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SUBCLASSES OF P-VALENT FUNCTIONS DEFINED BY CONVOLUTION INVOLVING THE Q-HYPERGEOMETRIC FUNCTION

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Abstract. Motivated by the familiar q-hypergeometric functions, we introduce here two general subclasses $S^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta)$ and $C^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta)$ of analytic p-valent functions with negative coefficients in the unit disk. These classes generalize well known classes of starlike and convex functions. For these classes, some results which include the coefficient estimates, the integral means inequalities and ν-ρ-neighborhood are obtained.

1. Introduction

The class of p-valent functions has been widely studied by many authors ever since the theory of univalent functions becomes one of the major attraction among the researchers. In fact, Aouf [1], and Hossen [2] studied the subclasses of p-valent functions of order α and type β which are an extension of the familiar classes studied earlier by Gupta and Jain [3]. There are many contributions on the class of prestarlike functions (see e.g. [4, 5]). Recently, Aouf and Silverman [6] studied some subclasses of p-valent γ-prestarlike functions of order α. Subsequently, Aouf [7] introduced and studied the classes of p-valent γ-prestarlike functions of order α and type β. In the present paper, motivated by the works of Raina and Choi [8], Raina and Srivastava [9], and Srivastava and Aouf [10], we introduce new subclasses of p-valent starlike and convex functions of order α type β defined by using the Hadamard product of certain fractional derivative operator of p-valent functions and q-hypergeometric functions.

Let $\mathcal{A}_p$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad p \in \mathbb{N}, \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}_p$ is called p-valent starlike of order α and type β if it satisfies

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\alpha} \right| < \beta, \quad z \in \mathcal{U}, \tag{2}$$

where $0 \le \alpha < p$, $0 < \beta \le 1$ and $p \in \mathbb{N}$. We denote by $S^*(p, \alpha, \beta)$ the class of p-valent starlike functions of order α type β. A function $f(z) \in \mathcal{A}_p$ is called p-valent convex of

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order α and type β if it satisfies

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}-p}{1+\frac{zf''(z)}{f'(z)}+p-2\alpha}\right|<\beta, \quad z\in\mathcal{U}, \tag{3}$$

where $0 \le \alpha < p$, $0 < \beta \le 1$ and $p \in \mathbb{N}$. We denote by $K(p, \alpha, \beta)$ the class of p-valent convex functions of order α type β. From (2) and (3), we note that

$$f(z) \in K(p,\alpha,\beta) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in S^*(p,\alpha,\beta). \tag{4}$$

The classes $S^*(p, \alpha, \beta)$ and $K(p, \alpha, \beta)$ were introduced by Aouf [1] and Hossen [2]. For $\beta = 1$, the class $S^*(p, \alpha, 1) = S^*(p, \alpha)$ was studied by Patil and Thakare [11], and the class $K(p, \alpha, 1) = K(p, \alpha)$ was introduced by Owa [12].

Let $\mathcal{T}_p$ denote the subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad a_{p+n} \ge 0, p \in \mathbb{N}. \tag{5}$$

We denote by $T^*(p, \alpha, \beta)$ and $C(p, \alpha, \beta)$, the classes obtained by taking intersections of the classes $S^*(p, \alpha, \beta)$ and $K(p, \alpha, \beta)$ respectively, with the class $\mathcal{T}_p$. Thus, we have

$$T^*(p,\alpha,\beta) = S^*(p,\alpha,\beta) \cap \mathcal{T}_p, \tag{6}$$

and

$$C(p,\alpha,\beta) = K(p,\alpha,\beta) \cap \mathcal{T}_p. \tag{7}$$

The classes $T^*(p, \alpha, \beta)$ and $C(p, \alpha, \beta)$ were studied by Aouf [1] and Hossen [2]. In particular, the classes $T^*(p, \alpha, 1) = T^*(p, \alpha)$ and $C(p, \alpha, 1) = C(p, \alpha)$ were introduced by Owa [12]. Also the classes $T^*(1, \alpha, 1) = T^*(\alpha)$ and $C(1, \alpha, 1) = C(\alpha)$ when $p = 1$ and $\beta = 1$, were studied by Silverman [13].

For functions $f(z)$ defined by (1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad p \in \mathbb{N},$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}, \quad p \in \mathbb{N}. \tag{8}$$

The function

$$S_\gamma^p = \frac{z^p}{(1-z)^{2(p-\gamma)}}, \quad 0 \le \gamma < p, p \in \mathbb{N}, \tag{9}$$

is the well-known extremal function for the class $S^*(p, \gamma)$. A function $f(z) \in \mathcal{A}_p$ is said to be in the class of p-valent γ-prestarlike of order α and type β, denoted by $R_\gamma^p(\alpha, \beta)$, if

$$(f * S_\gamma^p)(z) \in S^*(p,\alpha,\beta),$$

for $0 \le \alpha < p$, $0 < \beta \le 1$ and $p \in \mathbb{N}$. Moreover, the class $R_\gamma^p[\alpha, \beta] = R_\gamma^p(\alpha, \beta) \cap \mathcal{T}_p$. The classes $R_\gamma^p(\alpha, \beta)$ and $R_\gamma^p[\alpha, \beta]$ were introduced by Aouf [7].

Let h and g be analytic functions in the unit disk $\mathcal{U}$. The function g is subordinate to h, written as $g \prec h$, if g is univalent, $g(0) = h(0)$ and $g(\mathcal{U}) \subset h(\mathcal{U})$. In general, given two functions g and h which are analytic in $\mathcal{U}$, the function g is said to be subordinate to the function h if there exists a function w analytic in $\mathcal{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathcal{U}),$$

such that $g(z) = h(w(z)) \quad (z \in \mathcal{U})$.

A q-hypergeometric series is a power series in one complex variable z with power series coefficients which depend, apart from q, on r complex upper parameters $a_1, \ldots, a_r$ and s complex lower parameters b_1 as follows (see [18, p.4])

$$\phi^{r,s}(z) = \phi^{r,s}\left[\begin{array}{c} a_1, ..., a_r \\ ; q, z \\ b_1, ..., b_s \end{array}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1, q)_n ... (a_r, q)_n}{(q, q)_n (b_1, q)_n ... (b_s, q)_n} \left[(-1)^n q^{\left(\begin{array}{c} n \\ 2 \end{array}\right)}\right]^{1+s-r} z^n, \tag{10}$$

with $\left(\begin{array}{c} n \\ 2 \end{array}\right) = n(n-1)/2$, where $q \neq 0$ when $r > s+1$. Here $(a, q)_n$ is the q-shifted factorial defined by

$$(a, q)_n = \begin{cases} 1, & n = 0; \\ (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), & n \in N. \end{cases}$$

we note that

$$\lim_{q \to 1^-} \phi^{r,s}(q^{a_1}, ..., q^{a_r}; q^{b_1}, ..., q^{b_s}; q, (q-1)^{1+s-r} z) =_r F_s(a_1, ..., a_r; b_1, ..., b_s; z),$$

where ${}_rF_s(a_1, ..., a_r; b_1, ..., b_s; z)$ is the well known generalized hypergeometric function.

Many of the results for the classical hypergeometric functions can be generalized to the q-hypergeometric level. More recently, Purohit and Raina [19] introduced a generalized q-Taylor's formula in fractional q-calculus and derived certain q-generating functions for the q hypergeometric functions.

Remark 1.

i. The series $\phi^{r,s}$ converges absolutely for all z If $0 < |q| < 1$ and $r \leq s$, and for $|z| < 1$ if $r = s+1$. It also converges absolutely for $|z| < |b_1 \ldots b_s q|/|a_1 \ldots a_r|$ if $|q| > 1$.

ii. The series $\phi^{r,s}$ diverges for $z \neq 0$ if $0 < |q| < 1$ and $r > s+1$. It also diverges for $|z| < |b_1 \ldots b_s q|/|a_1 \ldots a_r|$ and $|q| < 1$, unless it terminates.

Now for $0 < |q| < 1$, $r = s+1$ and $p \in \mathbb{N}$, the q-hypergeometric function $\phi_p^{r,s}$ defined for $z \in \mathcal{U}$.

$$\phi_p^{r,s}(a_i; b_j; q, z) = z^p + \sum_{n=1}^{\infty} \frac{(a_1, q)_n ... (a_r, q)_n}{(q, q)_n (b_1, q)_n ... (b_s, q)_n} z^{p+n}. \tag{11}$$

Among several interesting definitions of fractional derivative operators given in the literature (see [8, 9, 10]), it is convenient to recall here the following definitions as follows:

Definition 1. *The fractional derivative operator of order λ is defined, for a function $f(z)$, by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi, 0 \leq \lambda < 1$$

where $f(z)$ is analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 2. *Let $0 \leq \lambda < 1$, and $\mu, \eta \in \mathbf{R}$. Then, in terms of the familiar Gauss's hypergeometric function ${}_rF_s$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is*

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi)_r F_s\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\xi}{z}\right) d\xi\right)$$

where $f(z)$ is analytic function in a simply connected region of the z-plane containing the origin with the order $f(z) = O(|z|^{\varepsilon}), z \to 0$, where $\varepsilon > max\{0, \mu - \eta\} - 1$ and the multiplicity of $(z-\xi)^{\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

Notice that, $J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z)$, $0 \le \lambda < 1$.

Next, we define the following new subclasses of p-valent functions as follows:

Definition 3. *A function $f(z) \in \mathcal{T}_p$ is said to be in the class $S_{\lambda,\mu,\eta}^{p,r,s}(a_i, b_j, q, \alpha, \beta)$ if and only if*

$$\left| \frac{\frac{z(\Omega_p^{\lambda,\mu,\eta} f(z))'}{\Omega_p^{\lambda,\mu,\eta} f(z)} - p}{\frac{z(\Omega_p^{\lambda,\mu,\eta} f(z))'}{\Omega_p^{\lambda,\mu,\eta} f(z)} + p - 2\alpha} \right| < \beta, \quad z \in \mathcal{U} \tag{12}$$

with

$$\Omega_p^{\lambda,\mu,\eta} f(z) = \phi_p^{r,s}(a_i, b_j; q; z) * M_{0,z}^{\lambda,\mu,\eta} f(z) \tag{13}$$

where $\phi_p^{r,s}(a_i; b_j; q, z)$ is given by (11) and $M_{0,z}^{\lambda,\mu,\eta} f(z)$ is the modification of the fractional derivative operator for a function $f(z)$ defined by

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(1-\mu+p)\Gamma(1+\eta-\lambda+p)}{\Gamma(1+p)\Gamma(1+\eta-\mu+p)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) \tag{14}$$

for $a_i \in \mathbb{C}$, $b_j \in \mathbb{C} - \{0, -1, -2, \ldots\}$ $(i = 1, \ldots, r; j = 1, \ldots, s)$,$0 \le \alpha < p$, $0 < \beta \le 1$; $\lambda \ge 0$; $\mu < p+1$; $\eta > max(\lambda, \mu) - p - 1$ and $p \in \mathbb{N}$.

Further, a function $f(z) \in \mathcal{T}_p$ is said to be in the class $C_{\lambda,\mu,\eta}^{p,r,s}(a_i, b_j, q, \alpha, \beta)$ if and only if

$$\frac{zf'(z)}{p} \in S_{\lambda,\mu,\eta}^{p,r,s}(a_i, b_j, q, \alpha, \beta) \tag{15}$$

By suitably specializing the values of $r, s, a_1, \ldots, a_r, b_1, \ldots, b_s, q, \lambda, \mu, \eta, \alpha$ and β, the classes $S_{\lambda,\mu,\eta}^{p,r,s}(a_i, b_j, q, \alpha, \beta)$ and $C_{\lambda,\mu,\eta}^{p,r,s}(a_i, b_j, q, \alpha, \beta)$ lead to various subclasses which were studied by various authors. As illustrations, we present some examples:

Example 1.

i. For $r = 1, s = 0, a_1 = q$ and $\lambda = \mu = 0$ we get

$$\begin{aligned} S_{0,0,\eta}^{p,1,0}(q, -, q, \alpha, \beta) &= T^*(p, \alpha, \beta) \\ &= \{f \in \mathcal{T}_p : \left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\alpha} \right| < \beta\}. \end{aligned}$$

ii. For $r = 1, s = 0, a_1 = q$ and $\lambda = \mu = 1$ we get

$$\begin{aligned} S_{1,1,\eta}^{p,1,0}(q, -, q, \alpha, \beta) &= C(p, \alpha, \beta) \\ &= \{f \in \mathcal{T}_p : \left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\alpha} \right| < \beta\}, \end{aligned}$$

where $T^*(p, \alpha, \beta)$ and $C(p, \alpha, \beta)$ denote the classes of p-valent starlike functions of order α and type β and p-valent convex functions of order α and type β respectively which were studied by Aouf [1] *and Hossen* [2].

Example 2.

i. For $r=1, s=0, a_1=q, p=1$ and $\lambda=\mu=0$ we get

$$\begin{aligned} S^{1,1,0}_{0,0,\eta}(q,-,q,\alpha,\beta) &= S^*(\alpha,\beta) \\ &= \{f\in\mathcal{T} : \left|\frac{\frac{zf'(z)}{f(z)}-1}{\frac{zf'(z)}{f(z)}+1-2\alpha}\right|<\beta\}. \end{aligned}$$

ii. For $r=1, s=0, a_1=q, p=1$ and $\lambda=\mu=1$ we get

$$\begin{aligned} C^{1,1,0}_{1,1,\eta}(q,-,q,\alpha,\beta) &= C^*(\alpha,\beta) \\ &= \{f\in\mathcal{T} : \left|\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf''(z)}{f'(z)}+2-2\alpha}\right|<\beta\}, \end{aligned}$$

where $S^*(\alpha,\beta)$ and $C^*(\alpha,\beta)$ denote the classes of starlike functions of order α and type β, and convex functions of order α and type β respectively which were studied by Gupta and Jain [3].

Example 3.

i. For $r=1, s=0, a_1=q, \beta=1$ and $\lambda=\mu=0$ we get

$$\begin{aligned} S^{p,1,0}_{0,0,\eta}(q,-,q,\alpha,1) &= T^*(p,\alpha) \\ &= \{f\in\mathcal{T}_p : \left|\frac{\frac{zf'(z)}{f(z)}-p}{\frac{zf'(z)}{f(z)}+p-2\alpha}\right|<1\}. \end{aligned}$$

ii. For $r=1, s=0, a_1=q, \beta=1$ and $\lambda=\mu=1$ we get

$$\begin{aligned} C^{p,1,0}_{1,1,\eta}(q,-,q,\alpha,1) &= C(p,\alpha) \\ &= \{f\in\mathcal{T}_p : \left|\frac{1+\frac{zf''(z)}{f'(z)}-p}{1+\frac{zf''(z)}{f'(z)}+p-2\alpha}\right|<1\}, \end{aligned}$$

where $T^*(p,\alpha)$ and $C(p,\alpha)$ denote the classes of p-valent starlike functions of order α and p-valent convex functions of order α and respectively studied by Owa [12].

Example 4.

i. For $r=1, s=0, a_1=q, p=1, \beta=1$ and $\lambda=\mu=0$ we get

$$\begin{aligned} S^{1,1,0}_{0,0,\eta}(q,-,q,\alpha,1) &= T^*(\alpha) \\ &= \{f\in\mathcal{T}_p : \left|\frac{zf'(z)}{f(z)}-1\right|<1-\alpha\}. \end{aligned}$$

ii. For $r=1, s=0, a_1=q, p=1, \beta=1$ and $\lambda=\mu=1$ we get

$$\begin{aligned} C^{1,1,0}_{1,1,\eta}(q,-,q,\alpha,1) &= C(\alpha) \\ &= \{f\in\mathcal{T}_p : \left|\frac{zf''(z)}{f'(z)}\right|<1-\alpha\}, \end{aligned}$$

where $T^*(\alpha)$ and $C(\alpha)$ denote the classes of starlike functions of order α and convex functions of order α respectively studied by Silverman [13].

Example 5. *For $r = 2, s = 1, a_1 = q, a_2 = q^{\alpha_1}, b_1 = q^{\beta_1}$, where $\alpha_1 = 2(p-\gamma)$, $0 \le \gamma < p$, $\beta_1 = 1$, $q \to 1$ and $\lambda = \mu = 0$ we get*

$$\begin{aligned} S^{p,2,1}_{0,0,\eta}(q, q^{\alpha_1}, q^{\beta_1}, \alpha, \beta) &= R^p_\gamma[\alpha, \beta] \\ &= \{f \in \mathcal{T}_p : (f * S^p_\gamma) \in S^*(p, \alpha, \beta)\}, \end{aligned}$$

where $R^p_\gamma[\alpha, \beta]$ denote the class of p-valent γ-prestarlike functions of order α and type β which was studied by Aouf [7].

Example 6. *For $r = 2, s = 1, a_1 = q, a_2 = q^{\alpha_1}, b_1 = q^{\beta_1}$, where $\alpha_1 = 2(p-\gamma)$, $0 \le \gamma < p$, $\beta_1 = 1$, $q \to 1$ and $\beta = 1$, $\quad \lambda = \mu = 0$ we get*

$$\begin{aligned} S^{p,2,1}_{0,0,\eta}(q, q^{\alpha_1}, q^{\beta_1}, \alpha, 1) &= R^p[\gamma, \alpha] \\ &= \{f \in \mathcal{T}_p : (f * S^p_\gamma) \in S^*(p, \alpha)\}, \end{aligned}$$

where $R^p[\gamma, \alpha]$ denote the class of p-valent γ-prestarlike functions of order α which were studied by Aouf and Silverman [6].

2. Coefficient Estimates

Theorem 1. *Let the function $f(z)$ be defined by* (5). *Then $f(z)$ belongs to the class $S^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta)$ if and only if*

$$\sum_{n=1}^{\infty} [n + \beta(n + 2p - 2\alpha)]\Upsilon_n \Delta^p_n(\lambda, \mu, \eta) a_{p+n} \le 2\beta(p - \alpha), \tag{16}$$

where

$$\Upsilon_n = \frac{(a_1, q)_n \dots (a_r, q)_n}{(q, q)_n (b_1, q)_n \dots (b_s, q)_n}, \tag{17}$$

and

$$\Delta^p_n(\lambda, \mu, \eta) = \frac{(1 + p)_n (1 + \eta - \mu + p)_n}{(1 - \mu + p)_n (1 + \eta - \lambda + p)_n}. \tag{18}$$

Corollary 1. *If the function $f(z)$ is in the class $S^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta)$, then*

$$a_{p+n} \le \frac{2\beta(p - \alpha)}{[n + \beta(n + 2p - 2\alpha)]\Upsilon_n \Delta^p_n(\lambda, \mu, \eta)}, \quad p; n \in \mathbb{N}, \tag{19}$$

where Υ_n and $\Delta^p_n(\lambda, \mu, \eta)$ are given by (17) *and* (18) *respectively. The result* (19) *is sharp for the function $f(z)$ of the form*

$$f(z) = z^p - \frac{2\beta(p - \alpha)}{[n + \beta(n + 2p - 2\alpha)]\Upsilon_n \Delta^p_n(\lambda, \mu, \eta)} z^{p+n}, \quad p; n \in \mathbb{N} \tag{20}$$

Theorem 2. *Let the function $f(z)$ be defined by* (5). *Then $f(z)$ belongs to the class $C^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta)$ if and only if*

$$\sum_{n=1}^{\infty} (p + n)[n + \beta(n + 2p - 2\alpha)]\Upsilon_n \Delta^p_n(\lambda, \mu, \eta) a_{p+n} \le 2\beta p(p - \alpha), \tag{21}$$

where Υ_n and $\Delta^p_n(\lambda, \mu, \eta)$ are given by (17) *and* (18) *respectively.*

Corollary 2. *If the function $f(z)$ is in the class $C^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta)$, then*

$$a_{p+n} \le \frac{2\beta p(p - \alpha)}{(p + n)[n + \beta(n + 2p - 2\alpha)]\Upsilon_n \Delta^p_n(\lambda, \mu, \eta)}, \quad p; n \in \mathbb{N}, \tag{22}$$

where Υ_n *and* $\Delta_n^p(\lambda,\mu,\eta)$ *are given by* (25) *and* (26) *respectively. The result* (22) *is sharp for the function* $f(z)$ *of the form*

$$f(z) = z^p - \frac{2\beta p(p-\alpha)}{(p+n)[n+\beta(n+2p-2\alpha)]\Upsilon_n\Delta_n^p(\lambda,\mu,\eta)}z^{p+n}, \quad p; n \in \mathbb{N}. \tag{23}$$

3. Integral means inequalities

Due to Littlewood [20], we obtain integral means inequalities for the functions belonging to the class $S_{\lambda,\mu,\eta}^{p,r,s}(a_i, b_j, q, \alpha, \beta)$.

Lemma 1 ([20]). *If the functions* f *and* g *are analytic in* $\mathcal{U}$ *with* $g \prec f$*, then for* $\varepsilon > 0$ *and* $0 < r < 1$*,*

$$\int_0^{2\pi} |g(re^{i\theta})|^\varepsilon d\theta \le \int_0^{2\pi} |f(re^{i\theta})|^\varepsilon d\theta.$$

Applying (16) and Lemma 1, we prove the following result.

Theorem 3. *Let* $f \in S_{\lambda,\mu,\eta}^{p,r,s}(a_i, b_j, q, \alpha, \beta)$,$0 \le \alpha < p$,$0 < \beta \le 1$,$p \in \mathbb{N}$, $\{\sigma(\alpha,\beta,n,p)\}_{n=1}^{\infty}$ *be a non decreasing sequence defined by*

$$\sigma(\alpha,\beta,n,p) = [n+\beta(n+2p-2\alpha)]\Upsilon_n\Delta_n^p(\lambda,\mu,\eta), \tag{24}$$

and $f_{p+1}(z)$ *be defined by*

$$f_{p+1}(z) = z^p - \frac{2\beta(p-\alpha)}{\sigma(\alpha,\beta,p+1,p)}z^{p+1}.$$

Then for $z = re^{i\theta}$, $0 < r < 1$*, we have*

$$\int_0^{2\pi} |f(re^{i\theta})|^\varepsilon d\theta \le \int_0^{2\pi} |f_{p+1}(re^{i\theta})|^\varepsilon d\theta. \tag{25}$$

Proof. For a function f of the form (5), the inequality (25) is equivalent to

$$\int_0^{2\pi} |1 - \sum_{n=1}^{\infty} a_{p+n}z^n|^\varepsilon d\theta \le \int_0^{2\pi} |1 - \frac{2\beta(p-\alpha)}{\sigma(\alpha,\beta,p+1,p)}z|^\varepsilon d\theta.$$

By Lemma 1, it suffices to show that

$$\sum_{n=1}^{\infty} |a_{p+n}|z^n \prec \frac{2\beta(p-\alpha)}{\sigma(\alpha,\beta,p+1,p)}z. \tag{26}$$

Setting $\sum_{n=1}^{\infty} a_{p+n}z^n = \frac{2\beta(p-\alpha)}{\sigma(\alpha,\beta,p+1,p)}w(z)$, from (24), we obtain

$$|w(z)| = \left|\sum_{n=1}^{\infty} \frac{\sigma(\alpha,\beta,p+1,p)}{2\beta(p-\alpha)}a_{p+n}z^n\right| \le |z|\sum_{n=1}^{\infty} \frac{\sigma(\alpha,\beta,p+1,p)}{2\beta(p-\alpha)}a_{p+n} \le |z| < 1.$$

By the definition of subordination, we have (26). This completes the proof. □

In the view of Examples 1 to 6, we state the following corollaries.

Corollary 3. *For* $0 \le \alpha < p$, $0 < \beta \le 1$, $p \in \mathbb{N}$ *and* $z = re^{i\theta}$, $0 < r < 1$*, we have the following*

i. *If* $f \in S_{0,0,\eta}^{p,1,0}(q, -, q, \alpha, \beta) = T^*(p, \alpha, \beta)$ *and* $f_{p+1}(z)$ *is defined by*

$$f_{p+1}(z) = z^p - \frac{2\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)]}z^{p+1}.$$

Then

$$\int_0^{2\pi} |f(re^{i\theta})|^{\varepsilon} d\theta \le \int_0^{2\pi} |f_{p+1}(re^{i\theta})|^{\varepsilon} d\theta. \tag{27}$$

ii. If $f \in C^{p,1,0}_{1,1,\eta}(q,-,q,\alpha,\beta) = C(p,\alpha,\beta)$ *and* $f_{p+1}(z)$ *is defined by*

$$f_{p+1}(z) = z^p - \frac{2\beta p(p-\alpha)}{(p+1)[1+\beta(1+2p-2\alpha)]} z^{p+1}.$$

Then (27) *holds true.*

Corollary 4. *For* $0 \le \alpha < 1$, $0 < \beta \le 1$, $p = 1$ *and* $z = re^{i\theta}$, $0 < r < 1$, *we have the following*

i. If $f \in S^{p,1,0}_{0,0,\eta}(q,-,q,\alpha,\beta) = S^*(\alpha,\beta)$ *and* $f_{p+1}(z) = f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{2\beta(1-\alpha)}{[1+\beta(3-2\alpha)]} z^2.$$

Then (27) *holds true.*

ii. If $f \in C^{p,1,0}_{1,1,\eta}(q,-,q,\alpha,\beta) = C^*(\alpha,\beta)$ *and* $f_{p+1}(z) = f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{\beta(1-\alpha)}{[1+\beta(3-2\alpha)]} z^2.$$

Then (27) *holds true.*

Corollary 5. *For* $0 \le \alpha < p$, $\beta = 1, p \in \mathbb{N}$ *and* $z = re^{i\theta}$, $0 < r < 1$, *we have the following*

i. If $f \in S^{p,1,0}_{0,0,\eta}(q,-,q,\alpha,\beta) = T^*(p,\alpha)$ *and* $f_{p+1}(z)$ *is defined by*

$$f_{p+1}(z) = z^p - \frac{(p-\alpha)}{(1+p-\alpha)} z^{p+1}.$$

Then (27) *holds true.*

ii. If $f \in C^{p,1,0}_{1,1,\eta}(q,-,q,\alpha,\beta) = C(p,\alpha)$ *and* $f_{p+1}(z)$ *is defined by*

$$f_{p+1}(z) = z^p - \frac{2p(p-\alpha)}{(p+1)(1+p-\alpha)} z^{p+1}.$$

Then (27) *holds true.*

Corollary 6. *For* $0 \le \alpha < 1$, $\beta = 1$, $p = 1$ *and* $z = re^{i\theta}$, $0 < r < 1$, *we have the following*

i. If $f \in S^{p,1,0}_{0,0,\eta}(q,-,q,\alpha,\beta) = T^*(\alpha)$ *and* $f_{p+1}(z) = f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{(1-\alpha)}{(2-\alpha)} z^2.$$

Then (27) *holds true.*

ii. If $f \in C^{p,1,0}_{1,1,\eta}(q,-,q,\alpha,\beta) = C(\alpha)$ *and* $f_{p+1}(z) = f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{(1-\alpha)}{2(2-\alpha)} z^2.$$

Then (27) *holds true.*

Corollary 7. *For $0 \leq \alpha < p$, $0 < \beta \leq 1$, $0 \leq \gamma < p$ and $z = re^{i\theta}$, $0 < r < 1$, we have the following. If $f \in S^{p,2,1}_{0,0,\eta}(q, q^{\alpha_1}, q^{\beta_1}, \alpha, \beta) = R^p_\gamma(\alpha, \beta)$ and $f_{p+1}(z)$ is defined by*

$$f_{p+1}(z) = z^p - \frac{2\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)](2(p-\gamma))_{p+1}} z^{p+1}.$$

Then (27) holds true.

Corollary 8. *For $0 < \alpha \leq p$, $0 \leq \gamma < p$ and $z = re^{i\theta}$, $0 < r < 1$, we have the following. If $f \in S^{p,2,1}_{0,0,\eta}(q, q^{\alpha_1}, q^{\beta_1}, \alpha, 1) = R^p(\gamma, \alpha)$ and $f_{p+1}(z)$ is defined by*

$$f_{p+1}(z) = z^p - \frac{(p-\alpha)}{(1+p-\alpha)(2(p-\gamma))_{p+1}} z^{p+1}.$$

Then (27) holds true.

Remark 2. *If we take $\alpha = 0$ in $T^*(\alpha)$ and $C(\alpha)$ of Corollary 6, we get the integral means obtained by Silverman [14].*

4. Neighborhoods of the class $S^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta)$

For $f \in \mathcal{T}_p$ of the form (5), and $\rho \geq 0$, we define the ν-ρ-neighborhood of f as the following:

$$M^\nu_\rho(f) = \left\{ g \in \mathcal{T}_p : g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad \sum_{n=1}^{\infty} n^{\nu+1} |a_{p+n} - b_{p+n}| \leq \rho \right\}, \qquad (28)$$

where ν is a fixed positive integer. It follows from (28), that if $e(z) = z$, then

$$M^\nu_\rho(e) = \left\{ g \in \mathcal{T}_p : g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad \sum_{n=1}^{\infty} n^{\nu+1} |b_{p+n}| \leq \rho \right\}. \qquad (29)$$

For $p = 1$ and $f \in \mathcal{T}$, we have the ν-ρ-neighborhood of f which was investigated by Farsin and Darus [17]. We note that $M^0_\rho(f) \equiv N_\rho(f)$, $M^1_\rho(f) \equiv M_\rho(f)$, where $N_\rho(f)$ is called a ρ-neighborhood of f introduced by Ruscheweyh [15] and $M_\rho(f)$ was defined by Silverman [16].

Now, we investigate ν-ρ-neighborhood for functions f in the class $S^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta)$.

Theorem 4. *If $\sigma(\alpha, \beta, n, p)/n^{\nu+1}$ defined by (24), then $S^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta) \subset M^\nu_\rho(e)$, where*

$$\rho = \frac{2^{\nu+1} 2\beta(p-\alpha)}{\sigma(\alpha, \beta, p+1, p)}.$$

Proof. It follows from (16) that if $f \in S^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta)$, then

$$\sum_{n=1}^{\infty} n^{\nu+1} |a_{p+n}| \leq \frac{2^{\nu+1} 2\beta(p-\alpha)}{\sigma(\alpha, \beta, p+1, p)}.$$

This gives that $S^{p,r,s}_{\lambda,\mu,\eta}(a_i, b_j, q, \alpha, \beta) \subset M^\nu_\rho(e)$. □

By taking different choices of $r, s, a_1, ..., a_r, b_1, ..., b_s, q, \alpha$ and β in the above theorem, we can state the following neighborhood results for various subclasses studied earlier by several researchers.

In view of the Examples 1 to 6 in Section 1 and Theorem 1, we have the following corollaries for the classes defined in these examples.

Corollary 9. *i.* $T^*(p,\alpha,\beta) \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}2\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)]}.$$

ii. $C(p,\alpha,\beta) \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}2\beta p(p-\alpha)}{(p+1)[1+\beta(1+2p-2\alpha)]}.$$

Corollary 10. *i.* $S^*(\alpha,\beta) \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}2\beta(1-\alpha)}{[1+\beta(3-2\alpha)]}.$$

ii. $C^*(\alpha,\beta) \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}\beta(1-\alpha)}{[1+\beta(3-2\alpha)]}.$$

Corollary 11. *i.* $T^*(p,\alpha) \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}(p-\alpha)}{(1+p-\alpha)}.$$

ii. $C(p,\alpha) \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}2p(p-\alpha)}{(p+1)(1+p-\alpha)}.$$

Corollary 12. *i.* $T^*(\alpha) \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}(1-\alpha)}{(2-\alpha)}.$$

ii. $C(\alpha) \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}(1-\alpha)}{2(2-\alpha)}.$$

Corollary 13. $R_\gamma^p[\alpha,\beta] \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}2\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)](2(p-\gamma))_{p+1}}.$$

Corollary 14. $R^p[\gamma,\alpha] \subset M_\rho^\nu(e)$, *where*

$$\rho = \frac{2^{\nu+1}(p-\alpha)}{(1+p-\alpha)(2(p-\gamma))_{p+1}}.$$

Remark 3. *A similar theorems of integral means inequalities and ν-ρ-neighborhood can be easily obtained for the class $C_{\lambda,\mu,\eta}^{p,r,s}(a_i,b_j,q,\alpha,\beta)$.*

References

[1] Aouf, M.K., *A generalization of multivalent functions with negative coefficients*, J. Korean Math. Soc. **25** (1) (1988), 53-66.

[2] Hossen, H.M., *Quasi-Hadamard product of Certain p-valent functions*, Demonstration Math. **33** (2) (2000), 277-281.

[3] Gupta, V.P. and Jain, P.K., *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc. **14** (1976), 409-416.

[4] Owa, S. and Uralegaddi, B.A., *A class of functions α-prestarlike of order β*, Bull. Korean Math. Soc. **21** (2) (1984), 77-85.

[5] Silverman, H. and Silvia, E.M., *Subclasses prestarlike functions*, Math. Japan. **29** (1984), 929-935.
[6] Aouf, M.K. and Silverman, H., *Subclasses of p-valent and prestarlike functions*, Internat. J. Contemp. Math. Soc. **2** (8) (2007), 357-372.
[7] Aouf, M.K., *Certain subclasses of multivalent prestarlike functions with negative coefficients*, J. Math. Inequal. **1** (4) (2007), 535-552.
[8] Raina, R.K. and Choi, J.H., *On a subclass of analytic and multivalent functions associated with a certain fractional calculus operator*, Indian J. Pure Appl. Math. **33** (1) (2002), 55-62.
[9] Raina, R.K. and Srivastava, H.M., *A certain subclass of analytic functions associated with operators of fractional calculus*, Computers and Mathematics with Applications **32** (1996), 13-19.
[10] Srivastava, H.M. and Aouf, M.K., *A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients*, J. Math. Anal. Appl. **171** (1992), 1-13.
[11] Partil, D.A. and Thakare, N.K., *On convex hulls and extreme points of p-valent starlike and convex classes with applications*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.) **27** (75) (1983), 145-160.
[12] Owa, S., *On certain classes of p-valent functions with negative coefficients*, Bull. Belg. Soc. Simon Stevin **59** (1985), 385-402.
[13] Silverman, H., *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 929-935.
[14] Silverman, H., *Integral means for univalent functions with negative coefficients*, Houston J. Math. **23** (1997), 169-174.
[15] Ruscheweyh, St., *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **81** (1981), 521-527.
[16] Silverman, H., *Neighborhoods of class of analytic functions*, Far East J. Math. Sci. **3** (1995), 165-169.
[17] Frasin, B. and Darus, M., *Integral means and neighborhoods for analytic univalent functions with negative coefficients*, Soochow J. Math. **30** (2004), 217-223.
[18] Gasper, G. and Rahman, M., *Basic Hypergeometric Series* (with a Foreword by Richard Askey), Encyclopedia of Mathematics and Its Application, Vol. 35, Cambridge University Press,Cambridge, New York, Port Chester, Melbourne and Sydney, 1990; Second edition, Encyclopedia of Mathematics and Its Application, Vol. 96, Cambridge University Press, Cambridge, London and New York, 2004.
[19] Purohit, S.D. and Raina, R.K., *Generalized q-Taylor's series and applications*, General Mathematics **18** (3) (2010), pp. 19-28.
[20] Littlewood, J.E., *On inequalities in theory of functions*, Proc. London Math. Soc. **23** (1925), 481-519.

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