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A CLASS OF NONLINEAR INTEGRAL EQUATIONS

MARIA DOBRIŢOIU

Abstract. Using *the Contraction Principle* and *the General Data Dependence Theorem*, several results of existence and uniqueness and of continuous dependence of data of the solution of a class of integral equations with modified argument from physics, that is a mathematical model reference with to the turbo-reactors working:

$$x(t) = \int_a^b K(t,s) \cdot h(s, x(s), x(a), x(b))ds + f(t), \;\; t \in [a,b],$$

where $a, b \in \mathbb{R}$, $a < b$, $K : [a,b] \times [a,b] \to \mathbb{R}$, $h : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ and $x : [a,b] \to \mathbb{R}$ are given. Also, an example is given.

1. Introduction

The integral equations, in general, and those with modified argument, in particular, form an important part of applied mathematics, with links with many theoretical fields, especially with practical fields.

In the '70, in the research on some problems from turbo-reactors industry, a nonlinear Fredholm integral equation with modified argument appears, and has the following form:

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t), \;\; t \in [a,b], \tag{1}$$

where $K : [a,b] \times [a,b] \times \mathbb{R}^3 \to \mathbb{R}$, $f : [a,b] \to \mathbb{R}$.

This integral equation is a mathematical model reference with to the turbo-reactors working.

Starting with this Fredholm integral equation, we have considered a modification of the argument through a continuous function $g : [a,b] \to [a,b]$, thus obtaining another integral equation with modified argument:

$$x(t) = \int_a^b K(t, s, x(s), x(g(s)), x(a), x(b))ds + f(t), \;\; t \in [a,b], \tag{2}$$

where $K : [a,b] \times [a,b] \times \mathbb{R}^4 \to \mathbb{R}$, $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to [a,b]$.

A generalization of the integral equation (2) is the following integral equation with modified argument

$$x(t) = \int_\Omega K(t, s, x(s), x(g(s)), x|_{\partial\Omega})ds + f(t), \;\; t \in \overline{\Omega}, \tag{3}$$

where $\Omega \subset \mathbb{R}^m$ is a bounded domain, $K : \overline{\Omega} \times \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^m \times C(\partial\Omega, \mathbb{R}^m) \to \mathbb{R}^m$, $f : \overline{\Omega} \to \mathbb{R}^m$, $g : \overline{\Omega} \to \overline{\Omega}$.

The results of the studies of the integral equations (1), (2) and (3) regarding the existence and uniqueness, the continuous dependence of data, the differentiability with

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respect to a and b, the differentiability with respect to a parameter, and the numerical methods for approximating the solution using the method of successive approximations with the trapezoids fomula, the Simpson's formula and the rectangle quadrature formula, respectively, have been published in [2], [5], [6], [7], [8], [9], [10] and [11]. Also, some properties of the solution of the integral equation (1) that were obtained using the Picard operators technique and the Abstract Gronwall lemma, have been studied in the papers mentioned above. In order to obtain these results were consulted several theorems, lemmas and basic results from [3], [4], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24] and [25].

This paper is focused mostly on the study of a class of integral equations with modified argument of type of integral equation (1) having the form:

$$x(t)=\int_a^b K(t,s)\cdot h(s,x(s),x(a),x(b))ds+f(t),\quad t\in[a,b], \tag{4}$$

where $K:[a,b]\times[a,b]\to\mathbb{R},\ \ h:[a,b]\times\mathbb{R}^3\to\mathbb{R},\ f:[a,b]\to\mathbb{R}$.

The purpose of this paper is to give several results of existence and uniqueness and continuous dependence of data of the solution of integral equation (4).

In order to establish these results, *the Contraction Principle* and *the General Data Dependence Theorem* have been used. Also some results presented in the treatises [4] and [20] are useful. Finally, an example is given.

2. Notations and preliminaries

Let X be a nonempty set, d a metric on X and $A:X\to X$ an operator. In this paper we shall use the following notations:

$P(X):=\{Y\subset X\ /\ Y\neq\varnothing\}$ - the set of all nonempty subsets of X

$I(A):=\{Y\in P(X)\ /\ A(Y)\subset Y\}$ - the family of the nonempty subsets of X, invariant for A

$F_A:=\{x\in X|\ A(x)=x\}$ - the fixed points set of A

$A^0:=1_X,\ A^1:=A,\ A^{n+1}:=A\circ A^n,\ n\in N$ - the iterate operators of A.

In order to study the existence and uniqueness of the solution of integral equation (4), in section 3 we use *the Contraction Principle* that we present below (see [22]).

Theorem 1. *(Contraction Principle) Let (X,d) be a complete metric space and $A:X\to X$ an α-contraction $(\alpha<1)$. In these conditions we have:*

(i) A has a unique fixed point, i.e. $F_A=\{x^*\}$;
(ii) $x^*=\lim\limits_{n\to\infty}A^n(x_0)$, for all $x_0\in X$;
(iii) $d\left(x^*,A^n(x_0)\right)\le\frac{\alpha^n}{1-\alpha}d\left(x_0,A(x_0)\right)$.

In order to study the continuous dependence of data of the solution of integral equation (4), we use also in section 3, *the General Data Dependence Theorem* that we present below. (see [22]).

Theorem 2. *(General Data Dependence Theorem) Let (X,d) be a complete metric space, $f,\ g:X\to X$ two operators and suppose that:*

(i) f is α-contraction and $F_f=\left\{x_f^*\right\}$;
(ii) $x_g^*\in F_g$;
(iii) there exists $\eta>0$ such that

$$d\left(f(x),g(x)\right)\le\eta,\quad for\ all\ x\in X.$$

In these conditions we have:

$$d\left(x_f^*, x_g^*\right) \leq \frac{\eta}{1-\alpha}.$$

3. The main results

In this section we present two theorems of existence and uniqueness of the solution of the integral equation with modified argument (4), that were obtained by applying *the Contraction Principle*. Also, we present one theorem of data dependence of the solution of this integral equation with modified argument, that was obtained by applying *the General Data Dependence Theorem*.

I. The existence and uniqueness of the solution

The existence and uniqueness of the solution of the integral equation (4) has been studied in the space $C[a,b]$ and also, in the sphere $\overline{B}\,(f;r) \subset C[a,b]$.

A. The solution in the space $C[a,b]$

We consider that the following conditions are fulfilled:

(a_1) $K \in C([a,b] \times [a,b])$;
(a_2) $h \in C([a,b] \times \mathbb{R}^3)$;
(a_3) $f \in C[a,b]$.

In addition, we denote by M_K a positive constant, such that

$$|K(t,s)| \leq M_K, \text{ for all } t,s \in [a,b].$$

Now, in order to obtain a theorem of existence and uniqueness of the solution of integral equation (4) in the space $C[a,b]$, we will reduce the problem of determination of the solutions of this integral equation to a fixed point problem. For this purpose we consider the operator $A : C[a,b] \to C[a,b]$, defined by the relation:

$$A(x)(t) := \int_a^b K(t,s) \cdot h(s, x(s), x(a), x(b))ds + f(t), \;\; t \in [a,b]. \tag{5}$$

The set of the solution of the integral equation (4) in the space $C[a,b]$ coincides with the fixed points set of the operator A defined by the relation (5).

Applying *the Contraction Principle*, we obtain:

Theorem 3. *Suppose that the conditions (a_1), (a_2) and (a_3) are fulfilled. In addition we suppose that:*

(i) there exists α, β, $\gamma > 0$ such that

$$|h(s,u_1,u_2,u_3) - h(s,v_1,v_2,v_3)| \leq \alpha|u_1 - v_1| + \beta\,|u_2 - v_2| + \gamma\,|u_3 - v_3|,$$

for all $s \in [a,b]$, u_i, $v_i \in \mathbb{R}$, $i = \overline{1,3}$;

(ii) $M_K\,(\alpha + \beta + \gamma)\,(b-a) < 1$.

Under these conditions the integral equation (4) has a unique solution $x^* \in C[a,b]$, which can be obtained by the successive approximations method starting at any element $x_0 \in C[a,b]$. Moreover, if x_n is the n-th successive approximation, then we have:

$$|x^* - x_n| \leq \frac{\left[M_K\,(\alpha + \beta + \gamma)\,(b-a)\right]^n}{1 - M_K\,(\alpha + \beta + \gamma)\,(b-a)} \cdot |x_0 - x_1|\,. \tag{6}$$

Proof. From the conditions (a_1), (a_2) and (a_3) it results that the operator A is correctly defined. Now we check if the conditions of *the Contraction Principle* are fulfilled. Let we prove that the operator A is a contraction.

Using the condition (i) we have:

$$\begin{aligned}&|A(x)(t)-A(y)(t)|\leq\\ &\leq\left|\int_a^b K(t,s)\,[h(s,x(s),x(a),x(b))-h(s,y(s),y(a),y(b))]\,ds\right|\leq\\ &\leq\left|\int_a^b |K(t,s)|\cdot|h(s,x(s),x(a),x(b))-h(s,y(s),y(a),y(b))|\,ds\right|\leq\\ &\leq M_K\left|\int_a^b [\alpha\,|x(s)-y(s)|+\beta\,|x(a)-y(a)|+\gamma\,|x(b)-y(b)|]\,ds\right|\end{aligned}$$

and using *the Chebyshev norm*, we obtain:

$$\|A(x)-A(y)\|_C\leq M_K\,(\alpha+\beta+\gamma)\,(b-a)\|x-y\|_C\,.$$

Therefore the operator A satisfies the Lipschitz condition with the constant $M_K(\alpha+\beta+\gamma)\,(b-a)>0$ and from condition (ii) it results that the operator A is a contraction with the coefficient $M_K\,(\alpha+\beta+\gamma)\,(b-a)$. Now, applying *the Contraction Principle* it results the conclusion of this theorem and the proof is complete. □

B. The solution in the sphere $\overline{B}\,(f;r)\subset C[a,b]$

We consider that the conditions (a_1) and (a_3) are fulfilled and we replace the condition (a_2) by the following condition:

(a_2') $h\in C([a,b]\times J^3)$, where $J\subset\mathbb{R}$ is a closed interval.

In addition, we denote with M_h a positive constant such that, for the restriction $h|_{[a,b]\times J^3}$, $J\subset\mathbb{R}$ compact, we have:

$$|h(s,u,v,w)|\leq M_h,\ \text{for all}\, s\in[a,b],\ \ u,v,w\in J\,.$$

The following result is a theorem of existence and uniqueness of the solution of the integral equation (4) in the sphere $\overline{B}\,(f;r)\subset C[a,b]$.

Theorem 4. *Suppose that the conditions* (a_1), (a_2') *and* (a_3) *are fulfilled. In addition we suppose that:*

(i) *there exists* $\alpha,\ \beta,\ \gamma>0$ *such that*

$$|h(s,u_1,u_2,u_3)-h(s,v_1,v_2,v_3)|\leq\alpha|u_1-v_1|+\beta\,|u_2-v_2|+\gamma\,|u_3-v_3|,$$

for all $s\in[a,b]$, $u_{i,}\ v_i\in J$, $i=\overline{1,3}$, $J\subset\mathbb{R}$ *is closed interval;*

(ii) $M_K\,(\alpha+\beta+\gamma)\,(b-a)<1$.

If there exists $r>0$ *such that*

$$\left[x\in\overline{B}\,(f;r)\right]\Longrightarrow[x(t)\in J\subset\mathbb{R}]\tag{7}$$

and the following condition is fulfilled:

(iii) $M_KM_h\,(b-a)<r$,

then the integral equation (4) *has a unique solution* $x^*\in\overline{B}\,(f;r)\subset C[a,b]$*, which can be obtained by the successive approximations method starting at any element from the sphere* $\overline{B}\,(f;r)$*. Moreover, if* x_n *is the n-th successive approximation, then the estimation* (6) *is met.*

Proof. We consider the operator $A:\overline{B}(f;r)\to C[a,b]$, defined by the relation (5), where r is a real positive number which satisfies the condition (7) and we suppose that there exists at least one number $r>0$ with this property.

From the conditions (i) and (ii) it results that this operator satisfies the contraction condition and from the condition (iii) it results that $A\left(\overline{B}\,(f;r)\right)\subset\overline{B}\,(f;r)$, i.e. $\overline{B}\,(f;r)\in I\,(A)$.

Now, we consider the operator $A:\overline{B}(f;r)\to\overline{B}(f;r)$, also denoted by A and defined by the same relation (5).

The set of the solution of the integral equation (4) in the sphere $\overline{B}(f;r) \subset C[a,b]$ coincides with the fixed points set of the operator A defined by the relation (5).

Since $\overline{B}(f;r) \subset C[a,b]$ is a closed subset in the Banach space $C[a,b]$, it can apply *the Contraction Principle* and the proof is complete. □

II. The data dependence of the solution

In order to study the data dependence of the solution of the integral equation (4) we consider the following perturbed integral equation:

$$y(t) = \int_a^b K(t,s) \cdot k(s, y(s), y(a), y(b))ds + g(t), \;\; t \in [a,b], \tag{8}$$

where $K : [a,b] \times [a,b] \to \mathbb{R}$, $\;k : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$.

Now we have the following data dependence theorem of the solution of the integral equation (4):

Theorem 5. *Suppose that:*

(i) the conditions of the theorem 3 are satisfied and we denote by $x^* \in C[a,b]$ the unique solution of the integral equation (4)*;*

(ii) $k \in C\left([a,b] \times \mathbb{R}^3\right)$, $g \in C[a,b]$;

(iii) there exists $\eta_1, \eta_2 > 0$ such that

$$|h(s,u,v,w) - k(s,u,v,w)| \le \eta_1, \quad \textit{for all } s \in [a,b],\; u,v,w \in \mathbb{R}$$

and

$$|f(t) - g(t)| \le \eta_2, \quad \textit{for all } t \in [a,b].$$

Under these conditions, if $y^* \in C[a,b]$ is a solution of the integral equation (8)*, then we have:*

$$\|x^* - y^*\|_C \le \frac{M_K \eta_1 (b-a) + \eta_2}{1 - M_K(\alpha + \beta + \gamma)(b-a)} \tag{9}$$

Proof. Let we consider the operator from the proof of the theorem 3, $A : C[a,b] \to C[a,b]$, attached to the integral equation (4) and defined by the relation (5):

$$A(x)(t) := \int_a^b K(t,s) \cdot h(s, x(s), x(a), x(b))ds + f(t), \;\; t \in [a,b].$$

Now, we attach to the integral equation (8) the operator $D : C[a,b] \to C[a,b]$, defined by the relation:

$$D(y)(t) := \int_a^b K(t,s) \cdot k(s, y(s), y(a), y(b))ds + g(t), \;\; t \in [a,b]. \tag{10}$$

From the conditions (a_1) and (ii) it results that the operator D is correctly defined.

The set of the solutions of the perturbed integral equation (8) in the space $C[a,b]$ coincides with the fixed points set of the operator D defined by the relation (10).

We have:

$$\begin{gathered} |A(x)(t) - D(x)(t)| \le \\ \le \left| \int_a^b |K(t,s)| \cdot |h(s,x(s),x(a),x(b)) - k(s,x(s),x(a),x(b))|\, ds \right| + \\ + |f(t) - g(t)| \end{gathered}$$

and from the condition (iii) it results that

$$|A(x)(t) - D(x)(t)| \le M_K \eta_1 (b-a) + \eta_2,$$

for all $t \in [a,b]$.

Now, using *the Chebyshev norm*, we obtain:

$$\|A(x) - D(x)\|_C \leq M_K \eta_1 (b-a) + \eta_2 \tag{11}$$

and applying *the General Data Dependence Theorem* it results the estimation (9):

$$\|x^* - y^*\|_C \leq \frac{M_K \eta_1 (b-a) + \eta_2}{1 - M_K (\alpha + \beta + \gamma)(b-a)}$$

and the proof is complete. □

4. Example

We consider the following integral equation with modified argument:

$$x(t) = \int_0^1 \frac{t+s}{2} \left[\frac{\sin(x(s))}{7} + \frac{x(0) + x(1)}{5} \right] ds + 2\cos t + 1, \quad t \in [0,1], \tag{12}$$

where $K \in C([0,1] \times [0,1])$, $K = \frac{t+s}{2}$, $h \in C([0,1] \times \mathbb{R}^3)$, $h(s,u,v,w) = \frac{\sin u}{7} + \frac{v+w}{5}$, $f \in C[0,1]$, $f(t) = 2\cos t + 1$, $x \in C[0,1]$ and the perturbed integral equation:

$$y(t) = \int_0^1 \frac{t+s}{2} \left[\frac{\sin(y(s))}{7} + \frac{y(0) + y(1)}{5} - s - 2 \right] ds + \cos t, \quad t \in [0,1], \tag{13}$$

where $K \in C([0,1] \times [0,1])$, $K = \frac{t+s}{2}$, $k \in C([0,1] \times \mathbb{R}^3)$, $k(s,u,v,w) = \frac{\sin u}{7} + \frac{v+w}{5} - s - 2$, $g \in C[0,1]$, $g(t) = \cos t$, $y \in C[0,1]$.

Using the theorem 3 we prove that the integral equation (12) has a unique solution $x^* \in C[0,1]$, and then we apply the theorem 5 in order to check the conditions of the continuous dependence of data of the solution of this integral equation.

The operator $A : C[0,1] \to C[0,1]$ attached to the equation (12) and defined by the relation:

$$A(x)(t) := \int_0^1 \frac{t+s}{2} \left[\frac{\sin(x(s))}{7} + \frac{x(0) + x(1)}{5} \right] ds + 2\cos t + 1, \quad t \in [0,1], \tag{14}$$

is an α_1-contraction with the coefficient $\alpha_1 = \frac{19}{35}$.

Since the conditions of the theorem 3 are satisfied, it results that the integral equation (12) has a unique solution, that we denote by $x^* \in C[0,1]$.

Now, from the following estimation:

$$|K(t,s)| = \frac{t+s}{2} \leq 1, \text{ for all } t,s \in [0,1]$$

we obtain the positive constant $M_K = 1$.

Also, we have:

$$|h(s,u,v,w) - k(s,u,v,w)| = |s+2| \leq 3$$

for all $s \in [0,1]$, $u,v,w \in \mathbb{R}$ and

$$|f(t) - g(t)| = |\cos t + 1| \leq 2$$

for all $t \in [0,1]$. Therefore it results that $\eta_1 = 3$, $\eta_2 = 2$.

Since the operator $D : C[0,1] \to C[0,1]$ attached to the integral equation (13) and defined by the relation:

$$D(y)(t) := \int_0^1 \frac{t+s}{2} \left[\frac{\sin(y(s))}{7} + \frac{y(0) + y(1)}{5} - s - 2 \right] ds + \cos t, \ t \in [0,1], \tag{15}$$

is also an α_2-contraction with the coefficient $\alpha_2 = \frac{19}{35}$, it results that the conditions of the theorem 3 are satisfied and therefore the integral equation (13) has a solution that we denote by $y^* \in C[0,1]$. It can see that this solution is even unique but it was not necessary.

Now, we observe that the conditions of the theorem 5 are satisfied and therefore the following estimation is met:

$$\|x^* - y^*\|_{C[0,1]} \leq \frac{175}{16}.$$

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University of Petroşani
Department of Mathematics and Computer Science
Universităţii 20, 332006 Petroşani, România
E-mail address: mariadobritoiu@yahoo.com