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SOME APPLICATIONS OF FEJÉR'S INEQUALITY FOR CONVEX FUNCTIONS (II)

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Abstract. Some applications of Fejér's inequality for convex functions are explored. Upper and lower bounds for the weighted integral

$$\int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx$$

under various assumptions for f with applications to the trapezoidal quadrature rule are given. Some inequalities for special means are derived.

1. Introduction

The *Hermite-Hadamard* integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{HH}$$

is well known in the literature and has many applications for special means.

For related results, see for instance the research papers [1], [11], [12], [13], [15], [14], [16], [17], [18], the monograph online [10] and the references therein.

In 1906, Fejér, while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where h is a convex function in the interval (a,b) and w is a positive function in the same interval such that*

$$w(a+t)=w(b-t),\quad 0\leq t\leq\frac{1}{2}(a+b),$$

i.e., $y=w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the $x-$axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx\leq\int_a^b h(x)\,w(x)\,dx\leq\frac{h(a)+h(b)}{2}\int_a^b w(x)\,dx. \tag{1}$$

If h is concave on (a,b), then the inequalities reverse in (1).

Clearly, for $w(x)\equiv 1$ on $[a,b]$ we get (HH).

We observe that, if we take $w(x)=\left(x-\frac{a+b}{2}\right)^2$, $x\in[a,b]$, then w satisfies the conditions in Theorem 1,

$$\int_a^b \left(x-\frac{a+b}{2}\right)^2 dx=\frac{1}{12}(b-a)^3$$

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and by (1) we have the following inequality

$$\frac{1}{12}h\left(\frac{a+b}{2}\right)(b-a)^3 \leq \int_a^b \left(x-\frac{a+b}{2}\right)^2 h(x)\,dx \leq \frac{h(a)+h(b)}{24}(b-a)^3, \tag{2}$$

that holds for any convex function $h:[a,b]\to\mathbb{R}$. If the function h is concave the inequalities in (2) reverse.

In this paper we establish amongst other results some better bounds for the weighted integral

$$\int_a^b \left(x-\frac{a+b}{2}\right)^2 h(x)\,dx$$

in the case of convex functions $h:[a,b]\to\mathbb{R}$. We also investigate the connection with the trapezoid rule and apply some of the obtained results for special means.

For some recent inequalities concerning the weighted integral

$$\int_a^b (b-x)(x-a)\,h(x)\,dx$$

under various assumptions for the function $h:[a,b]\to\mathbb{R}$, see the paper [8].

2. The Results

We start with the following equality that is of interest in itself.

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be such that the derivative f' is of bounded variation on $[a,b]$. Then we have the equality*

$$\frac{1}{8}(b-a)^2\left[f'(b)-f'(a)\right]-\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right] = \frac{1}{2}\int_a^b \left(x-\frac{a+b}{2}\right)^2 df'(x), \tag{3}$$

where the last integral is taken in the Riemann-Stieltjes sense.

Proof. Since $f'(\cdot)$ is of bounded variation and $\left(\cdot-\frac{a+b}{2}\right)^2$ is continuous on $[a,b]$ then the Riemann-Stieltjes integral from the right hand side of the equality (3) exists and utilizing the integration by parts rule we have

$$\begin{aligned}\int_a^b \left(x-\frac{a+b}{2}\right)^2 df'(x) &= \left(x-\frac{a+b}{2}\right)^2 f'(x)\Big|_a^b - 2\int_a^b \left(x-\frac{a+b}{2}\right) f'(x)\,dx \\ &= \frac{1}{8}(b-a)^2\left[f'(b)-f'(a)\right]-2\int_a^b \left(x-\frac{a+b}{2}\right) f'(x)\,dx.\end{aligned} \tag{4}$$

By the integration by parts rule for the Riemann integral we also have

$$\int_a^b \left(x-\frac{a+b}{2}\right) f'(x)\,dx = \frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx. \tag{5}$$

Utilising the equality (4) divided by 2 and the equality (5), we get the desired result (3). □

Remark 1. *If f' is absolutely continuous on $[a,b]$, then the equality* (3) *becomes*

$$\frac{1}{8}(b-a)^2\left[f'(b)-f'(a)\right]-\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right] \tag{6}$$
$$=\frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 f''(x)\,dx,$$

where the second integral is taken in the Lebesgue sense. This equality was obtained in a different way in [2].

Corollary 1. *If f is a convex function on $[a,b]$, then we have the inequality*

$$\frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right]\geq\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx. \tag{7}$$

Proof. If f is convex, then the derivative exists except at a countable number of points in $[a,b]$ and is increasing. The lateral derivatives $f'_-(b)$ and $f'_+(a)$ exist. If one is infinite then the inequality (7) holds trivially. If both of them are finite, then the function

$$g(x):=\begin{cases} f'_+(a), & x=a\\ f'_+(x), & x\in(a,b)\\ f_-(b), & x=b\end{cases}$$

is monotonic nondecreasing on $[a,b]$ and

$$\frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right]-\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right] \tag{8}$$
$$=\frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 dg(x).$$

Since

$$\int_a^b\left(x-\frac{a+b}{2}\right)^2 dg(x)\geq 0,$$

then (8) produces the desired result (7). □

Remark 2. *The inequality* (7) *has been obtained in a different way in* [6].

Theorem 2. *With the assumptions of Lemma 1 we have*

$$\left|\frac{1}{8}(b-a)^2\left[f'(b)-f'(a)\right]-\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right]\right| \tag{9}$$
$$\leq\frac{1}{8}(b-a)^2\bigvee_a^b(f').$$

Moreover, if f' is Lipschitzian with the constant $L>0$, then

$$\left|\frac{1}{8}(b-a)^2\left[f'(b)-f'(a)\right]-\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right]\right| \tag{10}$$
$$\leq\frac{1}{48}L(b-a)^2.$$

Proof. It is known that if $p : [c,d] \to \mathbb{C}$ is a continuous function and $v : [c,d] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and the following inequality holds

$$\left| \int_c^d p(t)\,dv(t) \right| \le \max_{t\in[c,d]} |p(t)| \bigvee_c^d (v)$$

where $\bigvee_c^d (v)$ denotes the total variation of v on $[c,d]$.

Utilising this property we have

$$\begin{aligned} \left| \int_a^b \left(x - \frac{a+b}{2} \right)^2 df'(x) \right| &\le \sup_{x\in[a,b]} \left(x - \frac{a+b}{2} \right)^2 \bigvee_a^b (f') \\ &= \frac{1}{4}(b-a)^2 \bigvee_a^b (f') \end{aligned}$$

and by the equality (3) we get (9).

It is well known that if $p : [a,b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, i.e.,

$$|v(s) - v(t)| \le M |s-t| \text{ for any } t,s \in [a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\,dv(t) \right| \le M \int_a^b |p(t)|\,dt.$$

Utilizing this property we have

$$\begin{aligned} \left| \int_a^b \left(x - \frac{a+b}{2} \right)^2 df'(x) \right| &\le L \int_a^b \left(x - \frac{a+b}{2} \right)^2 d(x) \\ &= \frac{1}{12} L (b-a)^3 \end{aligned}$$

and by the equality (3) we get (10). □

Now, when some convexity property is assumed for the second derivative, then following result holds.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function on (a,b) and such that the second derivative f'' is convex on (a,b). Then*

$$\begin{aligned} &\frac{1}{24} f''\left(\frac{a+b}{2} \right)(b-a)^3 \quad (11) \\ &\le \frac{1}{8}(b-a)^2 \left[f'_-(b) - f'_+(a) \right] - \left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)\,dx \right] \\ &\le \frac{f''(a)+f''(b)}{48}(b-a)^3 . \end{aligned}$$

Proof. We know from (6) that

$$\frac{1}{8}(b-a)^2\left[f'(b)-f'(a)\right]-\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right] \tag{12}$$
$$=\frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 f''(x)\,dx.$$

Since f'' is convex on (a,b), then by (2) we have

$$\frac{1}{12}f''\left(\frac{a+b}{2}\right)(b-a)^3\le\int_a^b\left(x-\frac{a+b}{2}\right)^2 f''(x)\,dx \tag{13}$$
$$\le\frac{f''(a)+f''(b)}{24}(b-a)^3.$$

Utilising (12) and (13) we deduce the desired result (11). □

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b).*

If there exists a real number m such that $f''(x)\ge m$ for any $x\in(a,b)$, then

$$\frac{1}{12}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{180}m(b-a)^5 \tag{14}$$
$$\le\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx$$
$$\le\frac{f(a)+f(b)}{24}(b-a)^3-\frac{7}{1440}m(b-a)^5.$$

If there exists a real number M such that $f''(x)\le M$ for any $x\in(a,b)$, then

$$\frac{f(a)+f(b)}{24}(b-a)^3-\frac{7}{1440}M(b-a)^5 \tag{15}$$
$$\le\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx$$
$$\le\frac{1}{12}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{180}M(b-a)^5.$$

Proof. Define the function $h_m:[a,b]\to\mathbb{R}$ by

$$h_m(x):=f(x)-\frac{1}{2}m\left(x-\frac{a+b}{2}\right)^2.$$

This function is twice differentiable and the second derivative is

$$h_m''(x)=f''(x)-m\ge 0,\ x\in(a,b)$$

showing that h_m is convex on $[a,b]$.

If we apply the inequality (2) for h_m, then we have

$$\frac{1}{12}f\left(\frac{a+b}{2}\right)(b-a)^3 \tag{16}$$
$$\le\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx-\frac{1}{2}m\int_a^b\left(x-\frac{a+b}{2}\right)^4 dx$$
$$\le\frac{f(a)-\frac{1}{8}m(b-a)^2+f(b)-\frac{1}{8}m(b-a)^2}{24}(b-a)^3$$
$$=\frac{f(a)+f(b)}{24}(b-a)^3-\frac{1}{96}m(b-a)^5.$$

We also have

$$\int_a^b \left(x - \frac{a+b}{2}\right)^4 dx = \frac{1}{90}(b-a)^5 .$$

Then (16) becomes

$$\begin{aligned}
&\frac{1}{12} f\left(\frac{a+b}{2}\right)(b-a)^3 + \frac{1}{180} m (b-a)^5 \\
&\leq \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\, dx \\
&\leq \frac{f(a)+f(b)}{24}(b-a)^3 - \frac{1}{96} m (b-a)^5 + \frac{1}{180} m (b-a)^5 \\
&= \frac{f(a)+f(b)}{24}(b-a)^3 - \frac{7}{1440} m (b-a)^5
\end{aligned}$$

which is equivalent to (14).

Now define the function $h_M : [a,b] \to \mathbb{R}$ by

$$h_M(x) := \frac{1}{2} M \left(x - \frac{a+b}{2}\right)^2 - f(x) .$$

This function is twice differentiable and

$$h_M''(x) := M - f''(x) \geq 0,\ x \in (a,b)$$

showing that h_M is convex on $[a,b]$.

If we apply the inequality (2) for h_M, then we have

$$\begin{aligned}
&\frac{1}{12}\left[-f\left(\frac{a+b}{2}\right)\right](b-a)^3 \\
&\leq \int_a^b \left(x - \frac{a+b}{2}\right)^2 \left[\frac{1}{2} M \left(x - \frac{a+b}{2}\right)^2 - f(x)\right] dx \\
&\leq \frac{\frac{1}{8} M (b-a)^2 - f(a) + \frac{1}{8} M (b-a)^2 - f(b)}{24}(b-a)^3 ,
\end{aligned}$$

which, by multiplication with -1, produces

$$\begin{aligned}
&\frac{1}{12} f\left(\frac{a+b}{2}\right)(b-a)^3 \\
&\geq \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\, dx - \frac{1}{180} M (b-a)^5 \\
&\geq \frac{f(a)+f(b) - \frac{1}{4} M (b-a)^2}{24}(b-a)^3 \\
&= \frac{f(a)+f(b)}{24} - \frac{1}{96} M (b-a)^5
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&\frac{f(a)+f(b)}{24} - \frac{1}{96} M (b-a)^5 + \frac{1}{180} M (b-a)^5 \\
&\leq \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\, dx \\
&\leq \frac{1}{12} f\left(\frac{a+b}{2}\right)(b-a)^3 + \frac{1}{180} M (b-a)^5
\end{aligned}$$

and the inequality (15) is proved. $\square$

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b). If there exists a $K>0$ such that $|f''(x)|\le K$ for any $x\in(a,b)$, then*

$$\left|\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx-\frac{1}{24}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right](b-a)^3\right| \le \frac{1}{192}K(b-a)^5. \tag{17}$$

Proof. If we write the inequality (14)) for $m=-K$ and the inequality (15) for $M=K$, then we have

$$\begin{aligned}&\frac{1}{12}f\left(\frac{a+b}{2}\right)(b-a)^3-\frac{1}{180}K(b-a)^5\\ &\le\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx\\ &\le\frac{f(a)+f(b)}{24}(b-a)^3+\frac{7}{1440}K(b-a)^5,\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\frac{f(a)+f(b)}{24}(b-a)^3-\frac{7}{1440}K(b-a)^5\\ &\le\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx\\ &\le\frac{1}{12}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{180}K(b-a)^5.\end{aligned} \tag{19}$$

If we add the inequality (18) with (19) and divide the sum by 2 we get the desired result (17). $\square$

Remark 3. *We observe that the case $m>0$ in the inequality* (14) *produces a better result than* (2)*.*

For twice differentiable functions we can provide the following *perturbed trapezoid quadrature rule*

$$\begin{aligned}\int_a^b f(x)\,dx &\simeq \frac{f(a)+f(b)}{2}(b-a)-\frac{1}{8}(b-a)^2\left[f'(b)-f'(a)\right]\\ &\quad+\frac{1}{24}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right].\end{aligned} \tag{20}$$

Denote $E_{P,T}(f;a,b)$ the error in approximating the integral as in (20), namely

$$\begin{aligned}E_{P,T}(f;a,b) &:= \int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}(b-a)+\frac{1}{8}(b-a)^2\left[f'(b)-f'(a)\right]\\ &\quad-\frac{1}{24}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right].\end{aligned}$$

The following result that provides an *a priory* error bound for functions whose fourth derivatives are bounded, holds.

Proposition 1. *Let $f:[a,b]\to\mathbb{R}$ be a four time differentiable function on (a,b). If there exists a $K>0$ such that $\left|f^{(4)}(x)\right|\leq K$ for any $x\in(a,b)$, then*

$$\left|E_{P,T}(f;a,b)\right|\leq\frac{1}{384}K(b-a)^5. \tag{21}$$

Proof. Writing the inequality (17) for the second derivative f'' we have

$$\left|\int_a^b\left(x-\frac{a+b}{2}\right)^2 f''(x)\,dx-\frac{1}{24}\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right](b-a)^3\right| \leq\frac{1}{192}K(b-a)^5.$$

Dividing this inequality by 2 and utilizing the representation (12) we have

$$\left|\frac{1}{8}(b-a)^2\left[f'(b)-f'(a)\right]-\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right]\right.$$
$$\left.-\frac{1}{48}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right]\right|$$
$$\leq\frac{1}{384}K(b-a)^5,$$

and the inequality (21) is proved. □

3. Applications for Special Means

Let us recall the following means for two positive numbers.

(1) *The Arithmetic mean*

$$A=A(a,b):=\frac{a+b}{2},\ a,b>0;$$

(2) *The Geometric mean*

$$G=G(a,b):=\sqrt{ab},\ a,b>0;$$

(3) *The Harmonic mean*

$$H=H(a,b):=\frac{2ab}{a+b},\ a,b>0;$$

(4) *The Logarithmic mean*

$$L=L(a,b):=\begin{cases} a & \text{if } a=b \\ \frac{b-a}{\ln b-\ln a} & \text{if } a\neq b; \end{cases},\ a,b>0,$$

(5) *The Identric mean*

$$I=I(a,b):=\begin{cases} a & \text{if } a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b \end{cases},\ a,b>0;$$

(6) *The p-Logarithmic mean*

$$L_p=L_p(a,b):=\begin{cases} a & \text{if } a=b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\neq b \end{cases},\ a,b>0.$$

The following inequality is well known in the literature:

$$H \leq G \leq L \leq I \leq A. \tag{22}$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Consider the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(x) = x^p$ for $p \geq 3$. We have the fourth derivative of the function given by

$$f^{(4)}(x) = p(p-1)(p-2)(p-3)x^{p-4},$$

which shows that the second derivative f'' is convex on $[a,b]$. Applying the inequality (11) we have

$$\begin{aligned} &\frac{p(p-1)}{24}A^{p-2}(a,b)(b-a)^2 \\ &\leq \frac{1}{8}p(p-1)(b-a)^2 L_{p-2}^{p-2}(a,b) - A(a^p,b^p) + L_p^p(a,b) \\ &\leq \frac{1}{24}p(p-1)A\left(a^{p-2},b^{p-2}\right)(b-a)^2 \end{aligned} \tag{23}$$

that holds for any $a,b > 0$ and $p \geq 3$.

Consider the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(x) = \frac{1}{x}$. Then $f''(x) = \frac{2}{x^3}$ and $f^{(4)}(x) = \frac{24}{x^5}$ showing that the second derivative is convex on $[a,b]$. Applying the inequality (11) we have

$$\begin{aligned} &\frac{1}{12}\left(\frac{a+b}{2}\right)^{-3}(b-a)^3 \\ &\leq \frac{1}{8}(b-a)^3\left(\frac{a+b}{a^2b^2}\right) - \left[\frac{\frac{1}{a}+\frac{1}{b}}{2}(b-a) - (\ln b - \ln a)\right] \\ &\leq \frac{\frac{1}{a^3}+\frac{1}{b^3}}{24}(b-a)^3. \end{aligned}$$

Dividing by $b-a > 0$ we have

$$\begin{aligned} &\frac{1}{12}A^{-3}(a,b)(b-a)^2 \\ &\leq \frac{1}{4}(b-a)^2\frac{A(a,b)}{G^4(a,b)} - H^{-1}(a,b) + L^{-1}(a,b) \\ &\leq \frac{1}{12}H^{-1}\left(a^3,b^3\right)(b-a)^2, \end{aligned} \tag{24}$$

that holds for any $a,b > 0$.

Consider the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(x) = -\ln x$. Then $f''(x) = \frac{1}{x^2}$ and $f^{(4)}(x) = \frac{6}{x^4}$ showing that the second derivative is convex on $[a,b]$. Applying the inequality (11) we have

$$\begin{aligned} &\frac{1}{24}\left(\frac{a+b}{2}\right)^{-2}(b-a)^3 \\ &\leq \frac{1}{8}(b-a)^2\left(\frac{b-a}{ab}\right) + \frac{\ln a + \ln b}{2}(b-a) - \int_a^b \ln x dx \\ &\leq \frac{\frac{1}{a^2}+\frac{1}{b^2}}{48}(b-a)^3. \end{aligned}$$

Dividing by $b-a>0$ we have

$$\begin{aligned}&\frac{1}{24}\left(\frac{a+b}{2}\right)^{-2}(b-a)^2\\&\leq\frac{1}{8}(b-a)^2\frac{1}{ab}+\frac{\ln a+\ln b}{2}-\frac{1}{(b-a)}\int_a^b \ln x dx\\&\leq\frac{\frac{1}{a^2}+\frac{1}{b^2}}{48}(b-a)^2.\end{aligned}$$

Observe that

$$\frac{1}{b-a}\int_a^b \ln x dx=\frac{1}{b-a}\left[x\ln x|_a^b-(b-a)\right]=\left[\ln\left(\frac{b^b}{a^a}\right)^{1/(b-a)}-1\right]=\ln I(a,b),$$

and

$$\frac{\ln a+\ln b}{2}=\ln G(a,b).$$

Then we get

$$\begin{aligned}&\frac{1}{24}A^{-2}(a,b)(b-a)^2\\&\leq\frac{1}{8}(b-a)^2G^{-2}(a,b)+\ln G(a,b)-\ln I(a,b)\\&\leq\frac{1}{24}H^{-1}\left(a^2,b^2\right)(b-a)^2\end{aligned}\tag{25}$$

that holds for any $a,b>0$.

The interested reader may apply the inequality (21) to obtain other similar results. However, the details are omitted here.

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