oscillator field and which pass into the last ones in the non-relativistic limit.

As we will demonstrate in the following, the second alternative is true.

2. A New Approach of the Relativistic Motion in a Central Field

The form (5) of the Hamilton function for a relativistic particle assumes that, in the relativistic classical mechanics, there exist forces independent of velocities, which derive from simple potentials. But, in the relativistic theory, the dependence of forces on velocities is unavoidable. Because of this, in general, the Hamilton function of a relativistic particle cannot be divided into a term dependent on velocity and a term independent of velocity. The form

$$H = \sqrt{m_0^2 c^4 + c^2 p^2 + 2m_0 c^2 U\left(r\right)}, \tag{7}$$

of the Hamilton function for a relativistic particle moving in a central field is an extension of the Hamilton function for a relativistic particle in one-dimensional motion, used in [19]. Such a Hamilton function with $U(r)$ having the form (4) is involved in the Hamilton operator of the Dirac three-dimensional oscillator [20, 21].

The Hamilton function (7) must satisfy the canonical equations of motion

$$\left.\begin{aligned} \dot{\mathbf{r}} &= \frac{\partial H}{\partial \mathbf{p}} = \frac{c^2 \mathbf{p}}{H}, \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{r}} = \frac{-m_0 c^2 \frac{\mathrm{d}U(r)}{\mathrm{d}r}}{H}\frac{\mathbf{r}}{r}. \end{aligned}\right\} \tag{8}$$

It follows from Eqs. (8) that the time derivative of the angular momentum $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ with respect to the field centre is

$$\dot{\mathbf{l}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = \frac{c^2}{H}\mathbf{p} \times \mathbf{p} - \frac{m_0 c^2 \frac{\mathrm{d}U(r)}{\mathrm{d}r}}{Hr}\mathbf{r} \times \mathbf{r} = \mathbf{0}. \tag{9}$$

Hence, the angular momentum is conserved.

This conservation law implies that the trajectory of the particle lies at a plane perpendicular to $\mathbf{l}$.

The total derivative of H with respect to the time is

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \mathbf{r}}\dot{\mathbf{r}} + \frac{\partial H}{\partial \mathbf{p}}\dot{\mathbf{p}} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \mathbf{r}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial H}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{r}} = \frac{\partial H}{\partial t}. \tag{10}$$

If the Hamilton function does not depend explicitly on the time, then it is a constant of motion. This constant is the total energy E of the particle: $H = E$.

The conservation laws of the angular momentum and energy involve the conservation of the areal velocity of the particle

$$\dot{\mathbf{S}} = \frac{1}{2}\mathbf{r} \times \dot{\mathbf{r}} = \frac{1}{2}\frac{c^2}{E}\mathbf{r} \times \mathbf{p} = \frac{1}{2}\frac{c^2 \mathbf{l}}{E}. \tag{11}$$

If we choose a system of Cartesian coordinates $Oxyz$ with the origin O at the field centre and with the axis Oz directed along the angular momentum, the motion takes place in the plane xOy. For reasons of field symmetry, we will use the polar coordinates (r, φ) as generalized coordinates in the study of this motion. The Hamilton function of the particle is

$$H = \sqrt{m_0^2 c^4 + c^2 p_r^2 + \frac{c^2 l^2}{r^2} + 2m_0 c^2 U\left(r\right)}, \tag{12}$$

where p_r is the radial component of the linear momentum.

The corresponding canonical equations are

$$\left.\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{c^2 p_r}{H}, \dot{\varphi} = \frac{\partial H}{\partial l} = \frac{\frac{c^2 l}{r^2}}{H}, \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{\frac{c^2 l^2}{r^3} - m_0 c^2 \frac{\mathrm{d}U(r)}{\mathrm{d}r}}{H}, \dot{l} = -\frac{\partial H}{\partial \varphi} = 0. \end{aligned}\right\} \tag{13}$$

For a weak external field, where

$$U\left(r\right)\ll\frac{m_0^2c^4+c^2p_r^2+\frac{c^2l^2}{r^2}}{2m_0c^2},$$

expanding the Hamilton function (12) into a power series of

$$\frac{2m_0c^2U\left(r\right)}{m_0^2c^4+c^2p_r^2+\frac{c^2l^2}{r^2}},$$

this function takes the form (6) with $U(r)$ replaced by

$$\frac{m_0c^2}{\sqrt{m_0^2c^4+c^2p_r^2+\frac{c^2l^2}{r^2}}}U\left(r\right)-\frac{1}{2}\frac{m_0^2c^4}{\left(m_0^2c^4+c^2p_r^2+\frac{c^2l^2}{r^2}\right)^{\frac{3}{2}}}U^2\left(r\right)+\dots$$

In the non-relativistic limit, where

$$\sqrt{p_r^2+\frac{l^2}{r^2}}\ll m_0c,$$

this expression reduces to the expression (2) with the approximation of the rest energy m_0c^2.

The Hamilton-Jacobi equation corresponding to the Hamilton function (12) is

$$\left(\frac{\partial S}{\partial t}\right)^2=m_0^2c^4+c^2\left(\frac{\partial S}{\partial r}\right)^2+\frac{c^2}{r^2}\left(\frac{\partial S}{\partial\varphi}\right)^2+2m_0c^2U\left(x\right), \tag{14}$$

where $S(r,\varphi,t)$ is the action integral of the particle.

The complete integral of Eq. (14) is

$$S\left(r,\varphi,t;l,E\right)=\frac{1}{c}\int\sqrt{E^2-m_0^2c^4-\frac{c^2l^2}{r^2}-2m_0c^2U\left(r\right)}\mathrm{d}r+l\varphi-Et. \tag{15}$$

Differentiating $S(r,\varphi,t;l,E)$ with respect to the constants l and E and equating the results with new constants φ_0 and t_0, we find the equation of the trajectory

$$cl\int\frac{\frac{\mathrm{d}r}{r^2}}{\sqrt{E^2-m_0^2c^4-\frac{c^2l^2}{r^2}-2m_0c^2U\left(r\right)}}=\varphi-\varphi_0 \tag{16}$$

and the equation of motion

$$\frac{E}{c}\int\frac{\mathrm{d}r}{\sqrt{E^2-m_0^2c^4-\frac{c^2l^2}{r^2}-2m_0c^2U\left(r\right)}}=t-t_0. \tag{17}$$

In the following, it is convenable to use the dimensionless quantities $u=\frac{l}{m_0cr}$, $\tau=\frac{m_0c^2t}{l}$, $\varepsilon=\frac{E}{m_0c^2}$ and $\upsilon(u)=\frac{U\left(\frac{l}{m_0cu}\right)}{m_0c^2}$.

Denote

$$\upsilon^*\left(u\right)=\sqrt{1+u^2+2\upsilon\left(u\right)}. \tag{18}$$

In accordance with the relation (12), the range of variation of u corresponding to the real motion of the particle is determined by the inequality

$$\varepsilon\geq\upsilon^*\left(u\right). \tag{19}$$

The relation (19) with the sign "=",

$$\varepsilon=\upsilon^*\left(u\right), \tag{20}$$

gives the limits of this range.

Suppose that, for a finite motion, this equation has two finite solutions $u_m = \frac{l}{m_0 c r_m}$ and $u_M = \frac{l}{m_0 c r_M}$ that give the polar radii r_m of the pericentre and r_M of the apocentre of the trajectory.

If, at $t = 0$, we have $r = r_m$ and $\varphi = 0$, then the equations of the trajectory and of motion assume, respectively, the forms

$$-\int_{u_m}^{u} \frac{\mathrm{d}u'}{\sqrt{\varepsilon^2 - 1 - u'^2 - 2\upsilon(u')}} = \varphi \tag{21}$$

and

$$-\varepsilon \int_{u_m}^{u} \frac{\frac{\mathrm{d}u'}{u'^2}}{\sqrt{\varepsilon^2 - 1 - u'^2 - 2\upsilon(u')}} = \tau. \tag{22}$$

While the particle moves from a pericentre to an apocentre and reversely, the radius vector rotates with the angle

$$\Delta\varphi = -2\int_{u_m}^{u_M} \frac{\mathrm{d}u}{\sqrt{\varepsilon^2 - 1 - u^2 - 2\upsilon(u)}}. \tag{23}$$

The trajectory is closed if $\Delta\varphi$ is a rational fraction s of 2π:

$$\Delta\varphi = \frac{2\pi}{s}. \tag{24}$$

3. The Bertrand Theorem in the Relativistic Classical Mechanics

In the following, we will determinate the form of the potential in which all finite motions have closed trajectories. To this end, let us suppose that $\upsilon^*(u)$ has a minimum for $u = u_c$ $(u_M < u_c < u_m)$ given by the conditions

$$\frac{\mathrm{d}\upsilon^*}{\mathrm{d}u}(u_c) = \frac{u_c + \frac{\mathrm{d}\upsilon}{\mathrm{d}u}(u_c)}{\sqrt{1 + u_c^2 + 2\upsilon(u_c)}} = 0$$

or

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}u}(u_c) = -u_c \tag{25}$$

and

$$\frac{\mathrm{d}^2\upsilon^*}{\mathrm{d}u^2}(u_c) = \frac{1 + \frac{\mathrm{d}^2\upsilon}{\mathrm{d}u^2}(u_c)}{\sqrt{1 + u_c^2 + 2\upsilon(u_c)}} > 0$$

or

$$1 + \frac{\mathrm{d}^2\upsilon}{\mathrm{d}u^2}(u_c) > 0. \tag{26}$$

Then, if the particle has the energy

$$\varepsilon_c = \upsilon^*(u_c), \tag{27}$$

its trajectory is a circle of radius $r_c = \frac{l}{m_0 c u_c}$. Any trajectory which only slightly differs from the circular one is assumed to be a closed curve. The angle $\Delta\varphi$ corresponding to this orbit can be calculated with the aid of the formula (23), developping the radicand into a power series of $(u - u_c)$ and taking into account the relations (18), (25) and (27). With $\varepsilon = \varepsilon_c + \varepsilon'$ $(\varepsilon' \ll 1)$, in the second order approximation, we have

$$\varepsilon^2 - 1 - u^2 - 2\upsilon(u) = \varepsilon'^2 + 2\varepsilon_c\varepsilon' - \left[1 + \frac{\mathrm{d}^2\upsilon}{\mathrm{d}u^2}(u_c)\right](u - u_c)^2. \tag{28}$$

Therefore,

$$\Delta\varphi = \frac{2\pi}{\sqrt{1 + \frac{\mathrm{d}^2\upsilon}{\mathrm{d}u^2}(u_c)}}. \tag{29}$$

It follows from (24) and (29) that

$$\frac{\mathrm{d}^2 \upsilon}{\mathrm{d}u^2}(u_c) = s^2 - 1. \tag{30}$$

Until now, we have discussed orbits which slightly go out a certain circular orbit. Evidently, as the potential $U(r)$ has been found, changing the values of the pair (E, l) which satisfy the conditions (25) and (27), we can pass from one circular orbit to another by a continuous variation of r_c. This signifies that the discrete parameter s must be the same for all circular orbits. Dividing the both sides of the relations (25) and (30), we obtain

$$\frac{\frac{\mathrm{d}^2 \upsilon}{\mathrm{d}u^2}}{\frac{\mathrm{d}\upsilon}{\mathrm{d}u}} = \frac{1 - s^2}{u}. \tag{31}$$

The relation (31) allows us to determine the potential $U(r)$. The solution of this equation is

$$\upsilon = C u^{2-s^2}, \tag{32}$$

where C is a constant of integration. Here, $s^2 \neq 2$ because $s = \sqrt{2}$ is not a rational number.

For $0 < s^2 < 2$, the constant C must be negative so that the function $\upsilon^*(u)$ may have a minimum. Then, $\upsilon^*(u)$ is equal to 1 when u is equal to u_m, which satisfies the relation $1 + 2Cu_m^{-s^2} = 0$ or $2C = -u_m^{s^2}$, and when u is equal to $u_M = 0$. Hence, when $\varepsilon = 1$, the trajectory has a pericentre with the polar radius $r_m = \frac{l}{m_0 c u_m}$ and an apocentre with the polar radius $r_M = \frac{l}{m_0 c u_M} \to \infty$. In these conditions, the formula (23) becomes

$$\Delta\varphi = 2\int_0^{u_m} \frac{\mathrm{d}u}{\sqrt{u_m^{s^2} u^{2-s^2} - u^2}}.$$

With the changes of variable $u = u_m x^{2/s^2}$, this integral reduces to

$$\Delta\varphi = \frac{4}{s^2}\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{2\pi}{s^2}. \tag{33}$$

Comparing the relations (24) and (33), we obtain $s = 1$. If we introduce this value of s into the expression (32), we find "the potential of the Coulomb field" (3).

For $s^2 > 2$, the constant C must be positive so that the function $\upsilon^*(u)$ may have a minimum. Then, $\upsilon^*(u)$ tends to ∞ when u is equal to $u_m \to \infty$. Hence, when $\varepsilon \to \infty$, the trajectory has a pericentre with the polar radius $r_m = \frac{l}{m_0 c u_m} \to 0$ and an apocentre with the polar radius $r_M = \frac{l}{m_0 c u_M}$. In these conditions, the formula (23) becomes

$$\Delta\varphi = 2\lim_{u_m \to \infty}\int_{u_M}^{u_m} \frac{\mathrm{d}u}{\sqrt{(u_m^2 - u^2) + 2C\left(u_m^{2-s^2} - u^{2-s^2}\right)}}.$$

With the changes of variable $u = u_m x$, this integral reduces to

$$\Delta\varphi = 2\lim_{u_m \to \infty}\int_{\frac{u_M}{u_m}}^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2) + 2Cu_m^{-s^2}\left(1 - x^{2-s^2}\right)}} = 2\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \pi. \tag{34}$$

Comparing the relations (24) and (34), we obtain $s = 2$. If we introduce this value of s into the expression (32), we find "the potential of the isotropic three-dimensional oscillator" (4).

The trajectories of the motion in "the Coulomb field" and in the field of "the isotropic three-dimensional oscillator" can be obtained by substituting the expressions (3) and (4) of the corresponding "potential energy" into Eq. (21). As in the non-relativistic classical mechanics, they are conic sections (Tables 1 and 2). All finite trajectories (circles and ellipses) are closed.

The equations of motion in the above-mentioned fields can easily be derived by substituting the expression (3) into Eq. (22) for "the Coulomb field" and the expression (4) into Eqs. (8) for "the isotropic three-dimensional oscillator". They have forms similar to the equations of motion known in the non-relativistic classical mechanics (Tables 1 and 2).

The period of one revolution of the particle on a closed trajectory is

$$T = 2\pi\sqrt{1 - \frac{\alpha}{m_0 c^2 a}}\sqrt{\frac{m_0 a^3}{\alpha}} \tag{35}$$

for "the Coulomb field" (the relativistic analogue of the Kepler third law) and

$$T = \frac{2\pi}{\omega_0}\frac{E}{m_0 c^2} \tag{36}$$

for "the isotropic three-dimensional oscillator". The formulas (35) and (36) contain supplementary factors in comparison with the corresponding non-relativistic formulas.

As in the non-relativistic classical mechanics, the "potentials" (3) and (4) admit extra constants of the motion. They can be obtained starting from Eqs. (8). In the case of "the Coulomb field", the extra constant is the vector

$$\mathbf{R} = \frac{\mathbf{p} \times \mathbf{l}}{m_0} - \alpha\frac{\mathbf{r}}{r} \tag{37}$$

(the relativistic analogue of the Runge-Lenz vector). In the case of "the isotropic three-dimensional oscillator", the extra constant is the symmetric second-rank tensor

$$E_{\alpha\beta} = \frac{p_\alpha p_\beta}{m_0} + m_0 \omega_0^2 x_\alpha x_\beta, \tag{38}$$

where $x_\alpha, x_\beta = x, y, z$ and $p_\alpha, p_\beta = p_x, p_y, p_z$. In the variables $\mathbf{r}$ and $\mathbf{p}$, these extra constant have the same forms as in the non-relativistic classical mechanics.

4. Conclusions

We have presented a new approach of the relativistic motion of a particle in a central field. It is based on a modified form of the Hamilton function of the particle.

This form has allowed us to extend the Bertrand theorem from the non-relativistic classical mechanics to the relativistic classical mechanics. We have found that, in the relativistic classical mechanics, there exist two central fields where all finite motions of a particle have closed orbits.

In the non-relativistic limit, the properties of these fields tend to the properties of the Coulomb field and the field of the isotropic three-dimensional oscillator.

TABLE 1. The properties of the relativistic motion in a "Coulomb field"

Energy E of the particle	Eccentricity e of the conic section	Type of the conic section	Equation of the conic section	Parameters of the conic section	Equations of motion
$E = m_0c^2\sqrt{1-\frac{\alpha^2}{c^2l^2}}$ $l > \frac{\alpha}{c}$	$e = 0$	Circle	$x^2 + y^2 = a^2$	$a = \frac{l^2}{m_0\alpha}$	$r = a$ $x = a\cos\chi$ $y = a\sin\chi$ $t = \sqrt{1-\frac{\alpha}{m_0c^2a}}\sqrt{\frac{m_0a^3}{\alpha}}\chi$
$m_0c^2\sqrt{1-\frac{\alpha^2}{c^2l^2}} < E < m_0c^2$ $l > \frac{\alpha}{c}$	$0 < e < 1$	Ellipse	$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1$	$a = \frac{\alpha m_0c^2}{m_0^2c^4-E^2}$ $b = \frac{cl}{\sqrt{m_0^2c^4-E^2}}$ $c = \sqrt{a^2-b^2}$	$r = a\left(1-e\cos\xi\right)$ $x = a\left(\cos\xi - e\right)$ $y = a\sqrt{1-e^2}\sin\xi$ $t = \frac{E}{m_0c^2}\sqrt{\frac{m_0a^3}{\alpha}}\left(\xi - e\sin\xi\right)$
$E = m_0c^2$	$e = 1$	Parabola	$y^2 = -2p\left(x-\frac{p}{2}\right)$	$p = \frac{l^2}{m_0\alpha}$	$r = \frac{p}{2}\left(1+\eta^2\right)$ $x = \frac{p}{2}\left(1-\eta^2\right)$ $y = p\eta$ $t = \frac{1}{2}\sqrt{\frac{m_0p^3}{\alpha}}\eta\left(1+\frac{\eta^2}{3}\right)$
$E > m_0c^2$	$e > 1$	Hyper-bola	$\frac{(x-c)^2}{a^2} - \frac{y^2}{b^2} = 1$	$a = \frac{\alpha m_0c^2}{E^2-m_0^2c^4}$ $b = \frac{cl}{\sqrt{E^2-m_0^2c^4}}$ $c = \sqrt{a^2+b^2}$	$r = a\left(e\cosh\zeta - 1\right)$ $x = a\left(e-\cosh\zeta\right)$ $y = a\sqrt{e^2-1}\sinh\zeta$ $t = \frac{E}{m_0c^2}\sqrt{\frac{m_0a^3}{\alpha}}\left(e\sinh\zeta - \zeta\right)$

TABLE 2. The properties of the relativistic motion in the field of an "isotropic three-dimensional oscillator"

Energy E of the particle	Eccentricity e of the conic section	Type of the conic section	Equation of the conic section	Parameters of the conic section	Equations of motion
$E = m_0c^2\sqrt{1+2\frac{\omega_0 l}{m_0c^2}}$	$e=0$	Circle	$x^2+y^2=a^2$	$a=\sqrt{\frac{l}{m_0\omega_0}}$	$x = a\cos\left(\frac{m_0c^2}{E}\omega_0 t\right)$ $y = a\sin\left(\frac{m_0c^2}{E}\omega_0 t\right)$
$E > m_0c^2\sqrt{1+2\frac{\omega_0 l}{m_0c^2}}$	$0<e<1$	Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$a=\sqrt{\frac{E^2-m_0^2c^4}{2m_0^2c^2\omega_0^2}}\sqrt{1-\sqrt{1-\left(\frac{2m_0c^2\omega_0 l}{E^2-m_0^2c^4}\right)^2}}$ $b=\sqrt{\frac{E^2-m_0^2c^4}{2m_0^2c^2\omega_0^2}}\sqrt{1+\sqrt{1-\left(\frac{2m_0c^2\omega_0 l}{E^2-m_0^2c^4}\right)^2}}$	$x = a\cos\left(\frac{m_0c^2}{E}\omega_0 t\right)$ $y = a\sin\left(\frac{m_0c^2}{E}\omega_0 t\right)$

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