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A GENERALIZATION OF COMPANION INEQUALITY OF OSTROWSKI'S TYPE FOR MAPPINGS WHOSE FIRST DERIVATIVES ARE BOUNDED AND APPLICATIONS IN NUMERICAL INTEGRATION

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ABSTRACT. An inequality for a companion of Ostrowski's integral inequality is proved. Application to a composite quadrature rule is considered.

1. INTRODUCTION

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \quad (1)$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In [12], Dragomir, Cerone and Roumeliotis proved the following generalization of Ostrowski's inequality.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$, differentiable on (a,b) and whose derivative f' is bounded on (a,b). Denote $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then,*

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)f(x)\right] - \int_a^b f(t)\,dt\right| \leq \left[\frac{(b-a)^2}{4}\left(\lambda^2 + (1-\lambda)^2\right) + \left(x - \frac{a+b}{2}\right)^2\right]\|f'\|_\infty . \quad (2)$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \leq x \leq b - \lambda\frac{b-a}{2}$.

Using (2), the authors obtained estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae. They also gave applications of the mentioned results in numerical integration and for special means, for more results about Ostrowski type inequalities see [1]–[14].

Companions of Ostrowski's integral inequality for absolutely continuous functions was considered by Dragomir in [10], as follows:

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Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. If f' is bounded on $[a, b]$, i.e., $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then the inequality*

$$\left|\frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{8} + 2\left(\frac{x - \frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, \quad (3)$$

for all $x \in [a, \frac{a+b}{2}]$.

Recently, in [2], proved a companion inequality for differentiable mappings whose derivatives are bounded.

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, and let $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$ and $\gamma \le f'(x) \le \Gamma$, $\forall x \in [a, b]$, then the following inequality holds,*

$$\left|\frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le (b-a)\left[\frac{1}{16} + \left(\frac{x - \frac{3a+b}{4}}{b-a}\right)^2\right]\cdot(\Gamma - \gamma), \quad (4)$$

for all $x \in [a, \frac{a+b}{2}]$.

Also, in [3] the following results was obtained:

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a, b \in I$ with $a < b$. If f' is bounded on $[a, b]$, i.e., $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then the inequality*

$$\left|(b-a)\left[\lambda\frac{f(a) + f(b)}{2} + (1-\lambda)\frac{f(x) + f(a+b-x)}{2}\right] - \int_a^b f(t)\,dt\right| \le \left[\frac{(b-a)^2}{8}\left(2\lambda^2 + (1-\lambda)^2\right) + 2\left(x - \frac{(3-\lambda)a + (1+\lambda)b}{4}\right)^2\right]\|f'\|_\infty. \quad (5)$$

holds, for all $\lambda \in [0, 1]$ and $a + \lambda\frac{b-a}{2} \le x \le \frac{a+b}{2}$.

In [11], Dragomir established some inequalities for this companion for mappings of bounded variation. In [13], Liu introduced some companions of an Ostrowski type inequality for functions whose second derivatives are absolutely continuous. Recently, Barnett, Dragomir and Gomma [5], have proved some companions for the Ostrowski inequality and the generalized trapezoid inequality. The aim of this paper is to study the companion inequality (2) which is of Ostrowski's type for differentiable bounded mappings.

2. Ostrowski inequality for mappings bounded variation

Theorem 6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable on I°, the interior of the interval I, where $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$ and $\gamma \le f'(x) \le \Gamma$, $\forall x \in [a, b]$, $\gamma, \Gamma > 0$, then* *the following inequality holds*

$$\left| \frac{b-a}{2} \left[\lambda \left(f(a) + f(b) \right) + (1-\lambda) \left(f(x) + f(a+b-x) \right) \right] - \int_a^b f(t)\, dt \right|$$
$$\leq \left[\frac{(b-a)^2}{16} \left(2\lambda^2 + (1-\lambda)^2 \right) + \left(x - \frac{(3-\lambda)a + (1+\lambda)b}{4} \right)^2 \right] (\Gamma - \gamma). \quad (6)$$

holds, for all $\lambda \in [0,1]$ *and* $a + \lambda\frac{b-a}{2} \leq x \leq \frac{a+b}{2}$.

Proof. Defining the mapping

$$K(x,t) = \begin{cases} t - \left(a + \lambda\frac{b-a}{2} \right), & t \in [a,x] \\ t - \frac{a+b}{2}, & t \in (x, a+b-x] \\ t - \left(b - \lambda\frac{b-a}{2} \right), & t \in (a+b-x, b] \end{cases} \quad (7)$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \leq x \leq \frac{a+b}{2}$.

Integrating by parts, we obtain

$$\int_a^b K(x,t)\, f'(t)\, dt$$
$$= \frac{b-a}{2} \left[\lambda \left(f(a) + f(b) \right) + (1-\lambda) \left(f(x) + f(a+b-x) \right) \right] - \int_a^b f(t)\, dt. \quad (8)$$

We have also,

$$\int_a^b K(x,t)\, dt = 0 \quad (9)$$

Let $C = \frac{\Gamma-\gamma}{2}$. From (8) and (9), it follows that

$$\int_a^b K(x,t) \left[f'(t) - C \right] dt$$
$$= \frac{b-a}{2} \left[\lambda \left(f(a) + f(b) \right) + (1-\lambda) \left(f(x) + f(a+b-x) \right) \right] - \int_a^b f(t)\, dt.$$

On the other hand,

$$\left| \int_a^b K(x,t) \left[f'(t) - C \right] dt \right| \leq \max_{t\in[a,b]} \left| f'(t) - C \right| \int_a^b \left| K(x,t) \right| dt. \quad (10)$$

Now, since

$$\int_p^r |t-q|\, dt = \int_p^q (q-t)\, dt + \int_q^r (t-q)\, dt = \frac{(q-p)^2 + (r-q)^2}{2}$$
$$= \frac{1}{4}(p-r)^2 + \left(q - \frac{r+p}{2} \right)^2, \quad (11)$$

for all r, p, q such that $p \leq q \leq r$. Then, we observe that

$$\int_a^x \left| t - \left(a + \lambda\frac{b-a}{2} \right) \right| dt = \frac{1}{4}(x-a)^2 + \left(\lambda\frac{b-a}{2} - \frac{x-a}{2} \right)^2,$$

$$\int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| dt = \left(x - \frac{a+b}{2} \right)^2,$$

and

$$\int_{a+b-x}^{b} \left| t - \left(b - \lambda \frac{b-a}{2} \right) \right| dt = \frac{1}{4}(x-a)^2 + \left(\frac{x-a}{2} - \lambda \frac{b-a}{2} \right)^2.$$

Then, we have

$$\begin{aligned}\int_a^b |K(x,t)|\, dt &= \frac{(x-a)^2 + ((x-a) - \lambda (b-a))^2}{2} + \left(x - \frac{a+b}{2} \right)^2 \\ &= \underbrace{\frac{1}{4}\lambda^2 (b-a)^2 + \left(x - \frac{(2-\lambda) a + \lambda b}{2} \right)^2}_{\text{by (10)}} + \left(x - \frac{a+b}{2} \right)^2 \\ &= \frac{\lambda^2}{4}(b-a)^2 + \underbrace{\frac{(1-\lambda)^2}{8}(b-a)^2 + 2\left(x - \frac{(3-\lambda) a + (1+\lambda) b}{4} \right)^2}_{\text{by (10)}}, \\ &= \frac{(b-a)^2}{8}\left(2\lambda^2 + (1-\lambda)^2 \right) + 2\left(x - \frac{(3-\lambda) a + (1+\lambda) b}{4} \right)^2,\end{aligned}$$

also, since $\max_{t\in[a,b]} |f'(t) - C| \le \frac{\Gamma-\gamma}{2}$, by (10) we have

$$\begin{aligned}&\left| \frac{b-a}{2} \left[\lambda (f(a) + f(b)) + (1-\lambda)(f(x) + f(a+b-x)) \right] - \int_a^b f(t)\, dt \right| \\ &\qquad \le \left[\frac{(b-a)^2}{16}\left(2\lambda^2 + (1-\lambda)^2 \right) + \left(x - \frac{(3-\lambda) a + (1+\lambda) b}{4} \right)^2 \right] (\Gamma - \gamma),\end{aligned}$$

for all $\lambda \in [0,1]$ and $a + \lambda \frac{b-a}{2} \le x \le \frac{a+b}{2}$, which gives the required result. □

Remark 1. *In Theorem 6, choose $\lambda = 0$ and $x = \frac{3a+b}{4}$, then we refer to the first inequality in* (3).

Corollary 1. *In Theorem 6, choose $x = \frac{a+b}{2}$, we get*

$$\begin{aligned}&\left| (b-a)\left[\lambda \frac{f(a) + f(b)}{2} + (1-\lambda) f\left(\frac{a+b}{2} \right) \right] - \int_a^b f(t)\, dt \right| \\ &\qquad \le \left(\lambda^2 + (1-\lambda)^2 \right) \frac{(b-a)^2}{8} (\Gamma - \gamma),\end{aligned}$$

Corollary 2. *In Corollary 1, if we choose*

(1) $\lambda = 0$, then we get

$$\left| (b-a) f\left(\frac{a+b}{2} \right) - \int_a^b f(t)\, dt \right| \le \frac{1}{8}(b-a)^2 (\Gamma - \gamma),$$

(2) $\lambda = \frac{1}{3}$, then we get

$$\left| \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right] - \int_a^b f(t)\, dt \right| \le \frac{5}{72}(b-a)^2 (\Gamma - \gamma),$$

(3) $\lambda = \frac{1}{2}$, *then we get*

$$\left| \frac{(b-a)}{2} \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(t)\,dt \right| \le \frac{1}{16}(b-a)^2(\Gamma-\gamma),$$

(4) $\lambda = 1$, *then we get*

$$\left| (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(t)\,dt \right| \le \frac{1}{8}(b-a)^2(\Gamma-\gamma),$$

Corollary 3. *In Theorem 6, Setting*

(1) $\lambda = \frac{1}{n}$, *for* $n = 1, 2, 3, \ldots$, *then we get*

$$\left| \frac{b-a}{2n}\left[f(a) + (n-1)\left[f(x) + f(a+b-x)\right] + f(b)\right] - \int_a^b f(t)\,dt \right|$$
$$\le \left[\frac{(b-a)^2}{16n^2}\left(2 + (n-1)^2\right) + \left(x - \frac{(3n-1)a + (n+1)b}{4n}\right)^2 \right](\Gamma-\gamma), \quad (12)$$

(2) $\lambda = \frac{n-1}{n}$, *for* $n = 1, 2, 3, \ldots$, *then we get*

$$\left| \frac{b-a}{2n}\left[(n-1)f(a) + f(x) + f(a+b-x) + (n-1)f(b)\right] - \int_a^b f(t)\,dt \right|$$
$$\le \left[\frac{(b-a)^2}{16n^2}\left(2(n-1)^2 + 1\right) + \left(x - \frac{(2n+1)a + (2n-1)b}{4n}\right)^2 \right](\Gamma-\gamma). \quad (13)$$

Corollary 4. *In* (12), *choose* $n = 4$ *and* $x = \frac{2a+b}{3}$, *then we get the following* 3/8*–Simpson's inequality*

$$\left| \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \int_a^b f(t)\,dt \right|$$
$$\le \frac{25}{576}(b-a)^2(\Gamma-\gamma).$$

3. A Composite Quadrature Formula

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a division of the interval $[a, b]$ and $h_i = x_{i+1} - x_i$, $(i = 0, 1, 2, \cdots, n-1)$.

Consider the general quadrature formula

$$Q_n(I_n, f)$$
$$:= \sum_{i=0}^{n-1} \frac{h_i}{2}\left[\lambda\left(f(x_i) + f(x_{i+1})\right) + (1-\lambda)\left(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i)\right)\right]. \quad (14)$$

for all $\lambda \in [0, 1]$ and $x_i + \lambda\frac{x_{i+1}-x_i}{2} \le \alpha_i \le \frac{x_i+x_{i+1}}{2}$.

The following result holds.

Theorem 7. *Let* f *as in Theorem 6, then we have*

$$\int_a^b f(t)\,dt = Q_n(I_n, f) + R_n(I_n, f).$$

where, $Q_n(I_n, f)$ is defined by formula (14)*, and the remainder satisfies the estimates*

$$|R_n(I_n, f)| \le (\Gamma - \gamma) \cdot \sum_{n=0}^{n-1} \left[\frac{h_i^2}{16} \left(2\lambda^2 + (1-\lambda)^2 \right) + \left(\alpha_i - \frac{(3-\lambda)x_i + (1+\lambda)x_{i+1}}{4} \right)^2 \right]$$

for all $\lambda \in [0,1]$ and $x_i + \lambda\frac{x_{i+1}-x_i}{2} \le \alpha_i \le \frac{x_i+x_{i+1}}{2}$.

Proof. Applying inequality (6) on the intervals $[x_i, x_{i+1}]$, we may state that

$$R_i(I_i, f) = \int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h_i}{2}\left[\lambda\left(f(x_i) + f(x_{i+1})\right) + (1-\lambda)\left(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i)\right)\right].$$

Summing the above inequality over i from 0 to $n-1$, we get

$$\begin{aligned} R_n(I_n, f) &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(t)\,dt - \sum_{i=0}^{n-1} \frac{h_i}{2}\left[\lambda\left(f(x_i) + f(x_{i+1})\right) + (1-\lambda)\left(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i)\right)\right] \\ &= \int_a^b f(t)\,dt - \sum_{i=0}^{n-1} \frac{h_i}{2}\left[\lambda\left(f(x_i) + f(x_{i+1})\right) + (1-\lambda)\left(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i)\right)\right], \end{aligned}$$

which follows from (6), that

$$\begin{aligned} |R_n(I_n, f)| &= \left| \int_a^b f(t)\,dt - \sum_{i=0}^{n-1} \frac{h_i}{2}\left[\lambda\left(f(x_i) + f(x_{i+1})\right) + (1-\lambda)\left(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i)\right)\right] \right| \\ &\le (\Gamma - \gamma) \cdot \sum_{n=0}^{n-1} \left[\frac{h_i^2}{16} \left(2\lambda^2 + (1-\lambda)^2 \right) + \left(\alpha_i - \frac{(3-\lambda)x_i + (1+\lambda)x_{i+1}}{4} \right)^2 \right] \end{aligned}$$

which completes the proof. □

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