

NEW OSTROWSKI TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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ABSTRACT. In this paper some new Ostrowski type inequalities for co-ordinated convex functions are obtained.

1. INTRODUCTION

In 1938, A. Ostrowski proved the following interesting inequality [11]:

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$.*

Then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \quad (1)$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

The inequality (1) can be rewritten in equivalent form as:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \|f'\|_\infty.$$

Since 1938 when A. Ostrowski proved his famous inequality, see [8], many mathematicians have been working about and around it, in many different directions and with a lot of applications in Numerical Analysis and Probability, etc.

Several generalizations of the Ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings and n -times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors. For recent results and generalizations concerning Ostrowski's inequality see [1]-[16] and the references therein.

Let us consider now a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in \mathbb{R}^2 with $a < b$ and $c < d$, a mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w),$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$. The mapping f is said to be concave on the co-ordinates on Δ if the above inequality holds in reversed direction, for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A modification for convex (concave) functions on Δ , which are also known as co-ordinated convex (concave) functions, was introduced by S. S. Dragomir [7] as follows:

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A function $f : \Delta \rightarrow \mathbb{R}$ is said to be convex (concave) on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex (concave) where defined for all $x \in [a, b], y \in [c, d]$.

A formal definition for co-ordinated convex (concave) functions may be stated in:

Definition 1. [9] *A mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)w) \\ & \leq tsf(x, u) + t(1-s)f(x, w) + s(1-t)f(y, u) + (1-t)(1-s)f(y, w), \end{aligned} \quad (2)$$

holds for all $t, s \in [0, 1]$ and $(x, u), (y, w) \in \Delta$. The mapping f is concave on the co-ordinates on Δ if the inequality (2) holds in reversed direction for all $t, s \in [0, 1]$ and $(x, y), (u, w) \in \Delta$.

Clearly, every convex (concave) mapping $f : \Delta \rightarrow \mathbb{R}$ is convex (concave) on the co-ordinates. Furthermore, there exists co-ordinated convex (concave) function which is not convex (concave), (see for instance [7]).

Here we also quote the following result form [8] (see also [7]) to be used in the sequel of the paper:

Theorem 2. [8] *Suppose that $f : \Delta \rightarrow \mathbb{R}$ is co-ordinated convex on Δ . Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ & \leq \frac{1}{4} \left[\frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\ & \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \quad (3)$$

The above inequalities are sharp. The inequalities in (3) hold in reverse direction if the mapping f is concave.

The main aim of this paper is to establish some new Ostrowski type inequalities for co-ordinated convex functions in Section 2.

2. MAIN RESULTS

To establish our main results we need the following identity:

Lemma 1. Let $f : \Delta \rightarrow \mathbb{R}$ be a twice partial differentiable mapping on Δ° . If $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$, then the following identity holds:

$$\begin{aligned} f(x, y) &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dvdu - A \\ &= \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) dsdt \\ &\quad - \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) dsdt \\ &\quad - \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) dsdt \\ &\quad + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) dsdt, \quad (4) \end{aligned}$$

for all $(x, y) \in \Delta$, where

$$A = \frac{1}{d-c} \int_c^d f(x, v) dv + \frac{1}{b-a} \int_a^b f(u, y) du.$$

Proof. Using integration by parts and the change of variables $u = tx + (1-t)a$, $v = sy + (1-s)c$, we have

$$\begin{aligned} &\frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) dsdt \\ &= \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 t \left[\int_0^1 s \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) ds \right] dt \\ &= \frac{(x-a)(y-c)}{(b-a)(d-c)} f(x, y) - \frac{x-a}{(b-a)(d-c)} \int_c^y f(x, v) dv \\ &\quad - \frac{y-c}{(b-a)(d-c)} \int_a^x f(u, y) du + \frac{1}{(b-a)(d-c)} \int_a^x \int_c^y f(u, v) dvdu \quad (5) \end{aligned}$$

Similarly, by integration by parts and suitable change of variables, we also have

$$\begin{aligned} &\frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) dsdt \\ &= -\frac{(x-a)(d-y)}{(b-a)(d-c)} f(x, y) - \frac{x-a}{(b-a)(d-c)} \int_d^y f(x, v) dv \\ &\quad + \frac{d-y}{(b-a)(d-c)} \int_a^x f(u, y) du + \frac{1}{(b-a)(d-c)} \int_a^x \int_d^y f(u, v) dvdu, \quad (6) \end{aligned}$$

$$\begin{aligned} &\frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) dsdt \\ &= -\frac{(b-x)(y-c)}{(b-a)(d-c)} f(x, y) + \frac{b-x}{(b-a)(d-c)} \int_c^y f(x, v) dv \\ &\quad - \frac{y-c}{(b-a)(d-c)} \int_b^x f(u, y) du + \frac{1}{(b-a)(d-c)} \int_b^x \int_c^y f(u, v) dvdu \quad (7) \end{aligned}$$

and

$$\begin{aligned} & \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) ds dt \\ &= \frac{(b-x)(d-y)}{(b-a)(d-c)} f(x, y) + \frac{b-x}{(b-a)(d-c)} \int_d^y f(x, v) dv \\ &+ \frac{d-y}{(b-a)(d-c)} \int_b^x f(u, y) du + \frac{1}{(b-a)(d-c)} \int_b^x \int_d^y f(u, v) dv du. \quad (8) \end{aligned}$$

From (5)-(8), we get (4). This completes the proof. \square

Theorem 3. Let $f : \Delta \rightarrow \mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ is convex on the co-ordinates on Δ and $\left| \frac{\partial^2}{\partial s \partial t} f(x, y) \right| \leq M$, $(x, y) \in \Delta$, then the following inequality holds:

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\ & \leq M \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \left[\frac{(y-c)^2 + (d-y)^2}{2(d-c)} \right], \quad (9) \end{aligned}$$

for all $(x, y) \in \Delta$, where A is defined in Lemma 1.

Proof. By Lemma 1, we have that the following inequality holds:

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\ & \leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds dt \\ & + \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right| ds dt \\ & + \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right| ds dt \\ & + \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right| ds dt, \quad (10) \end{aligned}$$

for all $(x, y) \in \Delta$.

Using the co-ordinated convexity of $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$, we get the following inequality holds:

$$\begin{aligned} & \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds dt \\ & \leq \left| \frac{\partial^2}{\partial s \partial t} f(x, y) \right| \int_0^1 \int_0^1 t^2 s^2 ds dt + \left| \frac{\partial^2}{\partial s \partial t} f(x, c) \right| \int_0^1 \int_0^1 t^2 s(1-s) ds dt \\ & \quad + \left| \frac{\partial^2}{\partial s \partial t} f(a, y) \right| \int_0^1 \int_0^1 s^2 t(1-t) ds dt \\ & \quad + \left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right| \int_0^1 \int_0^1 st(1-t)(1-s) ds dt. \quad (11) \end{aligned}$$

Since

$$\int_0^1 \int_0^1 t^2 s^2 ds dt = \frac{1}{9}, \int_0^1 \int_0^1 t^2 s(1-s) ds dt = \int_0^1 \int_0^1 s^2 t(1-t) ds dt = \frac{1}{18},$$

$$\int_0^1 \int_0^1 st(1-t)(1-s) ds dt = \frac{1}{36} \text{ and } \left| \frac{\partial^2}{\partial s \partial t} f(x, y) \right| \leq M, (x, y) \in \Delta.$$

Hence from (11), we obtain

$$\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds dt \leq \frac{1}{4}M. \tag{12}$$

Analogously, we also have

$$\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right| ds dt \leq \frac{1}{4}M, \tag{13}$$

$$\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right| ds dt \leq \frac{1}{4}M \tag{14}$$

and

$$\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right| ds dt \leq \frac{1}{4}M. \tag{15}$$

Now by making use of the inequalities (12)-(15) and the fact that

$$\begin{aligned} &(x-a)^2(y-c)^2 + (x-a)^2(d-y)^2 + (b-x)^2(y-c)^2 + (b-x)^2(d-y)^2 \\ &= \left[(x-a)^2 + (b-x)^2 \right] \left[(y-c)^2 + (d-y)^2 \right], \end{aligned}$$

we get the inequality (9). This completes the proof. □

The following result is about the powers of the absolute value of the partial derivatives:

Theorem 4. *Let $f : \Delta \rightarrow \mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ is convex on the co-ordinates on Δ , $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\left| \frac{\partial^2}{\partial s \partial t} f(x, y) \right| \leq M, (x, y) \in \Delta$, then the following inequality holds:*

$$\begin{aligned} &\left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\ &\leq \frac{M}{(1+p)^{\frac{2}{p}}} \left[\frac{(x-a)^2 + (b-x)^2}{b-a} \right] \left[\frac{(y-c)^2 + (d-y)^2}{d-c} \right], \end{aligned} \tag{16}$$

for all $(x, y) \in \Delta$, where A is defined in Lemma 1.

Proof. By Lemma 1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\begin{aligned}
& \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \leq \left(\int_0^1 \int_0^1 s^p t^p ds dt \right)^{\frac{1}{p}} \\
& \times \left[\frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}} \right. \\
& + \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \\
& + \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}} \\
& \left. + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \right], \quad (17)
\end{aligned}$$

for all $(x, y) \in \Delta$.

Since $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ is convex on the co-ordinates on Δ and $\left| \frac{\partial^2}{\partial s \partial t} f(x, y) \right| \leq M$, $(x, y) \in \Delta$, we have

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q ds dt \\
& \leq \left| \frac{\partial^2}{\partial s \partial t} f(a, y) \right|^q \int_0^1 \int_0^1 s(1-t) ds dt \\
& + \left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|^q \int_0^1 \int_0^1 (1-t)(1-s) ds dt \\
& + \left| \frac{\partial^2}{\partial s \partial t} f(x, y) \right|^q \int_0^1 \int_0^1 ts ds dt \\
& + \left| \frac{\partial^2}{\partial s \partial t} f(x, c) \right|^q \int_0^1 \int_0^1 t(1-s) ds dt = M^q.
\end{aligned}$$

Similarly, we also have the following inequalities:

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q ds dt \leq M^q, \\
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q ds dt \leq M^q
\end{aligned}$$

and

$$\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q ds dt \leq M^q.$$

Using the fact

$$\int_0^1 \int_0^1 s^p t^p ds dt = \frac{1}{(1+p)^{\frac{2}{p}}}$$

and the above inequalities in (17), we get (16). This completes the proof of the theorem. \square

A different approach leads us to the following result:

Theorem 5. Let $f : \Delta \rightarrow \mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ is convex on the co-ordinates on Δ , $q \geq 1$ and $\left| \frac{\partial^2}{\partial s \partial t} f(x, y) \right| \leq M$, $(x, y) \in \Delta$, then the following inequality holds:

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\ & \leq \frac{M}{4} \left[\frac{(x-a)^2 + (b-x)^2}{b-a} \right] \left[\frac{(y-c)^2 + (d-y)^2}{d-c} \right], \end{aligned} \tag{18}$$

for all $(x, y) \in \Delta$, where A is defined in Lemma 1.

Proof. Suppose $q \geq 1$. From Lemma 1 and using the power mean inequality for double integrals, we have

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \leq \left(\int_0^1 \int_0^1 st ds dt \right)^{1-\frac{1}{q}} \\ & \times \left[\frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}} \right. \\ & + \frac{(x-a)^2 (y-d)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \\ & + \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}} \\ & \left. + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \right], \end{aligned} \tag{19}$$

for all $(x, y) \in \Delta$.

By similar argument as in Theorem 4 that $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ is convex on the co-ordinates on Δ and $\left| \frac{\partial^2}{\partial s \partial t} f(x, y) \right| \leq M$, $(x, y) \in \Delta$, we have

$$\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q ds dt \leq \frac{1}{4} M^q,$$

$$\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q ds dt \leq \frac{1}{4} M^q,$$

$$\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q ds dt \leq \frac{1}{4} M^q$$

and

$$\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q ds dt \leq \frac{1}{4} M^q.$$

Now using the above inequalities and

$$\int_0^1 \int_0^1 st ds dt = \frac{1}{4}$$

in (19), we get the desired inequality (18). This completes the proof. \square

Remark 1. Since $(1+p)^{\frac{1}{p}} < 2$, $p > 1$ and accordingly, we have

$$\frac{1}{2} < \frac{1}{(1+p)^{\frac{1}{p}}}, p > 1$$

which gives

$$\frac{1}{4} < \frac{1}{(1+p)^{\frac{2}{p}}}, p > 1.$$

This reveals that the the inequality (18) gives tighter estimate than that of the inequality (16).

Remark 2. From the inequalities proved above in Theorem 3-Theorem 5, one can get several midpoint type inequalities by setting $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. However the details are left to the interested reader.

Now we drive some results with co-ordinated concavity property instead of co-ordinated convexity.

Theorem 6. Let $f : \Delta \rightarrow \mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ is concave on the co-ordinates on Δ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then the inequality

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\ & \leq \frac{1}{(1+p)^{\frac{2}{p}}(b-x)(d-c)} \left[(x-a)^2(y-c)^2 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{x+a}{2}, \frac{d+y}{2} \right) \right|^q \right. \\ & \quad + (x-a)^2(d-y)^2 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{x+a}{2}, \frac{d+y}{2} \right) \right|^q \\ & \quad + (b-x)^2(y-c)^2 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{b+a}{2}, \frac{y+c}{2} \right) \right|^q \\ & \quad \left. + (b-x)^2(d-y)^2 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{b+x}{2}, \frac{d+y}{2} \right) \right|^q \right], \quad (20) \end{aligned}$$

holds for all $(x, y) \in \Delta$, where A is defined in Lemma 1.

Proof. From Lemma 1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \leq \left(\int_0^1 \int_0^1 s^p t^p ds dt \right)^{\frac{1}{p}} \\ & \quad \times \left[\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}} \right. \\ & \quad + \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}} \\ & \quad \left. + \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \right], \quad (21) \end{aligned}$$

for all $(x, y) \in \Delta$.

Since $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ is concave on the co-ordinates on Δ , so an application of (3) with inequalities in reversed direction, gives us the following inequalities:

$$\begin{aligned} \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q ds dt & \\ & \leq \frac{1}{2} \left[\int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, \frac{y+c}{2}\right) \right|^q dt \right. \\ & \quad \left. + \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, sy + (1-s)c\right) \right|^q ds \right] \\ & \leq \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \right|^q, \end{aligned} \tag{22}$$

$$\begin{aligned} \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q ds dt & \\ & \leq \frac{1}{2} \left[\int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, \frac{d+y}{2}\right) \right|^q dt \right. \\ & \quad \left. + \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, sy + (1-s)c\right) \right|^q ds \right] \\ & \leq \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{x+a}{2}, \frac{d+y}{2}\right) \right|^q, \end{aligned} \tag{23}$$

$$\begin{aligned} \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q ds dt & \\ & \leq \frac{1}{2} \left[\int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, \frac{y+c}{2}\right) \right|^q dt \right. \\ & \quad \left. + \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{b+x}{2}, sy + (1-s)c\right) \right|^q ds \right] \\ & \leq \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{b+x}{2}, \frac{y+c}{2}\right) \right|^q \end{aligned} \tag{24}$$

and

$$\begin{aligned} \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q ds dt & \\ & \leq \frac{1}{2} \left[\int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)b, \frac{d+y}{2}\right) \right|^q dt \right. \\ & \quad \left. + \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{b+x}{2}, sy + (1-s)d\right) \right|^q ds \right] \\ & \leq \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{b+x}{2}, \frac{d+y}{2}\right) \right|^q. \end{aligned} \tag{25}$$

By making use of (22)-(25) in (21), we obtain (20). Thus the proof of the theorem is complete. \square

Theorem 7. Let $f : \Delta \rightarrow \mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ is concave on the co-ordinates on Δ and $q \geq 1$, then the inequality

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\ & \leq \frac{1}{4(b-a)(d-c)} \left[(x-a)^2 (y-c)^2 \left| \frac{\partial^2 f}{\partial s \partial t} \left(\frac{3x+a}{3}, \frac{3y+c}{3} \right) \right| \right. \\ & \quad + (x-a)^2 (d-y)^2 \left| \frac{\partial^2 f}{\partial s \partial t} \left(\frac{3x+a}{3}, \frac{3y+d}{3} \right) \right| \\ & \quad + (b-x)^2 (y-c)^2 \left| \frac{\partial^2 f}{\partial s \partial t} \left(\frac{3x+b}{3}, \frac{3y+c}{3} \right) \right| \\ & \quad \left. + (b-x)^2 (d-y)^2 \left| \frac{\partial^2 f}{\partial s \partial t} \left(\frac{3x+b}{3}, \frac{3y+d}{3} \right) \right| \right], \quad (26) \end{aligned}$$

holds for all $(x, y) \in \Delta$, where A is defined in Lemma 1.

Proof. By the concavity of $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ on the co-ordinates on Δ and power mean inequality, we note that the following inequality holds:

$$\begin{aligned} \left| \frac{\partial^2 f}{\partial s \partial t}(\lambda x + (1-\lambda)y, v) \right|^q & \geq \lambda \left| \frac{\partial^2 f}{\partial s \partial t}(x, v) \right|^q + (1-\lambda) \left| \frac{\partial^2 f}{\partial s \partial t}(y, v) \right|^q \\ & \geq \left(\lambda \left| \frac{\partial^2 f}{\partial s \partial t}(x, v) \right| + (1-\lambda) \left| \frac{\partial^2 f}{\partial s \partial t}(y, v) \right| \right)^q, \end{aligned}$$

for all $x, y \in [a, b]$, $\lambda \in [0, 1]$ and for fixed $v \in [c, d]$.

Hence,

$$\left| \frac{\partial^2 f}{\partial s \partial t}(\lambda x + (1-\lambda)y, v) \right| \geq \lambda \left| \frac{\partial^2 f}{\partial s \partial t}(x, v) \right| + (1-\lambda) \left| \frac{\partial^2 f}{\partial s \partial t}(y, v) \right|,$$

for all $x, y \in [a, b]$, $\lambda \in [0, 1]$ and for fixed $v \in [c, d]$.

Similarly, we can show that

$$\left| \frac{\partial^2 f}{\partial s \partial t}(u, \lambda z + (1-\lambda)w) \right| \geq \lambda \left| \frac{\partial^2 f}{\partial s \partial t}(u, z) \right| + (1-\lambda) \left| \frac{\partial^2 f}{\partial s \partial t}(u, w) \right|,$$

for all $z, w \in [c, d]$, $\lambda \in [0, 1]$ and for fixed $u \in [a, b]$, thus $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ is concave on the co-ordinates on Δ .

From Lemma 1, we have

$$\left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right|$$

$$\begin{aligned} &\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds dt \\ &\quad + \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right| ds dt \\ &\quad + \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q ds dt \\ &\quad + \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q ds dt, \quad (27) \end{aligned}$$

for all $(x, y) \in \Delta$.

Since $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ is concave on the co-ordinates on Δ , we have, by the Jensens's inequality for integrals, that

$$\begin{aligned} &\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds dt \\ &\quad = \int_0^1 t \left[\int_0^1 s \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds \right] dt \\ &\quad \leq \int_0^1 t \left[\left(\int_0^1 s ds \right) \left| \frac{\partial^2}{\partial s \partial t} f \left(tx + (1-t)a, \frac{\int_0^1 s(sy + (1-s)c) ds}{\int_0^1 s ds} \right) \right| \right] dt \\ &\quad = \frac{1}{2} \int_0^1 t \left| \frac{\partial^2}{\partial s \partial t} f \left(tx + (1-t)a, \frac{3y+c}{3} \right) \right| dt \\ &\quad \leq \frac{1}{2} \left(\int_0^1 t dt \right) \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{\int_0^1 t(tx + (1-t)a) dt}{2}, \frac{3y+c}{3} \right) \right| \\ &\quad = \frac{1}{4} \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{3x+a}{3}, \frac{3y+c}{3} \right) \right|. \quad (28) \end{aligned}$$

Analogously, we also have

$$\begin{aligned} &\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right| ds dt \\ &\quad \leq \frac{1}{4} \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{3x+a}{3}, \frac{3y+d}{3} \right) \right|, \quad (29) \end{aligned}$$

$$\begin{aligned} &\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right| ds dt \\ &\quad \leq \frac{1}{4} \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{3x+b}{3}, \frac{3y+c}{3} \right) \right| \quad (30) \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right| ds dt \\ &\quad \leq \frac{1}{4} \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{3x+b}{3}, \frac{3y+d}{3} \right) \right|. \quad (31) \end{aligned}$$

Using (28)-(31) in (27), we obtain the required inequality (26) and hence the theorem is complete. \square

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