SOME APPLICATIONS OF FEJÉR’S INEQUALITY FOR CONVEX FUNCTIONS (II)

S.S. DRAGOMIR AND I. GOMM

Abstract. Some applications of Fejér’s inequality for convex functions are explored. Upper and lower bounds for the weighted integral
\[ \int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) \, dx \]
under various assumptions for \( f \) with applications to the trapezoidal quadrature rule are given. Some inequalities for special means are derived.

1. Introduction

The Hermite-Hadamard integral inequality for convex functions \( f : [a, b] \to \mathbb{R} \)
\[ f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \quad (\text{HH}) \]
is well known in the literature and has many applications for special means. For related results, see for instance the research papers [1], [11], [12], [13], [15], [14], [16], [17], [18], the monograph online [10] and the references therein.

In 1906, Fejér, while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. Consider the integral \( \int_a^b h(x) w(x) \, dx \), where \( h \) is a convex function in the interval \( (a, b) \) and \( w \) is a positive function in the same interval such that
\[ w(a+t) = w(b-t), \quad 0 \leq t \leq \frac{1}{2} (a+b), \]
i.e., \( y = w(x) \) is a symmetric curve with respect to the straight line which contains the point \( \left( \frac{1}{2} (a+b), 0 \right) \) and is normal to the \( x \)-axis. Under those conditions the following inequalities are valid:
\[ h \left( \frac{a+b}{2} \right) \int_a^b w(x) \, dx \leq \int_a^b h(x) w(x) \, dx \leq \frac{h(a) + h(b)}{2} \int_a^b w(x) \, dx. \quad (1) \]
If \( h \) is concave on \( (a, b) \), then the inequalities reverse in \( (1) \).

Clearly, for \( w(x) \equiv 1 \) on \([a, b]\) we get \( \text{(HH)} \).

We observe that, if we take \( w(x) = \left( x - \frac{a+b}{2} \right)^2 \), \( x \in [a, b] \), then \( w \) satisfies the conditions in Theorem 1
\[ \int_a^b \left( x - \frac{a+b}{2} \right)^2 \, dx = \frac{1}{12} (b-a)^3 \]

2010 Mathematics Subject Classification. 26D15, 25D10.

Key words and phrases. convex functions, Hermite-Hadamard inequality, Fejér’s inequality, special means.

23
and by (1) we have the following inequality
\[ \frac{1}{12} h \left( \frac{a + b}{2} \right) (b - a)^3 \leq \int_a^b \left( x - \frac{a + b}{2} \right)^2 h(x) \, dx \]
that holds for any convex function \( h : [a, b] \rightarrow \mathbb{R} \). If the function \( h \) is concave the inequalities in (2) reverse.

In this paper we establish amongst other results some better bounds for the weighted integral
\[ \int_a^b \left( x - \frac{a + b}{2} \right)^2 h(x) \, dx \]
in the case of convex functions \( h : [a, b] \rightarrow \mathbb{R} \). We also investigate the connection with the trapezoid rule and apply some of the obtained results for special means.

For some recent inequalities concerning the weighted integral
\[ \int_a^b (b - x) (x - a) h(x) \, dx \]
under various assumptions for the function \( h : [a, b] \rightarrow \mathbb{R} \), see the paper [8].

2. The Results

We start with the following equality that is of interest in itself.

**Lemma 1.** Let \( f : [a, b] \rightarrow \mathbb{C} \) be such that the derivative \( f' \) is of bounded variation on \([a, b]\). Then we have the equality
\[ \frac{1}{8} (b - a)^2 \left[ f'(b) - f'(a) \right] - \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx \right] \]
\[ = \frac{1}{2} \int_a^b \left( x - \frac{a + b}{2} \right)^2 df'(x), \]
where the last integral is taken in the Riemann-Stieltjes sense.

**Proof.** Since \( f'(\cdot) \) is of bounded variation and \( \left( \cdot - \frac{a + b}{2} \right)^2 \) is continuous on \([a, b]\) then the Riemann-Stieltjes integral from the right hand side of the equality (3) exists and utilizing the integration by parts rule we have
\[ \int_a^b \left( x - \frac{a + b}{2} \right)^2 df'(x) \]
\[ = \left( x - \frac{a + b}{2} \right)^2 f'(x) \bigg|_a^b - 2 \int_a^b \left( x - \frac{a + b}{2} \right) f'(x) \, dx \]
\[ = \frac{1}{8} (b - a)^2 \left[ f'(b) - f'(a) \right] - 2 \int_a^b \left( x - \frac{a + b}{2} \right) f'(x) \, dx. \]

By the integration by parts rule for the Riemann integral we also have
\[ \int_a^b \left( x - \frac{a + b}{2} \right) f'(x) \, dx = \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx. \]

Utilising the equality (4) divided by 2 and the equality (5), we get the desired result (3). \( \square \)
Remark 1. If \( f' \) is absolutely continuous on \([a, b]\), then the equality (3) becomes
\[
\frac{1}{8} (b - a)^2 \left[ f'(b) - f'(a) \right] - \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx \right]
\]
where the second integral is taken in the Lebesgue sense. This equality was obtained in a different way in [2].

Corollary 1. If \( f \) is a convex function on \([a, b]\), then we have the inequality
\[
\frac{1}{8} (b - a)^2 \left[ f'(b) - f'(a) \right] - \left[ f(a) + f(b) \right] \geq f(a) + f(b),
\]
where the second integral is taken in the Lebesgue sense. This inequality was obtained in a different way in [6].

Proof. If \( f \) is convex, then the derivative exists except at a countable number of points in \([a, b]\) and is increasing. The lateral derivatives \( f'_-(b) \) and \( f'_+(a) \) exist. If one is infinite then the inequality (7) holds trivially. If both of them are finite, then the function
\[
g(x) := \begin{cases} 
    f'_+(a), & x = a \\
    f'_+(x), & x \in (a, b) \\
    f'_-(b), & x = b
\end{cases}
\]
is monotonic nondecreasing on \([a, b]\) and
\[
\frac{1}{8} (b - a)^2 \left[ f'_-(b) - f'_+(a) \right] - \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx \right]
\]
where the second integral is taken in the Lebesgue sense. This equality was obtained in a different way in [2].

Remark 2. The inequality (7) has been obtained in a different way in [6].

Theorem 2. With the assumptions of Lemma 7 we have
\[
\left| \frac{1}{8} (b - a)^2 \left[ f'(b) - f'(a) \right] - \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx \right] \right|
\]
Moreover, if \( f' \) is Lipschitzian with the constant \( L > 0 \), then
\[
\left| \frac{1}{8} (b - a)^2 \left[ f'(b) - f'(a) \right] - \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx \right] \right|
\]
Moreover, if \( f' \) is Lipschitzian with the constant \( L > 0 \), then
\[
\left| \frac{1}{48} L (b - a)^2 \right|
\]
Proof. It is known that if \( p : [c, d] \to \mathbb{C} \) is a continuous function and \( v : [c, d] \to \mathbb{C} \) is of bounded variation, then the Riemann-Stieltjes integral \( \int_c^d p(t) \, dv(t) \) exists and the following inequality holds

\[
\left| \int_c^d p(t) \, dv(t) \right| \leq \max_{t \in [c, d]} |p(t)| \sqrt{c} (v)
\]

where \( \sqrt{c} (v) \) denotes the total variation of \( v \) on \([c, d] \).

Utilising this property we have

\[
\left| \int_a^b \left( x - \frac{a + b}{2} \right)^2 df'(x) \right| \leq \sup_{x \in [a, b]} \left( x - \frac{a + b}{2} \right)^2 \sqrt{c} (f')
\]

\[
= \frac{1}{4} (b - a)^2 \sqrt{c} (f')
\]

and by the equality (3) we get (9).

It is well known that if \( p : [a, b] \to \mathbb{C} \) is a Riemann integrable function and \( v : [a, b] \to \mathbb{C} \) is Lipschitzian with the constant \( M > 0 \), i.e.,

\[
|v(s) - v(t)| \leq M |s - t| \quad \text{for any } t, s \in [a, b],
\]

then the Riemann-Stieltjes integral \( \int_a^b p(t) \, dv(t) \) exists and the following inequality holds

\[
\left| \int_a^b p(t) \, dv(t) \right| \leq M \int_a^b |p(t)| \, dt.
\]

Utilizing this property we have

\[
\left| \int_a^b \left( x - \frac{a + b}{2} \right)^2 df'(x) \right| \leq L \int_a^b \left( x - \frac{a + b}{2} \right)^2 d(x)
\]

\[
= \frac{1}{12} L (b - a)^3
\]

and by the equality (3) we get (10).

Now, when some convexity property is assumed for the second derivative, then following result holds.

**Theorem 3.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \((a, b)\) and such that the second derivative \( f'' \) is convex on \((a, b)\). Then

\[
\frac{1}{24} f'' \left( \frac{a + b}{2} \right) (b - a)^3 
\]

\[
\leq \frac{1}{8} (b - a)^2 \left[ f'_+ (b) - f'_+ (a) \right] - \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx \right] 
\]

\[
\leq \frac{f''(a) + f''(b)}{48} (b - a)^3.
\]
showing that

\[ M = \text{convex on } [a, b] \]

Thus, we can apply the inequality (2) for \( h \) to get

\[ \frac{1}{8} (b - a)^2 \left[f'(b) - f'(a)\right] = \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx \]

(12)

Since \( f'' \) is convex on \((a, b)\), then by (2) we have

\[ \frac{1}{12} f''(a + b) (b - a)^3 \leq \int_a^b \left(x - \frac{a + b}{2}\right)^2 f''(x) \, dx \]

(13)

Utilising (12) and (13) we deduce the desired result (11).

\[ \square \]

**Theorem 4.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \((a, b)\).

If there exists a real number \( m \) such that \( f''(x) \geq m \) for any \( x \in (a, b) \), then

\[ \frac{1}{12} f'' \left(\frac{a + b}{2}\right) (b - a)^3 + \frac{1}{180} m (b - a)^5 \]

(14)

\[ \leq \int_a^b \left(x - \frac{a + b}{2}\right)^2 f(x) \, dx \]

\[ \leq \frac{f(a) + f(b)}{24} (b - a)^3 - \frac{7}{1440} m (b - a)^5. \]

If there exists a real number \( M \) such that \( f''(x) \leq M \) for any \( x \in (a, b) \), then

\[ \frac{f(a) + f(b)}{24} (b - a)^3 - \frac{7}{1440} M (b - a)^5 \]

(15)

\[ \leq \int_a^b \left(x - \frac{a + b}{2}\right)^2 f(x) \, dx \]

\[ \leq \frac{1}{12} f'' \left(\frac{a + b}{2}\right) (b - a)^3 + \frac{1}{180} M (b - a)^5. \]

**Proof.** Define the function \( h_m : [a, b] \to \mathbb{R} \) by

\[ h_m(x) := f(x) - \frac{1}{2} m \left(x - \frac{a + b}{2}\right)^2. \]

This function is twice differentiable and the second derivative is

\[ h_m''(x) = f''(x) - m \geq 0, \ x \in (a, b) \]

showing that \( h_m \) is convex on \([a, b]\).

If we apply the inequality (2) for \( h_m \), then we have

\[ \frac{1}{12} f'' \left(\frac{a + b}{2}\right) (b - a)^3 \]

(16)

\[ \leq \int_a^b \left(x - \frac{a + b}{2}\right)^2 f(x) \, dx - \frac{1}{2} m \int_a^b \left(x - \frac{a + b}{2}\right)^4 \, dx \]

\[ \leq \frac{f(a) + f(b)}{24} (b - a)^3 + \frac{1}{96} m (b - a)^5. \]
We also have
\[ \int_a^b \left( x - \frac{a+b}{2} \right)^4 \, dx = \frac{1}{90} (b-a)^5. \]

Then (16) becomes
\[ \frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 + \frac{1}{180} m (b-a)^5 \]
\[ \leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f (x) \, dx \]
\[ \leq \frac{f(a) + f(b)}{24} (b-a)^3 - \frac{1}{96} m (b-a)^5 + \frac{1}{180} m (b-a)^5 \]
which is equivalent to (14).

Now define the function
\[ h_M : [a, b] \to \mathbb{R} \]
by
\[ h_M (x) := \frac{1}{2} M \left( x - \frac{a+b}{2} \right)^2 - f (x). \]
This function is twice differentiable and
\[ h''_M (x) := M - f'' (x) \geq 0, \quad x \in (a, b) \]
showing that \( h_M \) is convex on \([a, b]\).

If we apply the inequality (2) for \( h_M \), then we have
\[ \frac{1}{12} \left[ -f \left( \frac{a+b}{2} \right) \right] (b-a)^3 \]
\[ \leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 \left[ \frac{1}{2} M \left( x - \frac{a+b}{2} \right)^2 - f (x) \right] \, dx \]
\[ \leq \frac{1}{8} M (b-a)^2 - f (a) + \frac{1}{8} M (b-a)^2 - f (b) \]
\[ \leq \frac{24}{24} M (b-a)^2 - f (a) + \frac{1}{8} M (b-a)^2 - f (b) \]
[\( (b-a)^3 \), which, by multiplication with \( -1 \), produces
\[ \frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 \]
\[ \geq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f (x) \, dx - \frac{1}{180} M (b-a)^5 \]
\[ \geq \frac{f(a) + f(b)}{24} (b-a)^3 - \frac{1}{96} M (b-a)^5 \]
\[ = \frac{f(a) + f(b)}{24} - \frac{1}{96} M (b-a)^5 \]
that is equivalent to
\[ \frac{f(a) + f(b)}{24} - \frac{1}{96} M (b-a)^5 + \frac{1}{180} M (b-a)^5 \]
\[ \leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f (x) \, dx \]
\[ \leq \frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 + \frac{1}{180} M (b-a)^5 \]
and the inequality (15) is proved. □

**Corollary 2.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \((a, b)\). If there exists a \( K > 0 \) such that \(|f''(x)| \leq K\) for any \( x \in (a, b)\), then

\[
\left| \int_a^b \left( x - \frac{a + b}{2} \right)^2 f(x) \, dx - \frac{1}{24} \left[ f\left( \frac{a + b}{2} \right) + \frac{f(a) + f(b)}{2} \right] (b - a)^3 \right| \leq \frac{1}{192} K (b - a)^5. \tag{17}
\]

**Proof.** If we write the inequality (14) for \( m = -K \) and the inequality (15) for \( M = K \), then we have

\[
\frac{1}{12} f\left( \frac{a + b}{2} \right) (b - a)^3 - \frac{1}{180} K (b - a)^5 \tag{18}
\]

and

\[
\frac{f(a) + f(b)}{24} (b - a)^3 + \frac{7}{1440} K (b - a)^5 \tag{19}
\]

If we add the inequality (18) with (19) and divide the sum by 2 we get the desired result (17). □

**Remark 3.** We observe that the case \( m > 0 \) in the inequality (14) produces a better result than (2).

For twice differentiable functions we can provide the following perturbed trapezoid quadrature rule

\[
\int_a^b f(x) \, dx \simeq \frac{f(a) + f(b)}{2} (b - a) - \frac{1}{8} (b - a)^2 [f'(b) - f'(a)] + \frac{1}{24} (b - a)^3 \left[ f''\left( \frac{a + b}{2} \right) + \frac{f''(a) + f''(b)}{2} \right]. \tag{20}
\]

Denote \( E_{PT}(f; a, b) \) the error in approximating the integral as in (20), namely

\[
E_{PT}(f; a, b) := \int_a^b f(x) \, dx - \frac{f(a) + f(b)}{2} (b - a) - \frac{1}{8} (b - a)^2 [f'(b) - f'(a)] + \frac{1}{24} (b - a)^3 \left[ f''\left( \frac{a + b}{2} \right) + \frac{f''(a) + f''(b)}{2} \right].
\]

The following result that provides an a priori error bound for functions whose fourth derivatives are bounded, holds.
**Proposition 1.** Let $f : [a, b] \to \mathbb{R}$ be a four time differentiable function on $(a, b)$. If there exists a $K > 0$ such that $|f^{(4)}(x)| \leq K$ for any $x \in (a, b)$, then

$$|E_{P,T}(f; a, b)| \leq \frac{1}{384}K(b - a)^5.$$  \hspace{1cm} (21)

**Proof.** Writing the inequality (17) for the second derivative $f''$ we have

$$\left| \int_a^b \left( x - \frac{a + b}{2} \right)^2 f''(x) \, dx - \frac{1}{24} \left[ f''\left( \frac{a + b}{2} \right) + \frac{f''(a) + f''(b)}{2} \right](b - a)^3 \right| \leq \frac{1}{192}K(b - a)^5.$$  

Dividing this inequality by 2 and utilizing the representation (12) we have

$$\left| \frac{1}{8}(b - a)^2 \left[ f'(b) - f'(a) \right] - \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx \right] \right| \leq \frac{1}{384}K(b - a)^5,$$

and the inequality (21) is proved. \hfill \square

### 3. Applications for Special Means

Let us recall the following means for two positive numbers.

1. **The Arithmetic mean**

   $$A = A(a,b) := \frac{a + b}{2}, \ a,b > 0;$$

2. **The Geometric mean**

   $$G = G(a,b) := \sqrt{ab}, \ a,b > 0;$$

3. **The Harmonic mean**

   $$H = H(a,b) := \frac{2ab}{a + b}, \ a,b > 0;$$

4. **The Logarithmic mean**

   $$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \frac{b - a}{\ln b - \ln a} & \text{if } a \neq b; \end{cases}, \ a,b > 0,$$

5. **The Identric mean**

   $$I = I(a,b) := \begin{cases} a & \text{if } a = b \\ \left( \frac{b^a}{a^b} \right)^{\frac{1}{a - b}} & \text{if } a \neq b; \end{cases}, \ a,b > 0;$$

6. **The $p$-Logarithmic mean**

   $$L_p = L_p(a,b) := \begin{cases} a & \text{if } a = b \\ \left( \frac{a^{p+1} - b^{p+1}}{(p+1)(a - b)} \right)^{\frac{1}{p}} & \text{if } a \neq b. \end{cases}, \ a,b > 0.$$
The following inequality is well known in the literature:

\[ H \leq G \leq L \leq I \leq A. \]  

(22)

It is also known that \( L_p \) is monotonically increasing over \( p \in \mathbb{R} \), denoting \( L_0 = I \) and \( L_{-1} = L \).

Consider the function \( f : [a, b] \subset (0, \infty) \to (0, \infty) \), \( f(x) = x^p \) for \( p \geq 3 \). We have the fourth derivative of the function given by

\[ f^{(4)}(x) = p(p - 1)(p - 2)(p - 3)x^{b-4}, \]

which shows that the second derivative \( f'' \) is convex on \([a, b]\). Applying the inequality (11) we have

\[ \frac{p(p - 1)}{24} A^{p-2}(a, b) (b - a)^2 \]

(23)

\[ \leq \frac{1}{8} p(p - 1) (b - a)^2 L_{p-2}^p(a, b) - A(a^p, b^p) + L^p(a, b) \]

\[ \leq \frac{1}{24} p(p - 1) A(a^{p-2}, b^{p-2}) (b - a)^2 \]

that holds for any \( a, b > 0 \) and \( p \geq 3 \).

Consider the function \( f : [a, b] \subset (0, \infty) \to (0, \infty) \), \( f(x) = \frac{1}{x} \). Then \( f''(x) = \frac{2}{x^3} \) and \( f^{(4)}(x) = \frac{24}{x^4} \) showing that the second derivative is convex on \([a, b]\). Applying the inequality (11) we have

\[ \frac{1}{12} \left( \frac{a + b}{2} \right)^{-3} (b - a)^3 \]

\[ \leq \frac{1}{8} (b - a)^3 \left( \frac{a + b}{a^2b^2} \right) - \left[ \frac{1}{2} + \frac{1}{2} (b - a) - (\ln b - \ln a) \right] \]

\[ \leq \frac{a + b}{24} (b - a)^3. \]

Dividing by \( b - a > 0 \) we have

\[ \frac{1}{12} A^{-3}(a, b) (b - a)^2 \]

(24)

\[ \leq \frac{1}{4} (b - a)^2 \frac{A(a, b)}{G^4(a, b)} - H^{-1}(a, b) + L^{-1}(a, b) \]

\[ \leq \frac{1}{12} H^{-1}(a^3, b^3) (b - a)^2, \]

that holds for any \( a, b > 0 \).

Consider the function \( f : [a, b] \subset (0, \infty) \to (0, \infty) \), \( f(x) = -\ln x \). Then \( f''(x) = \frac{1}{x^2} \) and \( f^{(4)}(x) = \frac{24}{x^4} \) showing that the second derivative is convex on \([a, b]\). Applying the inequality (11) we have

\[ \frac{1}{24} \left( \frac{a + b}{2} \right)^{-2} (b - a)^3 \]

\[ \leq \frac{1}{8} (b - a)^2 \left( \frac{b - a}{ab} \right) + \frac{\ln a + \ln b}{2} (b - a) - \int_a^b \ln x dx \]

\[ \leq \frac{a + b}{48} (b - a)^3. \]
Dividing by $b - a > 0$ we have
\[
\frac{1}{24} \left( \frac{a + b}{2} \right) - 2 (b - a)^2 \\
\leq \frac{1}{8} (b - a)^2 \frac{1}{ab} + \frac{\ln a + \ln b}{2} - \frac{1}{(b - a)} \int_a^b \ln x \, dx \\
\leq \frac{1}{24} + \frac{1}{48} (b - a)^2.
\]

Observe that
\[
\frac{1}{b - a} \int_a^b \ln x \, dx = \frac{1}{b - a} \left[ x \ln x \bigg|_a^b - (b - a) \right] = \left[ \ln \left( \frac{b^b}{a^a} \right)^{1/(b-a)} - 1 \right] = \ln I(a, b),
\]
and
\[
\frac{\ln a + \ln b}{2} = \ln G(a, b).
\]

Then we get
\[
\frac{1}{24} A - 2 (a, b) (b - a)^2 \\
\leq \frac{1}{8} (b - a)^2 G - 2 (a, b) + \ln G(a, b) - \ln I(a, b) \\
\leq \frac{1}{24} H - 1 (a^2, b^2) (b - a)^2
\]
that holds for any $a, b > 0$.

The interested reader may apply the inequality (21) to obtain other similar results. However, the details are omitted here.

References


Mathematics, School of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia.

School of Computational & Applied Mathematics
University of the Witwatersrand, Private Bag 3
Johannesburg 2050, South Africa.

E-mail address: sever.dragomir@vu.edu.au
URL: [http://rgmia.org/dragomir](http://rgmia.org/dragomir)