TJMM
4 (2012), No. 2, 137–146

ANTIPERIODIC SOLUTIONS FOR SHUNTING INHIBITORY CELLULAR NEURAL NETWORKS WITH NONLINEAR BEHAVED FUNCTIONS AND MIXED DELAYS

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ABSTRACT. In this paper, a class of shunting inhibitory cellular neural networks (SICNNs) with nonlinear behaved functions and mixed delays are are considered. Sufficient conditions for the existence and globally exponentially stability of the antiperiodic solutions are established, which are new and complement previously known results. An example is employed to illustrate our feasible results.

1. INTRODUCTION

Recently, the dynamical behaviors of the shunting inhibitory cellular neural networks (SICNNs) with delays and constant coefficients have also been widely investigated. Many important results on the existence and uniqueness of equilibrium point, global asymptotic stability, and global exponential stability have been established and successfully applied to signal processing, pattern recognition, associative memories, and so on. There exist some results on the existence and stability of periodic and almost periodic solutions for the SICNNs with delays and constant coefficients. We refer readers to [1]–[10] and the references cited therein. Peng and Huang [11] and Wu and Zhou [12] obtained the existence and exponential stability of anti-periodic solutions for SICNNs with continuously distributed delays. To the best of author's knowledge, few authors have considered the existence and global exponential stability of antiperiodic solutions for SICNNs with nonlinear behaved functions and time-varying and continuously distributed delays (mixed delays). Obviously, SICNNs with nonlinear behaved functions and mixed delays is general and is worth to continue to investigate its dynamical properties such as existence and global exponential stability of antiperiodic solutions.

In this paper, we shall continue to consider the following SICNNs with mixed delays:

$$\begin{aligned} x'_{ij}(t) &= -a_{ij}(t,x_{ij}(t)) - \sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(t) f_{ij}(x_{kl}(t))x_{ij}(t) \\ &\quad - \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t) g_{ij}(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t) \\ &\quad - \sum_{D^{kl}\in N_r(i,j)} D_{ij}^{kl}(t)\int_0^\infty K_{ij}(u) h_{ij}(x_{kl}(t-u))du x_{ij}(t) + L_{ij}(t), \end{aligned} \tag{1}$$

where $i = 1,\ldots,m$, $j = 1,\ldots,n$, $\tau_{ij}(t)$ represents axonal signal transmission delay and is continuous with $0 \le \tau_{ij}(t) \le \tau$; $C_{ij}(t)$ denotes the cell at the (i,j) position of the lattice

2010 *Mathematics Subject Classification.* 34C25, 34K13.
Key words and phrases. Shunting inhibitory cellular neural networks, antiperiodic solutions, globally exponentially stability, time-varying and continuously distributed delays.

at the t; the r-neighborhood $N_r(i,j)$ of $C_{ij}(t)$ is

$$N_r(i,j) = \left\{C_{ij}^{kl} : \max(|k-i|, |l-j|) \le r, 1 \le k \le m, 1 \le l \le n\right\},$$

$x_{ij}(t)$ is the activity of the cell $C_{ij}(t)$ at time t, $L_{ij}(t)$ is the external input to $C_{ij}(t)$, $a_{ij}(t, x_{ij}(t))$ represents an appropriately behaved function of the cell $C_{ij}(t)$ at time t, which may be linear or nonlinear; nonnegative functions $B_{ij}^{kl}(t)$, $C_{ij}^{kl}(t)$ and $D_{ij}^{kl}(t)$ are the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell $C_{ij}(t)$ depending upon at time t, discrete delays and distributed delays, respectively; the activity functions $f_{ij}(\cdot)$, $g_{ij}(\cdot)$ and $h_{ij}(\cdot)$ are continuous function representing the output or firing rate of the cell $C_{ij}(t)$, respectively.

Let $u(t) : R \to R$ be continuous in t. $u(t)$ is said to be T-anti-periodic on R if $u(t+T) = -u(t)$ for all $t \in R$. The initial conditions associated with system (1) are of the form

$$x_{ij}(s) = \varphi_{ij}(s), \quad s \in (-\infty, 0], i = 1, \dots, m, j = 1, \dots, n,$$

where $\varphi_{ij}(t)$ denotes a real-valued bounded continuous function on $(-\infty, 0]$.

The objective of this paper is to give some sufficient conditions ensuring the existence and globally exponential stability of antiperiodic solution of system (1), which are new and complement the previously known results. Moreover, an example is provided to illustrate the effectiveness of our results.

For the sake of simplicity, we introduce some notations as follows.

$$\tau = \sup_{t\in R} \max_{(i,j)} \tau_{kl}(t), \quad \overline{B}_{ij}^{kl} = \sup_{t\in R} B_{ij}^{kl}(t), \quad \overline{C}_{ij}^{kl} = \sup_{t\in R} C_{ij}^{kl}(t), \quad \overline{D}_{ij}^{kl} = \sup_{t\in R} D_{ij}^{kl}(t).$$

Throughout this paper, we always consider system (1) together with the following assumptions.

- (H_1) $B_{ij}^{kl}, C_{ij}^{kl}, D_{ij}^{kl}, \tau_{ij} \in C(R, [0, +\infty))$, $I_{ij} \in C(R, R)$, $B_{ij}^{kl}(t+T) = B_{ij}^{kl}(t)$, $C_{ij}^{kl}(t+T) = C_{ij}^{kl}(t)$, $D_{ij}^{kl}(t+T) = D_{ij}^{kl}(t)$, $\tau_{ij}(t+T) = \tau_{ij}(t)$, $L_{ij}(t+T) = -L_{ij}(t)$, $i = 1, \dots, m$, $j = 1, \dots, n$.
- (H_2) $a_{ij} \in C(R^2, R)$, $a_{ij}(t+T, u) = -a_{ij}(t, -u)$. Furthermore, there exist positive constants $\underline{a}_{ij}$ such that $0 < \underline{a}_{ij} \le \partial a_{ij}(t,u)/\partial u$, $a_{ij}(t, 0) = 0$, $i = 1, \dots, m$, $j = 1, \dots, n$.
- (H_3) $f_{ij}, g_{ij}, h_{ij} \in C(R^2, R)$, $f_{ij}(0) = g_{ij}(0) = h_{ij}(0) = 0$, and there exist nonnegative constants $\mu_{ij}^f, \mu_{ij}^g, \mu_{ij}^h$ and $M_{ij}^f, M_{ij}^g, M_{ij}^h$ such that for $\forall u, v \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$,

$$f_{ij}(-u) = f_{ij}(u), \quad |f_{ij}(t,u) - f_{ij}(t,v)| \le \mu_{ij}^f |u - v|, \quad |f_{ij}(u)| \le M_{ij}^f,$$
$$g_{ij}(-u) = g_{ij}(u), \quad |g_{ij}(t,u) - g_{ij}(t,v)| \le \mu_{ij}^g |u - v|, \quad |g_{ij}(u)| \le M_{ij}^g,$$
$$h_{ij}(-u) = h_{ij}(u), \quad |h_{ij}(t,u) - h_{ij}(t,v)| \le \mu_{ij}^h |u - v|, \quad |h_{ij}(u)| \le M_{ij}^h.$$

- (H_4) We assume that there exists a constant L_{ij}^+ such that

$$L_{ij}^+ > \overline{L}_{ij} = \sup_{t\in R} |L_{ij}(t)|.$$

- (H_5) There exist constants $\delta_{ij} > 0$, $\eta > 0$ and $\lambda > 0$ such that for $i = 1, \dots, m$, $j = 1, \dots, n$,

$$\begin{aligned}\delta_{ij} \quad = \quad & \underline{a}_{ij} - \sum_{B^{kl}\in N_r(i,j)} \overline{B}_{ij}^{kl} M_{ij}^f - \sum_{C^{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_{ij}^g \\ & - \sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl} M_{ij}^h \int_0^\infty |K_{ij}(u)| du,\end{aligned}$$

$$\left(\lambda-\underline{a}_{ij}\right)+\sum_{B^{kl}\in N_r(i,j)}\overline{B}_{ij}^{kl}\left(M_{ij}^f+\mu_{ij}^f\frac{L_{ij}^+}{\delta_{ij}}\right)+\sum_{C^{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}\left(M_{ij}^g+e^{\lambda\tau}\mu_{ij}^g\frac{L_{ij}^+}{\delta_{ij}}\right)$$
$$+\sum_{D^{kl}\in N_r(i,j)}\overline{D}_{ij}^{kl}\left(M_{ij}^h\int_0^\infty|K_{ij}(u)|du+\mu_{ij}^h\frac{L_{ij}^+}{\delta_{ij}}\int_0^\infty|K_{ij}(u)|e^{\lambda u}du\right)<-\eta<0.$$

Definition 1. *Let $x^*(t)=\{x_{ij}^*(t)\}$ be an anti-periodic solution of system* (1) *with initial value $\varphi^*(t)=\{\varphi_{ij}^*\}$. If there exist constants $\lambda>0$ and $M>1$ such that for every solution $x(t)=\{x_{ij}(t)\}$ with initial value $\varphi(t)=\{\varphi_{ij}(t)\}$,*

$$|x_{ij}(t)-x_{ij}^*(t)|<M\|\varphi-\varphi^*\|_\infty e^{-\lambda t},\quad \forall t>0,\ i=1,\dots,m,\ j=1,\dots,n,$$

where $\|\varphi-\varphi^*\|_\infty=\sup_{-\infty<s\le 0}\max_{(i,j)}|\varphi_{ij}(s)-\varphi_{ij}^*(s)|$. Then $x^*(t)$ is said to be globally exponentially stable.

2. Main results

In this section, we will state and prove our main results of this paper.

System (1) can be written as the following system

$$\begin{aligned}x_{ij}'(t) &= -\alpha_{ij}(t,x_{ij}(t))x_{ij}(t)-\sum_{B^{kl}\in N_r(i,j)}B_{ij}^{kl}(t)f_{ij}(x_{kl}(t))x_{ij}(t)\\ &\quad-\sum_{C^{kl}\in N_r(i,j)}C_{ij}^{kl}(t)g_{ij}(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t)\\ &\quad-\sum_{D^{kl}\in N_r(i,j)}D_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)h_{ij}(x_{kl}(t-u))du\,x_{ij}(t)+L_{ij}(t),\end{aligned}\tag{2}$$

where $\alpha_{ij}(t,x_{ij}(t))\triangleq(\partial a_{ij}(t,u)/\partial u)|_{u=\xi_{ij}}$, $(\partial a_{ij}(t,u)/\partial u)|_{u=\xi_{ij}}$ denotes the derivative of $a_{ij}(t,u)$ at point $u=\xi_{ij}$, $e_{ij}\in R$, $0\le|\xi_{ij}|\le|x_{ij}(t)|$. From (H_2), we know that $a_{ij}(t,x_{ij}(t))$ is strictly monotone increasing about x_{ij}. Hence, $\alpha_{ij}(t,x_{ij}(t))$ is unique for any x_{ij}. Obviously, one can obtain that $\alpha_{ij}(t,x_{ij}(t))\ge\underline{a}_{ij}$.

Lemma 1. *Let (H_1)-(H_5) hold. Let $\tilde{x}(t)=\{\tilde{x}_{ij}(t)\}$ be a solution of system* (1) *with initial conditions*

$$\tilde{x}_{ij}(s)=\tilde{\varphi}_{ij}(s),\quad |\tilde{\varphi}_{ij}(s)|\le\frac{L_{ij}^+}{\delta_{ij}},\quad s\in(-\infty,0].$$

Then

$$\tilde{x}_{ij}(t)\le\frac{L_{ij}^+}{\delta_{ij}},\quad t\ge0,\ i=1,\dots,m,\ j=1,\dots,n.\tag{3}$$

Proof. Assume, by way of contradiction, that (3) does not hold. Then, there must exist $i\in\{1,2,\dots,m\}$, $j\in\{1,2,\dots,n\}$ and $t_0>0$ such that

$$\tilde{x}_{ij}(t_0)=\frac{L_{ij}^+}{\delta_{ij}},\ \text{and}\ \tilde{\varphi}_{ij}(t)\le\frac{L_{ij}^+}{\delta_{ij}},\quad t\in(-\infty,t_0].\tag{4}$$

Calculating the upper left derivative of $|\tilde{x}_{ij}(t_0)|$, together with (H_1)-(H_4), (2) and (4), we can obtain

$$\begin{aligned}0&\le D^+|x_{ij}(t_0)|\\ &\le -\alpha_{ij}(t_0,x_{ij}(t_0))|x_{ij}(t_0)|+\sum_{B^{kl}\in N_r(i,j)}B_{ij}^{kl}(t_0)|f_{ij}(x_{kl}(t_0))||x_{ij}(t_0)|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t_0)|g_{ij}(x_{kl}(t_0-\tau_{kl}(t_0)))||x_{ij}(t_0)| \\
&+\sum_{D^{kl}\in N_r(i,j)} D_{ij}^{kl}(t_0)\int_0^\infty |K_{ij}(u)||h_{ij}(x_{kl}(t_0-u))|du|x_{ij}(t_0)|+|L_{ij}(t_0)| \\
\le\ & -\underline{a}_{ij}\frac{L_{ij}^+}{\delta_{ij}}+\sum_{B^{kl}\in N_r(i,j)}\overline{B}_{ij}^{kl}M_{ij}^f\frac{L_{ij}^+}{\delta_{ij}}+\sum_{C^{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}M_{ij}^g\frac{L_{ij}^+}{\delta_{ij}} \\
&+\sum_{D^{kl}\in N_r(i,j)}\overline{D}_{ij}^{kl}M_{ij}^h\int_0^\infty |K_{ij}(u)|du\frac{L_{ij}^+}{\delta_{ij}}+\overline{L}_{ij} \\
=\ & -\left\{\underline{a}_{ij}-\sum_{B^{kl}\in N_r(i,j)}\overline{B}_{ij}^{kl}M_{ij}^f-\sum_{C^{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}M_{ij}^g\right. \\
&\left.-\sum_{D^{kl}\in N_r(i,j)}\overline{D}_{ij}^{kl}M_{ij}^h\int_0^\infty |K_{ij}(u)|du\right\}\frac{L_{ij}^+}{\delta_{ij}}+\overline{L}_{ij}<0.
\end{aligned}
$$

This is a contradiction and hence (3) holds. This completes the proof. □

Remark 1. *In view of the boundedness of this solution, from the theory of functional differential equations in* [13], *it follows that* $\tilde{x}(t)$ *can be defined on* $[0,+\infty)$.

Lemma 2. *Let* (H_1)*-*(H_5) *hold. Let* $x^*(t)=(x_{11}^*(t),x_{12}^*(t),\ldots,x_{mn}^*(t))^T$ *be the solution of system (1) with initial value* $\varphi^*(t)=(\varphi_{11}^*(t),\varphi_{12}^*(t),\ldots,\varphi_{mn}^*(t))^T$, $|\varphi_{ij}^*(t)|\le\frac{L_{ij}^+}{\delta_{ij}}$ *and* $x(t)=(x_{11}(t),x_{12}(t),\ldots,x_{mn}(t))^T$ *be the solution of system (1) with initial value* $\varphi(t)=(\varphi_{11}(t),\varphi_{12}(t),\ldots,\varphi_{mn}(t))^T$. *Then there exist constants* $\lambda>0$ *and* $M>1$ *such that*

$$|x_{ij}(t)-x_{ij}^*(t)|\le M\|\varphi-\varphi^*\|_\infty e^{\lambda t},\quad \forall t\ge 0,\ i=1,\ldots,m,\ j=1,\ldots,n.$$

Proof. Let $y(t)=\{y_{ij}(t)\}=\{x_{ij}(t)-x_{ij}^*(t)\}=x(t)-x^*(t)$. Then

$$
\begin{aligned}
& y_{ij}'(t) \\
=\ & -\left(a_{ij}(t,x_{ij}(t))-a_{ij}(t,x_{ij}^*(t))\right) \\
& -\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(t)\left(f_{ij}(x_{kl}(t))x_{ij}(t)-f_{ij}(x_{kl}^*(t))x_{ij}^*(t)\right) \\
& -\sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\left(g_{ij}(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t)-g_{ij}(x_{kl}^*(t-\tau_{kl}(t)))x_{ij}^*(t)\right) \\
& -\sum_{D^{kl}\in N_r(i,j)} D_{ij}^{kl}(t)\left(\int_0^\infty K_{ij}(u)h_{ij}(x_{kl}(t-u))dux_{ij}(t)\right. \\
& \qquad\qquad\left.-\int_0^\infty K_{ij}(u)h_{ij}(x_{kl}^*(t-u))dux_{ij}^*(t)\right) \\
=\ & -\beta_{ij}(t,y_{ij}(t))y_{ij}(t)-\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(t)\left(f_{ij}(x_{kl}(t))x_{ij}(t)-f_{ij}(x_{kl}^*(t))x_{ij}^*(t)\right) \\
& -\sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\left(g_{ij}(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t)-g_{ij}(x_{kl}^*(t-\tau_{kl}(t)))x_{ij}^*(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{D^{kl}\in N_r(i,j)} D_{ij}^{kl}(t)\left(\int_0^\infty K_{ij}(u)h_{ij}(x_{kl}(t-u))du x_{ij}(t)\right.\\
&\qquad\qquad \left. - \int_0^\infty K_{ij}(u)h_{ij}(x_{kl}^*(t-u))du x_{ij}^*(t)\right),\\
&\qquad\qquad i=1,\ldots,m,\ j=1,\ldots,n, \qquad (5)
\end{aligned}
$$

where $\beta_{ij}(t,y_{ij}(t)) \triangleq (\partial a_{ij}(t,u)/\partial u)|_{u=x_{ij}^*(t)+\theta y_{ij}(t)}$, $0\le\theta\le 1$. Similarly to (2), we have $\beta_{ij}(t,y_{ij}(t)) \ge \overline{a}_{ij}$.

We consider the Lyapunov functional

$$V_{ij}(t) = |y_{ij}(t)|e^{\lambda t}, \quad i=1,\ldots,m,\ j=1,\ldots,n. \qquad (6)$$

Calculating the upper right derivative of $V_{ij}(t)$ along the solution $y(t)=\{y_{ij}(t)\}$ of system (5) with the initial value $\bar{\varphi}=\varphi-\varphi^*$, we have

$$
\begin{aligned}
& D^+(y_{ij}(t))\\
\le\ & \lambda|y_{ij}(t)|e^{\lambda t} - \beta_{ij}(t,y_{ij}(t))|y_{ij}(t)|e^{\lambda t}\\
& + e^{\lambda t}\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(t)\left|f_{ij}(x_{kl}(t))x_{ij}(t) - f_{ij}(x_{kl}^*(t))x_{ij}^*(t)\right|\\
& + e^{\lambda t}\sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\left|g_{ij}(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t) - g_{ij}(x_{kl}^*(t-\tau_{kl}(t)))x_{ij}^*(t)\right|\\
& + e^{\lambda t}\sum_{D^{kl}\in N_r(i,j)} D_{ij}^{kl}(t)\left|\int_0^\infty K_{ij}(u)h_{ij}(x_{kl}(t-u))du x_{ij}(t)\right.\\
& \qquad\qquad \left. - \int_0^\infty K_{ij}(u)h_{ij}(x_{kl}^*(t-u))du x_{ij}^*(t)\right|\\
\le\ & \left(\lambda - \underline{a}_{ij}\right)|y_{ij}(t)|e^{\lambda t}\\
& + e^{\lambda t}\sum_{B^{kl}\in N_r(i,j)} \overline{B}_{ij}^{kl}\left(|f_{ij}(x_{kl}(t))||y_{ij}(t)| + |f_{ij}(x_{kl}(t)) - f_{ij}(x_{kl}^*(t))||x_{ij}^*(t)|\right)\\
& + e^{\lambda t}\sum_{C^{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\left(|g_{ij}(x_{kl}(t-\tau_{kl}(t)))||y_{ij}(t)|\right.\\
& \qquad\qquad \left. + |g_{ij}(x_{kl}(t-\tau_{kl}(t))) - g_{ij}(x_{kl}^*(t-\tau_{kl}(t)))||x_{ij}^*(t)|\right)\\
& + e^{\lambda t}\sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl}\int_0^\infty |K_{ij}(u)||h_{ij}(x_{kl}(t-u)) - h_{ij}(x_{kl}^*(t-u))|du|x_{ij}^*(t)|\\
& + e^{\lambda t}\sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl}\int_0^\infty |K_{ij}(u)||h_{ij}(x_{kl}(t-u))|du|y_{ij}(t)|,\\
& \qquad\qquad i=1,\ldots,m,\ j=1,\ldots,n. \qquad (7)
\end{aligned}
$$

Let $M>1$ denote an arbitrary real number and set

$$\|\varphi-\varphi^*\|_\infty = \sup_{-\infty<s\le 0}\max_{(i,j)}|\varphi_{ij}-\varphi_{ij}^*| > 0.$$

It follows from (6) that

$$V_{ij}(t) = |y_{ij}(t)|e^{\lambda t} < M\|\varphi-\varphi^*\|_\infty, \quad \forall t\in(-\infty,0],\ i=1,\ldots,m,\ j=1,\ldots,n.$$

We claim that the following statement is true:

$$V_{ij}(t) = |y_{ij}(t)|e^{\lambda t} < M\|\varphi-\varphi^*\|_\infty, \quad \forall t>0,\ i=1,\ldots,m,\ j=1,\ldots,n. \qquad (8)$$

Contrarily, there must exist some $i \in \{1,2,\ldots,m\}$, $j \in \{1,2,\ldots,n\}$ and $\tilde{t} > 0$ such that

$$V_{ij}(\tilde{t}) = M\|\varphi - \varphi^*\|_\infty, \;\; V_{kl}(t) = M\|\varphi - \varphi^*\|_\infty \quad \forall t \in (-\infty, \tilde{t}), \tag{9}$$

for $k = 1,\ldots,m$, $l = 1,\ldots,n$. Together with (7) and (9), we obtain

$$\begin{aligned}
&0 \le D^+(y_{ij}(\tilde{t}) - M\|\varphi - \varphi^*\|_\infty) = D^+(y_{ij}(\tilde{t})) \\
\le\;& \left(\lambda - \underline{a}_{ij}\right)|y_{ij}(\tilde{t})|e^{\lambda\tilde{t}} \\
&+e^{\lambda\tilde{t}} \sum_{B^{kl}\in N_r(i,j)} \overline{B}_{ij}^{kl}\left(|f_{ij}(x_{kl}(\tilde{t}))||y_{ij}(\tilde{t})| + |f_{ij}(x_{kl}(\tilde{t})) - f_{ij}(x^*_{kl}(\tilde{t}))||x^*_{ij}(\tilde{t})|\right) \\
&+e^{\lambda\tilde{t}} \sum_{C^{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\left(|g_{ij}(x_{kl}(\tilde{t}-\tau_{kl}(\tilde{t})))||y_{ij}(\tilde{t})|\right. \\
&\qquad\qquad \left.+|g_{ij}(x_{kl}(\tilde{t}-\tau_{kl}(\tilde{t}))) - g_{ij}(x^*_{kl}(\tilde{t}-\tau_{kl}(\tilde{t})))||x^*_{ij}(\tilde{t})|\right) \\
&+e^{\lambda\tilde{t}} \sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl}\int_0^\infty |K_{ij}(u)||h_{ij}(x_{kl}(\tilde{t}-u)) - h_{ij}(x^*_{kl}(\tilde{t}-u))|du|x^*_{ij}(\tilde{t})| \\
&+e^{\lambda\tilde{t}} \sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl}\int_0^\infty |K_{ij}(u)||h_{ij}(x_{kl}(\tilde{t}-u))|du|y_{ij}(\tilde{t})| \\
\le\;& \left(\lambda - \underline{a}_{ij}\right)|y_{ij}(\tilde{t})|e^{\lambda\tilde{t}} + \sum_{B^{kl}\in N_r(i,j)} \overline{B}_{ij}^{kl}\left(M^f_{ij}|y_{ij}(\tilde{t})|e^{\lambda\tilde{t}} + \mu^f_{ij}|y_{kl}(\tilde{t})|e^{\lambda\tilde{t}}|x^*_{ij}(\tilde{t})|\right) \\
&+ \sum_{C^{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\left(M^g_{ij}|y_{ij}(\tilde{t})|e^{\lambda\tilde{t}} + \mu^g_{ij}e^{\lambda\tau}|y_{kl}(\tilde{t}-\tau_{kl}(\tilde{t})))|e^{\lambda(\tilde{t}-\tau_{kl}(\tilde{t}))}|x^*_{ij}(\tilde{t})|\right) \\
&+ \sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl}\left(M^h_{ij}\int_0^\infty |K_{ij}(u)|du|y_{ij}(\tilde{t})|e^{\lambda\tilde{t}}\right. \\
&\qquad\qquad \left.+\mu^h_{ij}\int_0^\infty |K_{ij}(u)|e^{\lambda u}|y_{kl}(\tilde{t}-u))|e^{\lambda(\tilde{t}-u)}du|x^*_{ij}(\tilde{t})|\right) \\
\le\;& M\|\varphi - \varphi^*\|_\infty\left\{\left(\lambda - \underline{a}_{ij}\right)\right. \\
&+ \sum_{B^{kl}\in N_r(i,j)} \overline{B}_{ij}^{kl}\left(M^f_{ij} + \mu^f_{ij}\frac{L^+_{ij}}{\delta_{ij}}\right) + \sum_{C^{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\left(M^g_{ij} + e^{\lambda\tau}\mu^g_{ij}\frac{L^+_{ij}}{\delta_{ij}}\right) \\
&\left.+ \sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl}\left(M^h_{ij}\int_0^\infty |K_{ij}(u)|du + \mu^h_{ij}\frac{L^+_{ij}}{\delta_{ij}}\int_0^\infty |K_{ij}(u)|e^{\lambda u}du\right)\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(\lambda - \underline{a}_{ij}\right) + \sum_{B^{kl}\in N_r(i,j)} \overline{B}_{ij}^{kl}\left(M^f_{ij} + \mu^f_{ij}\frac{L^+_{ij}}{\delta_{ij}}\right) + \sum_{C^{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\left(M^g_{ij} + e^{\lambda\tau}\mu^g_{ij}\frac{L^+_{ij}}{\delta_{ij}}\right) \\
&\quad+ \sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl}\left(M^h_{ij}\int_0^\infty |K_{ij}(u)|du + \mu^h_{ij}\frac{L^+_{ij}}{\delta_{ij}}\int_0^\infty |K_{ij}(u)|e^{\lambda u}du\right) \ge 0,
\end{aligned}$$

which contradicts (H_5). Hence, (8) holds. It follows that

$$|x_{ij}(t) - x^*_{ij}(t)| = |y_{ij}(t)| < M\|\varphi - \varphi^*\|_\infty e^{-\lambda t}, \quad \forall t > 0, \;\; i = 1,\ldots,m, \;\; j = 1,\ldots,n.$$

This completes the proof of Lemma 2. □

Remark 2. *If* $x^*(t) = (x_{11}^*(t), x_{12}^*(t), \ldots, x_{mn}^*(t))^T$ *be the* T *-anti-periodic solution of system* (1)*, it follows from Lemma 2 and Definition 1 that* $x^*(t)$ *is globally exponentially stable.*

Theorem 1. *Let* (H_1)-(H_5) *hold. Then system* (1) *has exactly one* T*-antiperiodic solution* $x^*(t)$*, which is globally exponentially stable.*

Proof. Let $v(t) = \{v_{ij}(t)\}$ be a solution of system (1) with initial conditions

$$v_{ij}(s) = \varphi_{ij}^v(s), \quad |\varphi_{ij}^v(s)| \le \frac{L_{ij}^+}{\delta_{ij}}, \quad s \in (-\infty, 0],\ i = 1, \ldots, m,\ j = 1, \ldots, n.$$

According to Remark 1, $v(t)$ exists on $[0, +\infty)$. Moreover, by Lemma 1, the solution $v(t)$ is bounded and

$$|v_{ij}(t)| \le \frac{L_{ij}^+}{\delta_{ij}}, \quad t \in R,\ i = 1, \ldots, m,\ j = 1, \ldots, n..$$

From (1) and (H_1)-(H_3), we have

$$\begin{aligned}
& [(-1)^{p+1} v_{ij}(t + (p+1)T)]' = (-1)^{p+1} v_{ij}'(t + (p+1)T) \\
= \ & (-1)^{p+1} \{-a_{ij}(t + (p+1)T, v_{ij}(t + (p+1)T)) \\
& - \sum_{B^{kl} \in N_r(i,j)} B_{ij}^{kl}(t + (p+1)T) f_{ij}(v_{kl}(t + (p+1)T)) v_{ij}(t + (p+1)T) \\
& - \sum_{C^{kl} \in N_r(i,j)} C_{ij}^{kl}(t + (p+1)T) \\
& \qquad \times g_{ij}(v_{kl}(t + (p+1)T - \tau_{kl}(t + (p+1)T))) v_{ij}(t + (p+1)T) \\
& - \sum_{D^{kl} \in N_r(i,j)} D_{ij}^{kl}(t + (p+1)T) \int_0^\infty K_{ij}(u) h_{ij}(v_{kl}(t + (p+1)T - u)) du \\
& \qquad \times v_{ij}(t + (p+1)T) + L_{ij}(t + (p+1)T)\} \\
= \ & -a_{ij}(t, (-1)^{p+1} v_{ij}(t + (p+1)T)) \\
& - \sum_{B^{kl} \in N_r(i,j)} B_{ij}^{kl}(t) f_{ij}((-1)^{p+1} v_{kl}(t + (p+1)T)) (-1)^{p+1} v_{ij}(t + (p+1)T) \\
& - \sum_{C^{kl} \in N_r(i,j)} C_{ij}^{kl}(t) g_{ij}((-1)^{p+1} v_{kl}(t + (p+1)T - \tau_{kl}(t))) \\
& \qquad \times (-1)^{p+1} v_{ij}(t + (p+1)T) \\
& - \sum_{D^{kl} \in N_r(i,j)} D_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) h_{ij}((-1)^{p+1} v_{kl}(t + (p+1)T - u)) du \\
& \qquad \times (-1)^{p+1} v_{ij}(t + (p+1)T) + L_{ij}(t),
\end{aligned} \tag{10}$$

where $i = 1, \ldots, m$, $j = 1, \ldots, n$. Thus, for any natural number p, $(-1)^{p+1} v_{ij}(t+(p+1)T)$ are the solutions of system (1). Then, by Lemma 2, there exists a constant $M > 0$ such that

$$\begin{aligned}
& |(-1)^{p+1} v_{ij}(t + (p+1)T) - (-1)^p v_{ij}(t + pT)| \\
\le \ & M e^{\lambda(t+pT)} \sup_{-\infty < s \le 0} \max_{(i,j)} |v_{ij}(s + T) - v_{ij}(s)| \\
\le \ & 2M e^{\lambda(t+pT)} \max_{(i,j)} \left\{ \frac{L_{ij}^+}{\delta_{ij}} \right\}, \quad \forall t + pT > 0,\ i = 1, \ldots, m,\ j = 1, \ldots, n.
\end{aligned} \tag{11}$$

Thus, for any natural number q, we obtain

$$(-1)^{q+1}v_{ij}(t+(q+1)T) = v_{ij}(t) + \sum_{p=0}^{q}[(-1)^{p+1}v_{ij}(t+(p+1)T) - (-1)^p v_{ij}(t+pT)].$$

Then, for $i = 1, \dots, m$, $j = 1, \dots, n$, we have

$$\begin{aligned} & |(-1)^{q+1}v_{ij}(t+(q+1)T)| \\ \leq \quad & |v_{ij}(t)| + \sum_{p=0}^{q} |(-1)^{p+1}v_{ij}(t+(p+1)T) - (-1)^p v_{ij}(t+pT)|. \end{aligned} \tag{12}$$

In view of (11), we can choose a sufficiently large constant $N > 0$ and a positive constant γ such that

$$|(-1)^{p+1}v_{ij}(t+(p+1)T) - (-1)^p v_{ij}(t+pT)| \leq \gamma(e^{\lambda T})^p, \tag{13}$$

for any $p > N$, $i = 1, \dots, m$, $j = 1, \dots, n$, on any compact set of R. It follows from (12) and (13) that $\{(-1)^q v_{ij}(t+qT)\}$ uniformly converges to a continuous function $x^*(t) = (x_{11}^*(t), x_{12}^*(t), \dots, x_{mn}^*(t))^T$ on any compact set of R.

Now we will show that $x^*(t)$ is the T-anti-periodic solution of system (1). First, $x^*(t)$ is T-anti-periodic, since

$$x^*(t+T) = \lim_{q\to\infty} v^*(t+T+qT) = -\lim_{(q+1)\to\infty}(-1)^{q+1}v^*(t+(q+1)T) = -x^*(t).$$

Next, we prove that $x^*(t)$ is a solution of (1). In fact, together with the continuity of the right side of (1) and (10) implies that $\{((-1)^{q+1}v_{ij}(t+(q+1)T))'\}$ uniformly converges to a continuous function on any compact set of R. Thus, letting $q \to \infty$, we obtain

$$\begin{aligned} \frac{d}{dt}(x_{ij}^*(t)) \quad = \quad & -a_{ij}(t, x_{ij}^*(t)) - \sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(t) f_{ij}(x_{kl}^*(t)) x_{ij}^*(t) \\ & - \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t) g_{ij}(x_{kl}^*(t-\tau_{kl}(t))) x_{ij}^*(t) \\ & - \sum_{D^{kl}\in N_r(i,j)} D_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) h_{ij}(x_{kl}^*(t-u)) du x_{ij}^*(t) + L_{ij}(t), \end{aligned}$$

Therefore, due to the above equation, we have $x^*(t)$ is a solution of (1). At last, by Lemma 2, we can prove that $x^*(t)$ is globally exponentially stable. This completes the proof. □

3. An illustrative example

In this section, we shall give an illustrative example to show the effectiveness of the main results. We consider the following SICNNs with nonlinear behaved functions and mixed delays:

$$\begin{aligned} x_{ij}'(t) \quad = \quad & -a_{ij}(t, x_{ij}(t)) - \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t) g_{ij}(x_{kl}(t-\tau_{kl}(t))) x_{ij}(t) \\ & - \sum_{D^{kl}\in N_r(i,j)} D_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) h_{ij}(x_{kl}(t-u)) du x_{ij}(t) + L_{ij}(t), \end{aligned} \tag{14}$$

where $r = 1$, $\tau_{ij}(t) = 0.2\cos^2 t$, $g_{ij}(u) = 0.6|\sin u|$, $h_{ij}(u) = 0.8|\arctan u|$, $K_{ij}(u) = |\sin u|e^{-2u}$, $i, j = 1, 2, 3$. Take

$$(a_{ij}(t,x))_{3\times 3} = \begin{pmatrix} 6x+\sin x - x\cos t & 6x-\sin x - x\cos t & 5x+\sin x + x\cos t \\ 5x+\sin x + x\cos t & 7x-\sin x + x\cos t & 6x+\sin x - x\cos t \\ 7x-\sin x + x\cos t & 6x+\sin x - x\cos t & 5x-\sin x + x\cos t \end{pmatrix},$$

$$(C_{ij}(t))_{3\times 3} = \begin{pmatrix} 0.1|\cos t| & 0.2|\sin t| & 0.2|\cos t| \\ 0.2|\sin t| & 0 & 0.3|\sin t| \\ 0.2|\sin t| & 0.1|\sin t| & 0.4|\cos t| \end{pmatrix},$$

$$(D_{ij}(t))_{3\times 3} = \begin{pmatrix} 0.1|\sin t| & 0.3|\cos t| & 0.1|\sin t| \\ 0.2|\cos t| & 0 & 0.2|\cos t| \\ 0.3|\sin t| & 0.1|\cos t| & 0.3|\sin t| \end{pmatrix},$$

$$(L_{ij}(t))_{3\times 3} = \begin{pmatrix} 2.3\sin t & 1.2\sin t & 2.4\sin t \\ 2.1\sin t & 2.7\sin t & 2.5\sin t \\ 2.6\sin t & 1.3\sin t & 1.9\sin t \end{pmatrix}, (L_{ij}^{+})_{3\times 3} = \begin{pmatrix} 2.5 & 1.5 & 2.5 \\ 2.3 & 3.1 & 2.8 \\ 2.8 & 1.5 & 2.1 \end{pmatrix}.$$

Through simple computation, we can easily obtain that $\tau = 0.2$, $M_{ij}^{g} = \mu_{ij}^{g} = 0.6$, $M_{ij}^{h} = \mu_{ij}^{h} = 0.8$, $\underline{a}_{13} = \underline{a}_{21} = \underline{a}_{33} = 3$, $\underline{a}_{11} = \underline{a}_{12} = \underline{a}_{23} = \underline{a}_{32} = 4$, $\underline{a}_{22} = \underline{a}_{31} = 5$,

$$\left(\sum_{C^{kl}\in N_1(i,j)} \overline{C}_{ij}^{kl}\right)_{3\times 3} = \begin{pmatrix} 0.5 & 1.0 & 0.7 \\ 0.8 & 1.7 & 1.2 \\ 0.5 & 1.2 & 0.8 \end{pmatrix},$$

$$\left(\sum_{D^{kl}\in N_1(i,j)} \overline{D}_{ij}^{kl}\right)_{3\times 3} = \begin{pmatrix} 0.6 & 0.9 & 0.5 \\ 1.0 & 1.6 & 1.0 \\ 0.6 & 1.1 & 0.6 \end{pmatrix}.$$

Define continuous function $\Gamma_{ij}(w)$ by setting

$$\begin{aligned} \Gamma_{ij}(w) &= \left(w - \underline{a}_{ij}\right) + \sum_{C^{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\left(M_{ij}^{g} + e^{w\tau}\mu_{ij}^{g}\frac{L_{ij}^{+}}{\delta_{ij}}\right) \\ &\quad + \sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl}\left(M_{ij}^{h}\int_0^\infty |K_{ij}(u)|du + \mu_{ij}^{h}\frac{L_{ij}^{+}}{\delta_{ij}}\int_0^\infty |K_{ij}(u)|e^{wu}du\right), \end{aligned}$$

where $w \in [0, 1.5]$, $i, j = 1, 2, 3$. Then, we obtain

$$\begin{aligned} &\max_{(i,j)}\{\Gamma_{ij}(0)\} \\ = \; &\max_{(i,j)}\left\{-\underline{a}_{ij} + \sum_{C^{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\left(M_{ij}^{g} + \mu_{ij}^{g}\frac{L_{ij}^{+}}{\delta_{ij}}\right)\right. \\ &\left. + \sum_{D^{kl}\in N_r(i,j)} \overline{D}_{ij}^{kl}\left(M_{ij}^{h}\int_0^\infty |K_{ij}(u)|du + \mu_{ij}^{h}\frac{L_{ij}^{+}}{\delta_{ij}}\int_0^\infty |K_{ij}(u)|du\right)\right\} < -1.5. \end{aligned}$$

Thus, there exists $\lambda \in [0, 1.5]$ such that $\Gamma_{ij}(\lambda) < 0$ for $i, j = 1, 2, 3$. It follows that system (14) satisfies all the conditions in Theorem 1. Hence, system (14) has exactly one π-anti-periodic solution. Moreover, the π-antiperiodic solution is globally exponentially stable.

4. Conclusion

In this paper, the shunting inhibitory cellular neural networks (SICNNs) with nonlinear behaved functions and time-varying and distributed delays (mixed delays) are investigated. For this model, we have given some sufficient conditions ensuring the existence and globally exponential stability of antiperiodic solution. Behaved functions $a_{ij}(t, x_{ij}(t))$ is linear in [11, 12], but in our model, behaved functions $a_{ij}(t, x_{ij}(t))$ may be nonlinear. This show that our results extend and improve some earlier publications. Moreover, a simple example is given to illustrate the effectiveness of our results. Thus, our results are valuable in the design of SICNNs.

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