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GROWTH OF UNIVERSAL ENTIRE HARMONIC FUNCTIONS

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Abstract. In this paper we obtained some bounds on growth parameters order and type of J-universal entire harmonic functions on $\mathbb{R}^N$. The results are expressed in terms of the derivatives of function at origin on $\mathbb{R}^N$.

1. Introduction

Let ξ denote the space of all entire (holomorphic) functions on $\mathbb{C}$ and let H_N denote the space of all functions harmonic on $\mathbb{R}^N$, where $N \geq 2$. MacLane [8] constructed an entire function f whose sequence of derivatives $(f^{(n)})$ is dense in ξ. Such a function f is known as universal entire function. M.P. Aldred and D.H. Armitage [1] studied analogous in the space H_N and obtained the permissible growth rates of universal functions in H_N and show that the set of all such functions is very large.

Let $\mathbb{N}$ denote the set of non-negative integers, and let $x = (x_1, \ldots, x_N)$ be a cartesian coordinate system in $\mathbb{R}^N$. If $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, then $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and

$$D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}.$$

The function $h \in H_N$ is said to be a universal harmonic function if the set $\{D^\alpha h : \alpha \in \mathbb{N}^N\}$ is dense in H_N.

When considering measures of growth of functions $f \in \xi$ (resp. H_N) we use the classical measures of order and type. Let $M(r, f)$ is the maximum value of $|f|$ on the circle (resp. sphere) of radius r centered at the origin of $\mathbb{C}$ (resp. $\mathbb{R}^N$). The order $\rho(f)$ of f is defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log \log M(r, f)}{\log r}. \tag{1}$$

In the case where $0 < \rho(f) < \infty$, the type $T(f)$ of f is defined as

$$T(f) = \limsup_{r\to\infty} \frac{\log M(r, f)}{r^\rho}. \tag{2}$$

Blair and Rubel [4] studied universal entire functions. Their proof is easily modified to yield the result on subsequence : if (n_k) is a strictly increasing sequence in N, then there exists $f \in \xi$ such that (f^{n_k}) is dense in ξ. This observation suggests the possibility that corresponding to any infinite subset J of N^N there exists a function $h \in H_N$ such that $\{D^\alpha h : \alpha \in J\}$ is dense in H_N. A function with this property is known as $J-$universal function.

In this paper we shall obtain some bounds on growth parameters order and type of J-universal entire harmonic function for any infinite set J.

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Let S denote the unit sphere in $\mathbb{R}^N$, and let σ be $(N-1)$ dimensional measure on S, normalized so that $\sigma(S) = 1$. An inner product and norm on H_N are defined as

$$< g, h >= \int_S gh d\sigma, \|h\| = \sqrt{< h, h >}.$$

Let $H_{n,N}$ denote the space of all homogeneous harmonic polynomials of degree n on R^N. Then $H_{n,N}$ is a vector space of dimension

$$d_n = (N + 2n - 2)\frac{(N + n - 3)!}{n!(N - 2)!}$$

2. Preliminary Results

In this section we shall prove some results which will be used in sequel.

Lemma 1. *Let $h_n(x) \in H_{n,N}$. Then*

$$\|h_n\|_\infty \leq \sqrt{d_n}\|h_n\|_2.$$

Proof. The proof follows from [6, pp. 107]. □

We define the norm of the m^{th} gradient of h at origin on R^N by

$$|\nabla_m h(0)| = \left(\frac{m!}{2^m} \sum_{|\alpha|=m} \frac{[D^\alpha h(0)]^2}{\alpha!}\right)^{1/2} \tag{3}$$

where $\alpha! = \alpha_1!\alpha_2! \ldots \alpha_N!$.

Note that if $h_j(x)$ is a spherical harmonic (a harmonic homogeneous polynomial) of degree $j \neq m$, then $|\nabla_m h_j(0)| = 0$.

Lemma 2. *Let $h = h(x)$ be a harmonic polynomial of degree n on $\mathbb{R}^N$. Then*

$$M(r, h) \leq M_1(r, h)$$

where

$$M(r, h) = \max_{|x|=r} |h(x)|$$

and

$$M_1(r, h) = \sqrt{\Gamma\frac{N}{2}} \sum_{m=0}^{\infty} \left[d_m(|\nabla_m h(0)|^2/(m!\Gamma(m + \frac{N}{2})))\right]^{1/2} r^m.$$

Proof. Since the convergence of the spherical harmonic expansion

$$h(x) = \sum_{m=0}^{\infty} h_m(x)$$

is absolute and uniform for $|x| = r$, it follows that

$$\begin{aligned} M(r, h) &= \max_{|x|=r} |h(x)| \\ &\leq \sum_{m=0}^{\infty} \|h_m\|_\infty r^m. \end{aligned}$$

Using Lemma 1, we get

$$\begin{aligned} &\leq \sum_{m=0}^{\infty} \sqrt{d_m}\|h_m\|_2 r^m \\ &= \sqrt{\Gamma\frac{N}{2}} \sum_{m=0}^{\infty} \left[d_m \left(|\nabla_m h(0)|^2 / \left(m!\Gamma\left(m+\frac{N}{2}\right)\right)\right)\right]^{1/2} r^m. \end{aligned}$$

□

Lemma 3. *Let $\phi : (0,\infty) \to (0,\infty)$ be given by*

$$\phi(r) = \sum_{m=0}^{\infty} a_m r^m$$

where a_m is non-negative for all positive integer m, and ϕ is non-constant, then the order of ϕ is

$$\rho(\phi) = \limsup_{m\infty} \frac{m \log m}{\log a_m^{-1}} \tag{4}$$

if $a_m = 0$, we interpret $(m \log m)/(\log a_m^{-1})$ as 0. The type of ϕ is given by

$$T(\phi) = M \limsup_{m\to\infty} m(a_m)^{\rho(\phi)/m} \tag{5}$$

where $M = \left[2^{\rho(\phi)/2}(e\rho(\phi))\right]^{-1}$.

Proof. The proof follows on the lines of R.P. Boas [5, Thm. 2.2.2]. □

Lemma 4. *Let h be a harmonic polynomial of degree n on R^N. For each $\alpha \in \mathbb{N}^N$ with $|\alpha| \geq 1$ there exists a harmonic polynomial h_α such that $D^\alpha h_\alpha = h$ and*

$$M(r, h_\alpha) \leq A|\alpha|^A (n+1)^{(N-1)/2}(|\alpha|!)^{-1}(C_N r^N)^{|\alpha|} M(r,h)$$

for each positive number r.

Proof. The proof is essentially same as [2, Lemma 4]. □

3. Main Results

In this section we shall prove our main results.

Theorem 1. *If J is an infinite subset of $\mathbb{N}^N$, then there exists a $J-$universal function of order λ_N and type C_N.*

Proof. Let (h_n) be a dense sequence of polynomials in H_N and the degree of h_n is $\delta(n)$. In view of Lemma 4, for each $\alpha \in \mathbb{N}^N$ with $|\alpha| \geq 1$, there exists a harmonic polynomial $h_{n,\alpha}$ such that $D^\alpha h_{n,\alpha} = h_n$ and

$$M(r, h_{n,\alpha}) \leq C_n |\alpha|^A (|\alpha|!)^{-1} (C_N r^{\lambda N})^{|\alpha|}(1+r)^{\delta(n)} \tag{6}$$

where e_n depends only on n and N. Fix $\eta > 0$. Then

$$\frac{M(r,h_{n,\alpha})}{e^{r^{(1+\eta)\lambda_N}}} \leq \frac{C_n|\alpha|^A (C_N)^{|\lambda|}(r^{\lambda_N})^{|\alpha|}(1+r)^{\delta(n)}}{|\alpha|! e^{(r^{(1+\eta/2)\lambda_N}; r^{\eta/2\lambda_N})}}. \tag{7}$$

The function $(t^{\lambda_N})^k e^{-t^{(1+\eta/2)\lambda_N}}$, takes its minimum value for positive t when $t = (k/(1+\eta/2))^{1/(1+\eta/2)\lambda_N}$. Thus, from (7) we have

$$\begin{aligned}\frac{M(r,h_{n,\alpha})}{e^{r^{(1+\eta)\lambda_N}}} &\leq \frac{C_n|\alpha|^{A+|\alpha|/(+\eta/2)}(C_N)^{|\alpha|}(1+\eta/2)^{-\frac{|\alpha|}{1+\eta/2}}(1+r)^{\delta(n)}}{|\alpha|!e^{|\alpha|/(1+\eta/2)}(|\alpha|/1+\eta/2)^{(\eta/2/1+\eta/2)}} \\ &\leq \frac{C_{n,\eta}|\alpha|^{(A+|\alpha|/(1+\eta/2))}}{|\alpha|!e^{|\alpha|}}.\frac{(1+\eta/2)^{-|\alpha|/(1+\eta/2)}(C_N)^{|\alpha|}}{}\end{aligned}$$

where $C_{n,\eta}$ depends only on n, N and η. For sufficiently large $|\alpha|$ we have

$$M(r,h_{n,\alpha}) \leq 2^{-n}e^{r^{(1+1/n)\lambda_N}} \forall r > 0.$$

From (6) it is clear that for sufficiently large $|\alpha|, M(r,h_{n,\alpha})$ is small as we please. Using Cauchy's estimates [3, pp. 33], we observe that for each $k \in \mathbb{N}$ there exists a number q, depending on k and n such that

$$|D^\beta h_{n,\alpha}| \leq 2^{-n} \text{ on } B(n)$$

whenever $|\beta| \leq k$ and $|\alpha| \geq q$, here $B(n)$ denotes the open ball of radius n centered at the origin of $\mathbb{R}^N$.

Now choosing a seuqence (α_n) in J. First taking $\alpha_1 \in J$ such that

$$M(r,h_1,\alpha_1) \leq 2^{-1}e^{r^{2\lambda_N}}.$$

Having chosen $\alpha_1,\ldots,\alpha_{n-1}$ for some $n \geq 2$, choose $\alpha_n \in J$ so that

$$\begin{aligned}|\alpha_n| &> \max\{|\alpha_{n-1}|, deg\ h_1,\alpha_1,\ldots,deg\ h_{n-1}\alpha_{n-1}\} \\ M(r,h_n,\alpha_n) &\leq 2^{-n}e^{r^{(1+1/n)\lambda_N}}\end{aligned} \tag{8}$$

and we have

$$|D^\beta h_n,\alpha_n| \leq 2^{-n} \text{ on } B(n)(|\beta| \leq |\alpha_{n-1}|).$$

Now proceeding on the lines of [2, pp. 390] we can prove that h is $J-$universal.

In order to prove that h is of order λ_N, we proceed as follows:

Let p be a positive integer. Since h_n, α_n are polynomials, therefore from (8) we get

$$\begin{aligned}M(r,h) &\leq \sum_{n=1}^{\infty} M(r,h_n,\alpha_n) \\ &\leq 0(e^{r^{\lambda_N}}) + \sum_{n=p}^{\infty} 2^{-n}e^{r^{(1+1/n)\lambda_N}} \\ &= 0(e^{r^{(1+1/p)\lambda_N}}).\end{aligned}$$

Taking p sufficiently large, and using (1) it follows that h is of order λ_N.

Now we show that h is of type C_N. Using (6) we have

$$\frac{M(r,h_{n,\alpha})}{e^{C_N(1+\eta)r^{\lambda_N}}} \leq \frac{C_N|\alpha|^A(C_N r^{\lambda_N})^{|\alpha|}(1+r)^{\delta(n)}}{|\alpha|!e^{C_N(1+n/2)r^{\lambda_N}}.e^{(c_N\eta r^{\lambda_N})/2}}. \tag{9}$$

The function $(t^{\lambda_N})^k e^{-(1+n/2)t^{\lambda_N}}$ takes its minimum value for positive t when $t = [k/(1+n/2)]^{1/\lambda_N}$. Hence (9) gives that

$$\begin{aligned}\frac{M(r,h_{n,\alpha})}{e^{C_N(1+\eta)r^{\lambda_N}}} &\leq \frac{C_n|\alpha|^{A+|\alpha|}(C_N)^{|\alpha|}(1+r)^{\delta(n)}}{e^{|\alpha|(1+C_N+C_N(\eta/2(1+n/2)))}.(1+n/2)^{|\alpha|}|\alpha|!} \\ &\leq \frac{C_{n,\eta}|\alpha|^{A+|\alpha|}(C_N)^{|\alpha|}}{e^{|\alpha|(1+C_N+C_N(\eta/2(1+n/2)))}.(1+n/2)^{|\alpha|}|\alpha|!}.\end{aligned}$$

For sufficiently large $\alpha|$, we have

$$M(r, h_{n,\alpha}) \leq 2^{-n} e^{C_N(1+1/n)r^{\lambda_N}} \forall r > 0.$$

Now we proceed as in the proof of order of h and bearing in mind (2), we obtain that the type of $J-$universal function is C_N. □

Theorem 2. *Let $h \in H_N$ and h is $J-$universal function of order λ_N, then*

$$L \leq \lambda_N(h) \leq \lambda_N(M_1)$$

where

$$L = \limsup_{m\to\infty} \frac{\log mN}{\log \left[\frac{D^{\beta_m} h(0)}{\sqrt{2^{mN}(m!)^N(mN)!}}\right]^{-\frac{1}{mN}}}.$$

Proof. For each $m \in N$, let $\beta_m = (m, \ldots, m) \in \mathbb{N}^N$. Let $J = \{\beta_m : m \in \mathbb{N}\}$ and suppose that h is a $J-$universal function. Using Fryant and Shankar result [6, Thm. 4] for order, we have

$$\begin{aligned}
\lambda_N(h) &= \limsup_{m\to\infty} \frac{m \log m}{\log(m!/|\nabla_m h(0)|)} \\
&\geq \limsup_{m\to\infty} \frac{mN \log mN}{\log((mN)!/|\nabla_{mN} h(0)|)} \\
&\geq \limsup_{m\to\infty} \frac{mN \log mN}{\log\left(\frac{2^{mN}(mN)!(m!)^N}{(D^{\beta_m}h(0))^2}\right)^{1/2}} \\
&= \limsup_{m\to\infty} \frac{2mN \log mN}{\log\left(\frac{2^{mN}(mN)!(m!)^N}{(D^{\beta_m}h(0))^2}\right)} \\
&= \limsup_{m\to\infty} \frac{\log mN}{\log\left(\frac{D^{\beta_m}h(0)}{\sqrt{2^{mN}(mN)!(m!)^N}}\right)^{-\frac{1}{mN}}}.
\end{aligned}$$

Using the relation

$$\log\left(\frac{d_m|\nabla_m h(0)|^2}{m!\sqrt{m+N/2}}\right)^{1/2} \simeq \log\frac{|\nabla_m h(0)|}{m!} \quad \text{as } m \to \infty, \tag{10}$$

in Lemma 2 and applying the Lemma 3 for the order of $M_1(r, h)$ and since the order of $M(r, h)$ is less than or equal to the order of $M_1(r, h)$, we get

$$\lambda_N(M_1) \geq \lambda_N(h) \geq L.$$

□

Theorem 3. *Let $h \in H_N$ be a $J-$universal function for each $m \in \mathbb{N}$, $\beta_m = (m, \ldots, m) \in \mathbb{N}^N$, $J = (\beta_m : m \in \mathbb{N})$ and $0 < \lambda_N(h) < \infty$. Then the type C_N is given by*

$$L^* \leq C_N(h) \leq C_N(M_1)$$

where

$$L^* = 2^{-\lambda_N(h)/2}(e\lambda_N(h))^{-1} \limsup_{m\to\infty} m_N \left(\frac{D^{\beta_m}h(0)}{\sqrt{2^{mN}(mN)!(m!)^N}}\right)^{\frac{\lambda_N}{mN}}.$$

Proof. In view of Theorem 2.6 of Fugard [7] that if $h \in H_N$ and h is of type $C_N(h)$, then

$$C_N(h) = 2^{-\lambda_N(h)/2}(e\lambda_N(h))^{-1} \limsup_{m\to\infty} m \left(\frac{|\nabla_m h(0)|}{m!}\right)^{\frac{\lambda_N(h)}{m}}.$$

For each $m \in \mathbb{N}, \beta_m = (m,\ldots,m) \in \mathbb{N}^N$. Let $J = (\beta_m : m \in \mathbb{N})$ and suppose that h is a $J-$universal function. We see that

$$\begin{aligned}\limsup_{m\to\infty\infty} m \left(\frac{|\nabla_m h(0)|}{m!}\right)^{\lambda_N/m} &\geq \limsup_{m\to\infty} mN \left(\frac{\nabla_{mN} h(0)}{(mN)!}\right)^{\frac{\lambda_N}{mN}} \\ &\geq \limsup_{m\to\infty} \left(\frac{D^{\beta_m}h(0)}{\sqrt{2^{mN}(mN)!(m!)^N}}\right)^{\frac{\lambda_N}{mN}}.\end{aligned}$$

Hence we get

$$C_N(h) \geq L^*.$$

Since the type of $M(r,h)$ is less than or equal to the type of $M_1(r,h)$. Now applying Lemma 3 for the type of $M_1(r,h)$ in Lemma 2 with the relation (10), we obtain

$$C_N(M_1) \geq C_N(h) \geq L^*.$$

□

Remark 1. *For $J = \{(m,\ldots,m) \in \mathbb{N}^N : m \in \mathbb{N}\}$ there is no $J-$universal function of order less than L and type less than L^*.*

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