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THERMOELASTOSTATIC EQUILIBRIUM OF SOME SPHERICAL SEMI-WEDGES: GREEN'S FUNCTIONS AND INTEGRAL FORMULAS

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Abstract. In this study new exact Green's functions and new exact Green-type integral formula for a boundary value problem (BVP) in thermoelasticity for some spherical semi-wedges with mixed homogeneous mechanical boundary conditions (zero normal stresses and tangential displacements on quarter-plane $\Gamma_{\varphi 0}$ and on the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}$; zero normal displacements and tangential stresses on quarter-plane $\Gamma_{\varphi\alpha}$) are derived. The thermoelastic displacements are subjected to a heat source applied in the inner points of the spherical semi-wedges and to a mixed non-homogeneous boundary heat conditions (temperatures on quarter-plane $\Gamma_{\varphi 0}$ and on the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}$; heat flux on quarter-plane $\Gamma_{\varphi\alpha}$). When thermoelastic Green's functions are derived, the thermoelastic displacements are created by an inner unit point heat source, described by δ-Dirac's function. All results are obtained in terms of elementary functions that are formulated in a special theorem.

1. Introduction

The main objective of this paper is to prove a theorem (section 2) on derivation of thermoelastostatic Green functions (subsection 2.2) and Green-type integral formula (subsection 2.3) for a homogeneous isotropic 3D spherical semi-wedges $V(0 \leq r < \infty;\ 0 \leq \varphi \leq \alpha,\ 0 \leq \beta \leq \pi/2)$; $\alpha = \pi/n$; $n = 2, 3, 4\ldots$, which is bounded by the quarter-planes $\Gamma_{\varphi 0}(0 \leq r < \infty;\ \varphi = 0;\ 0 \leq \beta \leq \pi/2)$, $\Gamma_{\varphi\alpha}(0 \leq r < \infty;\ \varphi = \alpha;\ 0 \leq \beta \leq \pi/2)$, and by the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}(0 \leq r < \infty;\ 0 \leq \varphi \leq \alpha;\ \beta = \pi/2)$, where spherical coordinates r, φ,vβ are used. When deriving the Green-type integral formula, on the boundary quarter-planes $\Gamma_{\varphi 0}$, $\Gamma_{\varphi\alpha}$ the homogeneous mixed mechanical boundary conditions are given (zero normal stresses and tangential displacements on quarter-plane $\Gamma_{\varphi 0}$ and on the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}$; zero normal displacements and tangential stresses on quarter-plane $\Gamma_{\varphi\alpha}$). The searched thermoelastic field of displacements is created by the inner heat source $F(M)$, $M(r, \varphi, \beta) \in V$, by non-homogeneous Neumann's boundary conditions for the heat flux and by non-homogeneous Dirichlet's boundary conditions for the temperature T, given on the boundary quarter-planes $\Gamma_{\varphi\alpha}$, $\Gamma_{\varphi 0}$ and on the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}$, respectively. The mentioned-above Green-type integral formula is derived on the base of the main thermoelastic Green's functions $U_q(M, N)$ (see subsection 2.2). These functions represent the influence of an inner unit point heat source, described by the δ-Dirac's function and applied in the point $M(r, \varphi, \beta) \in V$, on the thermoelastic displacement applied in the point $N(\rho, \psi, \vartheta) \in V$ in the direction of the axis $q = \rho, \psi, \vartheta$. As it is shown below (see subsection 2.2) to derive the main thermoelastic Green's functions $U_q(M, N)$, we need:

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(1) to construct two Green's functions: the function of influence of an inner unit point heat source on temperature - $G(M,N)$ for boundary value problem (BVP) in heat conduction and the function of influence of an inner unit point body force, applied in the point $N(\rho,\psi,\vartheta)\in V$ in the direction $q=\rho,\psi,\vartheta$, on volume dilatation - $\Theta^{(q)}(M,N)$ in the point $M(r,\varphi,\beta)\in V$ for BVP in elasticity;
(2) to compute a convolution over volume of product of these two functions (the mathematical procedures to compute this integral over volume are presented in Appendix A).

In the subsection 2.1 a special method is used to derive the functions $\Theta^{(q)}(M,N)$. This method is based on proposed integral representations of solution for fundamental Lame's elasticity equations via Green's functions for Poisson's equation that permit to determine volume dilatation on the boundaries of the spherical semi-wedges. Then using the integral representation for solution of Poisson's type equation for volume dilatation via its already known boundaries values and via the respective Green's function we obtained the final expressions for functions $\Theta^{(q)}(M,N)$ inside of spherical semi-wedges. The next objectives of this paper are: a. to analyze the advantages, usefulness and importance of the obtained results in comparison with other traditional methods for solving the BVP of thermoelasticity (section 4), and b. to analyze the possibilities of deriving Green's integral formula in thermoelasticity for other spherical canonical domains (section 3).

To achieve these objectives, as it was stated above, first we need to prove a theorem (section 2) on derivation of thermoelastostatic Green's functions and Green-type integral formula for the spherical semi-wedges V. To accomplish this we use the general Green's integral formula in stationary thermoelasticity (subsection 1.1) and the integral formula for thermoelastic influence functions (subsection 1.2) suggested and published by the author earlier.

1.1. General Green's Integral Formula in Stationary Thermoelasticity.

Green's function plays the leading role in finding solutions in integrals for BVPs in different fields of mathematical physics. The theory of thermoelasticity, which is a synthesis of the theory of heat conduction and elasticity theory, is one of such fields. To date, a number of theories of thermoelasticity have been developed and described in classical scientific literature [1, 2, 3, 4, 5, 6]. However, many new developments of thermoelasticity and many references are included in [7]. The best developed theory widely used in practical calculations is the theory of stationary thermal stresses, i.e., the theory of uncoupled thermoelasticity when the temperature field does not depend on the field of elastic displacements and when inertial terms can be ignored. According to this theory, the formulation of the BVP consists of non-homogeneous Lame's equations, written in an arbitrary inner point $M\equiv(r,\varphi,\beta)$:

$$\begin{gathered}\mu\left[\nabla^2 u_r-\tfrac{2}{r^2}\left(u_r+\tfrac{1}{\sin\beta}\tfrac{\partial}{\partial\beta}\left(u_\beta\sin\beta\right)+\tfrac{1}{\sin\beta}\tfrac{\partial u_\varphi}{\partial\varphi}\right)\right]+(\lambda+\mu)\tfrac{\partial\theta}{\partial r}+\gamma\tfrac{\partial T}{\partial r}=0\,;\\ \mu\left[\nabla^2 u_\beta-\tfrac{2}{r^2}\left(\tfrac{\partial u_r}{\partial\beta}-\tfrac{u_\beta}{2\sin^2\beta}-\tfrac{\cos\beta}{\sin^2\beta}\tfrac{\partial u_\varphi}{\partial\varphi}\right)\right]+\tfrac{\lambda+\mu}{r}\cdot\tfrac{\partial\theta}{\partial\beta}+\tfrac{\gamma}{r}\cdot\tfrac{\partial T}{\partial\beta}=0\,;\\ \mu\left[\nabla^2 u_\varphi+\tfrac{2}{r^2\sin\beta}\left(\tfrac{\partial u_r}{\partial\varphi}+ctg\beta\tfrac{\partial u_\beta}{\partial\varphi}-\tfrac{u_\varphi}{r\sin\beta}\right)\right]+\tfrac{\lambda+\mu}{r\sin\beta}\cdot\tfrac{\partial\theta}{\partial\varphi}+\tfrac{\gamma}{r\sin\beta}\cdot\tfrac{\partial T}{\partial\varphi}=0\,,\end{gathered}\tag{1}$$

subject to suitable homogeneous mechanical boundary conditions, where

$$\theta=\frac{\partial u_r}{\partial r}+\frac{1}{r}\left(2u_r+\frac{1}{\sin\beta}\cdot\frac{\partial u_\varphi}{\partial\varphi}+\frac{\partial u_\beta}{\partial\beta}+ctg\beta u_\beta\right)\tag{2}$$

is the thermoelastic volume dilatation, created by temperature T; and

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\beta}\frac{\partial}{\partial\beta}\left(\sin\beta\frac{\partial}{\partial\beta}\right)+\frac{1}{r^2\sin\beta}\frac{\partial^2}{\partial\varphi^2}\tag{3}$$

is the Laplace's differential operator, written in spherical coordinates; λ, μ are the Lame's constants of elasticity; $\gamma = \alpha_t (2\mu + 3\lambda)$ is the thermoelastic constant; and α_t is the coefficient of the linear thermal expansion. The temperature field T in Eq. (1) has to be determined from the BVP in heat conduction that consists of the Poisson's equation

$$\nabla^2 T(r, \varphi, \beta) = -a^{-1} F(r, \varphi, \beta) \tag{4}$$

subject to inner heat source F and to non-homogeneous boundary conditions when temperature $T(M)$, heat flux $a(\partial T(M)/\partial n_M)$ or heat exchange law $\alpha T(M) + a(\partial T(M)/\partial n_M)$ between exterior medium and surface of the body are prescribed. In (4) a is thermal conductivity; α is coefficient of convective heat conductivity. To solve the BVP of stationary thermoelasticity in Eqs (1)-(4) using traditional methods, at the first stage we need to solve the BVP of heat conduction in Eq. (4) with the given boundary conditions and to find the temperature field. At the second stage we need to solve BVP of thermoelasticity in Eq. (1) with the already known temperature field and with the given mechanical boundary conditions. But using the influence functions $U_q = U_q(M, N)$ introduced in [8-14] the solution (the field of thrmoelastic displacements) of the BVP of thermoelasticity in Eqs (1)-(4) can be written directly by the following Green-type integral formula:

$$\begin{aligned} u_q(N) = a^{-1} \int\limits_V F(M) U_q(M, N) dV(M) - \int\limits_{\Gamma_D} T\left(\tilde{M}\right) \frac{\partial U_q\left(\tilde{M}, N\right)}{\partial n_{\tilde{M}}} d\Gamma_D\left(\tilde{M}\right) + \\ + \int\limits_{\Gamma_N} \frac{\partial T\left(\tilde{M}\right)}{\partial n_{\tilde{M}}} U_q\left(\tilde{M}, N\right) d\Gamma_N\left(\tilde{M}\right) + \\ + \int\limits_{\Gamma_M} \left[T\left(\tilde{M}\right) + \alpha^{-1} a \frac{\partial T\left(\tilde{M}\right)}{\partial n_{\tilde{M}}}\right] U_q\left(\tilde{M}, N\right) d\Gamma_M\left(\tilde{M}\right), \end{aligned} \tag{5}$$

where Γ_D, Γ_N and Γ_M denote the surfaces on which the boundary conditions of Dirichlet's, Neumann's and mixed types, respectively for thermal data are prescribed. One of the advantages of this formula is that the searched thermoelastic displacements $u_q(N)$ are determined in the integral form directly via the prescribed inner heat source and other thermal data, given on the boundary. Analogous results regarding thermoelastic Green's functions and general Green-type integral formula in dynamic uncoupled thermoelasticity are presented in [15].

After determinating the thermoelastic displacements using the Green's type integral formula, the respective thermal stresses are determined by the Duhamel-Neumann law:

$$\sigma_{ql} = 2\mu\varepsilon_{ql} + \delta_{ql}(\lambda\theta - \gamma T); \quad q, l = \rho, \psi, \xi, \tag{6}$$

where δ_{ql} is the Kronecher's symbol.

1.2. **Thermoelastic Influence Functions.** The influence functions $U_q = U_q(M, N)$ occurring in (4) and corresponding to a unit heat source are determined by the following integral formula [11, 12, 14]:

$$U_q(M, N) = \gamma \int_V G(M, N')\, \Theta^{(q)}(N', N)\, dV(N'); M, N, N' \in V, \tag{7}$$

where G is the Green's function for a heat conduction BVP corresponding to an internal point heat source, and

$$\Theta^{(q)} = \frac{\partial U_r^{(q)}}{\partial r} + \frac{1}{r}\left(2U_r^{(q)} + \frac{1}{\sin\beta}\cdot\frac{\partial U_\varphi^{(q)}}{\partial\varphi} + \frac{\partial U_\beta^{(q)}}{\partial\beta} + ctg\beta U_\beta^{(q)}\right) \tag{8}$$

are the functions of influence of inner unit concentrated body forces on elastic volume dilatation, where $U_s^{(q)}(M,N)$; $s = r,\varphi,\beta$; $q = \rho,\psi,\vartheta$ are the displacements components of elastic Green's tensor, i.e. the displacement at the point M in the direction s due to the unit force applied at the point N in the direction q. They satisfy the following Lame's equations:

$$\begin{gathered}\mu\left[\nabla^2 U_r^{(q)} - \tfrac{2}{r^2}\left(U_r^{(q)} + \tfrac{1}{\sin\beta}\tfrac{\partial}{\partial\beta}\left(U_\beta^{(q)}\sin\beta\right) + \tfrac{1}{\sin\beta}\tfrac{\partial U_\varphi^{(q)}}{\partial\varphi}\right)\right] + \delta_{rq}\delta(M-N) = 0;\\ \mu\left[\nabla^2 U_\beta^{(q)} - \tfrac{2}{r^2}\left(\tfrac{\partial U_r^{(q)}}{\partial\beta} - \tfrac{U_\beta^{(q)}}{2\sin^2\beta} - \tfrac{\cos\beta}{\sin^2\beta(q)}\tfrac{\partial U_\varphi^{(q)}}{\partial\varphi}\right)\right] + \delta_{\beta q}\delta(M-N) = 0\,;\\ \mu\left[\nabla^2 U_\varphi^{(q)} + \tfrac{2}{r^2\sin\beta}\left(\tfrac{\partial U_\varphi^{(q)}}{\partial\varphi} + ctg\beta\tfrac{\partial U_\beta^{(q)}}{\partial\varphi} - \tfrac{U_\varphi^{(q)}}{r\sin\beta}\right)\right] + \delta_{\varphi q}\delta(M-N) = 0\,,\end{gathered} \tag{9}$$

subject to suitable homogeneous mechanical boundary conditions. In Eqs (6)-(8), $M,N\in V$, $M\equiv(r,\varphi,\beta)$; $N\equiv(\rho,\psi,\vartheta)$; the expression $\delta(M-N)$ is the Dirac's function and δ_{sq} is the Kronecker's symbol. The Green's function G in Eq. (6) has to be determined from the following heat conduction equation:

$$\nabla_M^2 G(M,N) = -\delta(M-N)\ ;\quad M,N\in V,\ M\equiv(r,\varphi,\beta)\ ;\ N\equiv(\rho,\psi,\vartheta)\ , \tag{10}$$

subject to the respective homogeneous boundary conditions. The considered influence functions in Eq. (6) have physical sense as displacements at an inner point of observation $M\equiv(r,\varphi,\beta)$, generated by a unit heat source, applied at an inner point $N\equiv(\rho,\psi,\vartheta)$ and described by the δ-Dirac's function. According to Eq. (6) they are determined by a convolution of two influence functions over the body V. The first influence function is the Green's function G for the BVP in heat conduction. The second function is the influence functions $\Theta^{(q)}(M,N)$ for BVP in elasticity.

Finally, the influence functions are functions of double influence [11-13], which take in consideration both physical phenomena (heat conduction and elasticity) in a solid body:

(1) *over the coordinates of the point of observation $M\equiv(r,\varphi,\beta)$ for thermoelastic displacements, they satisfy the equations of the BVP for determining Green's functions in the theory of heat conduction* (9)*, in which the unit heat source is replaced by $\gamma\Theta^{(q)}(M,N)$*

$$\nabla_M^2 U_q(M,N) = -\gamma\,\Theta^{(q)}(M,N)\,;\quad M,N\in V,\ M\equiv(r,\varphi,\beta)\,;\ N\equiv(\rho,\psi,\vartheta) \tag{11}$$

and suitable boundary conditions are imposed on $U_k(x,\xi)$;

(2) *over the coordinates of the point of application $N\equiv(\rho,\psi,\gamma)$ of the unit point heat source, they satisfy the equations of BVP for determining components of the Green's matrix* (8)*, in which the unit concentrated body forces are replaced by the derivatives of Green's functions for the heat conduction problem and $U_s^{(q)}(M,N)$ are replaced by $U_q(M,N)$:*

$$\begin{gathered}\mu\left[\nabla^2 U_\rho - \tfrac{2}{\rho^2}\left(U_\rho + \tfrac{1}{\sin\vartheta}\tfrac{\partial}{\partial\vartheta}(U_\vartheta\sin\vartheta) + \tfrac{1}{\sin\vartheta}\tfrac{\partial U_\psi}{\partial\psi}\right)\right] + (\lambda+\mu)\tfrac{\partial\Theta}{\partial\rho} + \gamma\tfrac{\partial G}{\partial\rho} = 0\,;\\ \mu\left[\nabla^2 U_\vartheta - \tfrac{2}{\rho^2}\left(\tfrac{\partial U_\rho}{\partial\vartheta} - \tfrac{U_\vartheta}{2\sin^2\vartheta} - \tfrac{\cos\vartheta}{\sin^2\vartheta}\tfrac{\partial U_\psi}{\partial\psi}\right)\right] + \tfrac{\lambda+\mu}{\rho}\cdot\tfrac{\partial\Theta}{\partial\vartheta} + \tfrac{\gamma}{\rho}\cdot\tfrac{\partial G}{\partial\vartheta} = 0\,;\\ \mu\left[\nabla^2 U_\psi + \tfrac{2}{\rho^2\sin\vartheta}\left(\tfrac{\partial U_\rho}{\partial\psi} + ctg\vartheta\tfrac{\partial U_\vartheta}{\partial\psi} - \tfrac{U_\psi}{\rho\sin\vartheta}\right)\right] + \tfrac{\lambda+\mu}{\rho\sin\vartheta}\cdot\tfrac{\partial\Theta}{\partial\psi} + \tfrac{\gamma}{\rho\sin\vartheta}\cdot\tfrac{\partial G}{\partial\psi} = 0\,,\end{gathered} \tag{12}$$

with respective homogeneous mechanical boundary conditions. In Eq. (11) Θ is the thermoelastic volume dilatation created by themperature G. Take note that all influence functions in Eq. (4) are determined on the boundary independently or using the respective limits from the main influence functions $U_k(x,\xi)$:

a) formula for influence functions corresponding to a unit point heat flux

$$a\left[\partial T\left(\tilde{M}\right)/\partial n_{\tilde{M}}\right]=\delta\left(\tilde{M}-N\right)$$

on the surface Γ_N and representing the thermoelastic displacements

$$\begin{gathered}U_q\left(\tilde{M},N\right)=\lim_{M\to\tilde{M}}U_q\left(M,N\right)=\\=\lim_{M\to\tilde{M}}\gamma\int\limits_V G\left(M,\bar{N}\right)\Theta^{(q)}\left(\bar{N},N\right)dV\left(\bar{N}\right)\ ;\quad M,N,\bar{N}\in V;\quad\tilde{M}\in\Gamma_N;\end{gathered}\tag{13}$$

b) formula for influence functions corresponding to a unit point temperature $T\left(\tilde{M}\right)=\delta\left(\tilde{M}-N\right)$ on the surface Γ_D and representing the thermoelastic displacements

$$\begin{gathered}\frac{\partial U_q\left(\tilde{M},N\right)}{\partial n_{\tilde{M}}}=\lim_{M\to\tilde{M}}\frac{\partial U_q\left(M,N\right)}{\partial n_M}=\\=\gamma\int\limits_V\frac{\partial G\left(\tilde{M},\bar{N}\right)}{\partial n_{\tilde{M}}}\Theta^{(q)}\left(\bar{N},N\right)dV\left(\bar{N}\right);\ M,N,\bar{N}\in V,\ \tilde{M}\in\Gamma_D\end{gathered}\tag{14}$$

c) formula for influence functions corresponding to a unit point heat exchange of the body with exterior medium $\alpha T\left(\tilde{M},\tau\right)+a\left[\partial T\left(\tilde{M}\right)/\partial n_{\tilde{M}}\right]=\delta\left(\tilde{M}-N\right)$ through the surface Γ_M and representing the thermoelastic displacements

$$\begin{gathered}U_q\left(\tilde{M},N\right)=\lim_{M\to\tilde{M}}U_q\left(M,N\right)=\\=\lim_{M\to\tilde{M}}\int\limits_V G\left(M,\bar{N}\right)\Theta^{(q)}\left(\bar{N},N\right)dV\left(\bar{N}\right);\quad M,N,\bar{N}\in V;\quad\tilde{M}\in\Gamma_M\,.\end{gathered}\tag{15}$$

The formula in Eq. (4) can be treated also as a generalization of the Maysel's formula [4-6] for those cases when the temperature field satisfies the BVP of heat conduction. Temperature field in this case is caused by the inner heat source and by the prescribed on the boundary temperature, heat flux or certain law of heat exchange between exterior medium and surface of the body. The advantage of the proposed integral formula in Eq. (4) is that it allows us to unite the two-staged process of solving the BVP in the theory of thermoelasticity (the first stage comprises finding temperature fields and the second stage comprises finding thermoelastic displacements) into one single stage. Also, the advantage of the integral formula in Eq (4) in comparison with the well-known Maysel's integral formula is that the thermoelastic displacements are determined directly via given heat actions. Besides, for any particular type of BVP we can obtain all possible solutions for different laws describing the above mentioned heat actions. The main difficulties for practical realization of the integral formula in Eqs (4) and (5) are to derive the functions of influence of a unit concentrated forces on elastic volume dilatation $\Theta^{(q)}$ and the Green's functions in heat conduction G. In addition we need to compute some volume integrals of the product of mentioned above functions. These difficulties, especially deriving the functions $\Theta^{(q)}$, were overcome successfully for Cartesian canonical domains

[11]. For cylindrical and spherical domains only general integral representations for $\Theta^{(k)}$ and Green's matrices were proposed in [16, 17].

2. Deriving The Green's Functions And Integral Formula For Thermoelastic Spherical Semi-Wedges

In this section we give a theorem for determining the thermoelastic displacements for spherical semi-wedges in the form of volume and surface integrals, which is a particular case of the general integral formula in Eq. (4). To do this, first, on the basis of the theory described above we construct the functions of influence of the inner unit point heat source on the thermoelastic displacements $U_k(x,\xi)$. At the second step we have to calculate (on the basis of the main influence functions $U_k(x,\xi)$ the other influence functions $U_q\left(\tilde{M},N\right)$, $\partial U_q\left(\tilde{M},N\right)/\partial n_{\varphi\alpha}$- on the boundary quarter-planes $\Gamma_{\varphi 0}$, $\Gamma_{\varphi\alpha}$ of the spherical semi-wedges; and to write the Green-type integral formula for respective BVP of thermoelasticity. At the last step it is necessary to show that the obtained influence functions and Green's type integral formulas satisfy the respective BVPs.

Theorem 1. *Let the field of displacements $u_q(N)$ at inner points $N \equiv (\rho,\psi,\vartheta)$ of the thermoelastic spherical semi-wedges $V\,(0 \le r < \infty;\ 0 \le \varphi \le \alpha;\ (\pi/2) \le \beta \le \pi)$; $\alpha = \pi/n$; $n = 2,2,3,\ldots$, be determined by non-homogeneous Lame's equations* (1). *Let in the points $M \equiv (r,0,\beta) \in \Gamma_{\varphi 0}$, $M \equiv (r,\alpha,\beta) \in \Gamma_{\varphi\alpha}$ and $M \equiv (r,\varphi,\pi/2) \in \Gamma_{\beta(\pi/2)}$ of the boundary quarter - planes $\Gamma_{\varphi 0}\,(0 \le r < \infty;\ \varphi = 0;\ 0 \le \beta \le \pi/2)$, $\Gamma_{\varphi\alpha}(0 \le r < \infty;\ \varphi = \alpha;\ 0 \le \beta \le \pi/2)$ and of the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}\,(\,0 \le r < \infty;\ 0 \le \varphi \le \pi;\ \beta = \pi/2)$, the following homogeneous mechanical conditions are given:*
a) locally mixed boundary conditions on the boundary quarter-plane $\Gamma_{\varphi 0}$:

$$u_r(r,0,\beta) = 0;\ \sigma_{\varphi\varphi}(r,0,\beta) = 0;\ u_\beta(r,0,\beta) = 0;\ \varphi = 0;\ 0 \le \beta \le \pi/2;\ 0 \le r < \infty \quad (16)$$

b) locally mixed boundary conditions on the boundary quarter-plane $\Gamma_{\varphi\alpha}$:

$$\sigma_{\varphi\, r}(r,\alpha,\beta) = 0;\ u_\varphi(r,\alpha,\beta) = 0;\ \sigma_{\varphi\beta}(r,\alpha,\beta) =;\ \varphi = \alpha;\ 0 \le \beta \le \pi/2;\ 0 \le r < \infty \quad (17)$$

and displacements boundary conditions on the boundary circular infinite sector $\Gamma_{\beta(\pi/2)}$:

$$\begin{aligned} &u_r(r,\phi,\pi/2) = 0;\ \sigma_{\beta\phi}(r,\phi,\pi/2) = 0; \\ &\sigma_{\beta\beta}(r,\phi,\pi/2) = 0;\ 0 \leqslant r < \infty;\ 0 \leqslant \phi \leqslant \alpha;\ \beta = \pi/2 \end{aligned} \quad (18)$$

where $\sigma_{\varphi r} = \sigma_{r\varphi}$, $\sigma_{\varphi\beta} = \sigma_{\beta\varphi r}$ and $\sigma_{\varphi\varphi}$, $\sigma_{\beta\beta}$ are the tangential and normal stresses, which are determined by the well-known Duhamel-Neumann law in Eq. (6).

Let also the temperature field $T(M)$ in Eq. (1)*, generated by the inner heat source $F(M)$, heat flux $S_{\varphi\alpha}(r,\alpha,\beta)$ (Neumann's boundary condition) and temperature $T_{\varphi 0}(r,0,\beta)$, $T_{\beta(\pi/2)}(r,\varphi,\pi/2)$ (Dirichlet'sboundary condition) satisfy the following BVP of heat conduction*

$$\begin{gathered} \nabla^2 T(M) = -a^{-1}F(M)\,,\ M \equiv (r,\varphi,\beta) \in V; \\ T(r,0,\beta) = f_{\varphi 0}(r,0,\beta)\,;\varphi = 0;\ 0 \le \beta \le \pi/2;\ 0 \le r < \infty; \\ \partial T(r,\alpha,\beta)/\partial n_{\Gamma_{\varphi\alpha}} = a^{-1}S_{\varphi\alpha}(r,\alpha,\beta)\,;\ \varphi = \alpha; 0 \le \beta \le \pi/2;\ 0 \le r < \infty; \\ T(r,\varphi,\pi/2) = f_{\beta(\pi/2)}(r,\varphi,\pi/2)\,;\ \beta = \pi/2;\ 0 \le \varphi \le \pi;\ 0 \le r < \infty. \end{gathered} \quad (19)$$

If the inner heat source, boundary heat flux and temperature satisfy the conditions:

$$\begin{gathered} \textstyle\int_0^\infty \int_0^\alpha \int_0^{\frac{\pi}{2}} |F(r,\varphi,\beta)|\, r^2 dr\, d\varphi\, d\beta < \infty\,; \quad \int_0^\infty \int_0^{\frac{\pi}{2}} |S_{\varphi\alpha}(r,\alpha,\beta)|\, r\, dr\, d\beta < \infty\,; \\ \textstyle\int_0^\infty \int_0^{\frac{\pi}{2}} |f_{\varphi\, 0}(r,0,\beta)|\, r\, dr\, d\beta < \infty\,;\ \int_0^\infty \int_0^\alpha \left|f_{\beta(\pi/2)}(r,\varphi,\pi/2)\right| r\, dr\, d\varphi < \infty, \end{gathered} \quad (20)$$

then the solution of the BVP in Eqs. (1) *and* (16) – (20) *of thermoelasticity for searched displacements* $u_q(\mathrm{N})$ *for the considered spherical semi-wedges exists and it can be presented by the following Green's type integral formula, written in the matrix form:*

$$u(N) = \frac{1}{a}\left[\int_0^\infty\int_0^\alpha\int_0^{\pi/2} F(r,\phi,\beta)\,U(r,\phi,\beta;N)\,r^2\,dr\,d\phi\,d\beta + \int_0^\infty\int_0^{\pi/2} S_{\phi\,\alpha}(r,\alpha,\beta)\,Q_{\phi\,\alpha}(r,\alpha,\beta;N)\,r\,dr\,d\beta\right] - \\ -\int_0^\infty\int_0^{\pi/2} f_{\phi\,0}(r,0,\beta)\,Q_{\phi\,0}(r,0,\beta;N)\,r\,dr\,d\beta - \\ \int_0^\infty\int_0^\alpha f_{\beta(\pi/2)}(r,\phi,\pi/2)\,Q_{\beta(\pi/2)}(r,\phi,\pi/2;N)\,r\,dr\,d\phi;\ N\equiv(\rho,\psi,\vartheta), \tag{21}$$

where $|u(N)| < \infty$ *everywhere and vanishes at infinity* $\lim\limits_{\rho\to\infty} u_q(N) \to 0$. *The matrices of influence of an inner unit point heat source, of a unit point heat flux on the boundary quarter-plane* $\Gamma_{\varphi\alpha}$ *and of the unit point temperature on the boundaries* $\Gamma_{\varphi 0}$, $\Gamma_{\beta(\pi/2)}$ *onto thermoelastic displacements:* $U(M,\mathrm{N})$, $Q_{\varphi\alpha}(r,\alpha,\beta;N) = U(r,\alpha,\beta;N)$, $Q_{\varphi 0}(r,0,\beta;N) = U(r,0,\beta;N)/\partial n_{\varphi 0}$, $Q_{\beta(\pi/2)}(r,\varphi,\pi/2;N) = U(r,\varphi,\pi/2;N)/\partial n_{\beta(\pi/2)}$, *also the matrix of searched displacements* $u(N)$ *in Eq.* (21), *are determined as follows:*

(1) The matrix $U(M,\mathrm{N})$

$$U(M,N) = \begin{pmatrix} U_\rho \\ U_\psi \\ U_\vartheta \end{pmatrix} = \\ = -m\sum_{k=0}^{n-1}(-1)^k \begin{pmatrix} f_k R_k^{-1} - f_{k\psi}R_{k\psi}^{-1} - f_{k\vartheta}R_{k\vartheta}^{-1} + f_{k\psi\vartheta}R_{k\psi\vartheta}^{-1} \\ \phi_k\left(R_{k\vartheta}^{-1} - R_k^{-1}\right) - \phi_{k\psi}\left(R_{k\psi\vartheta}^{-1} - R_{k\psi}^{-1}\right) \\ \eta_k R_k^{-1} - \eta_{k\psi}R_{k\psi}^{-1} - \eta_{k\vartheta}R_k^{-1} + \eta_{k\psi\vartheta}R_{k\psi}^{-1} \end{pmatrix}; \\ m = \gamma[8\pi(\lambda+2\mu)]^{-1} \tag{22}$$

where

$$R_k = \sqrt{r^2+\rho^2-2r\rho\cos(\phi-\omega_k)};\ R_{k\psi} = \sqrt{r^2+\rho^2-2r\rho\cos(\phi+\omega_k)}; \\ R_{k\vartheta} = \sqrt{r^2+\rho^2-2r\rho\cos(\phi-\omega_{k\vartheta})};\ R_{k\psi\vartheta} = \sqrt{r^2+\rho^2-2r\rho\cos(\phi+\omega_{k\vartheta})}; \\ \cos(\phi-\omega_k) = \sin\beta\sin\vartheta\cos(\varphi-\psi-2k\pi/n) + \cos\beta\cos\vartheta; \\ \cos(\phi+\omega_k) = \sin\beta\sin\vartheta\cos(\varphi+\psi-2k\pi/n) + \cos\beta\cos\vartheta; \\ \cos(\phi-\omega_{k\vartheta}) = \sin\beta\sin\vartheta\cos(\varphi-\psi-2k\pi/n) - \cos\beta\cos\vartheta; \\ \cos(\phi+\omega_{k\vartheta}) = \sin\beta\sin\vartheta\cos(\varphi+\psi-2k\pi/n) - \cos\beta\cos\vartheta; \tag{23}$$

and

$$\begin{array}{c} f_k = \rho - rcos\left(\phi - \omega_k\right);\ f_{k\psi} = \rho - rcos\left(\phi + \omega_k\right); \\ f_{k\vartheta} = \rho - rcos\left(\phi - \omega_{k\vartheta}\right);\ f_{k\psi\vartheta} = \rho - rcos\left(\phi + \omega_{k\vartheta}\right); \\ \varphi_k = r\sin\beta sin\left(\varphi - \psi - (2k\pi/n)\right); \varphi_{k\psi} = r\sin\beta sin\left(\varphi + \psi - (2k\pi/n)\right); \\ \eta_k = -r\left(\sin\beta\cos\vartheta\cos\left(\varphi - \psi - 2k\pi/n\right) - \cos\beta\sin\vartheta\right); \\ \eta_{k\vartheta} = -r\left(\sin\beta\cos\vartheta\cos\left(\varphi - \psi - 2k\pi/n\right) + \cos\beta\sin\vartheta\right); \\ \eta_{k\psi} = -r\left(\sin\beta\cos\vartheta\cos\left(\varphi + \psi - 2k\pi/n\right) - \cos\beta\sin\vartheta\right); \\ \eta_{k\psi\vartheta} = -r\left(\sin\beta\cos\vartheta\cos\left(\varphi + \psi - 2k\pi/n\right) + \cos\beta\sin\vartheta\right); \end{array} \tag{24}$$

(2) The matrix $Q_{\varphi 0}(r,0,\beta;N)$ on the boundary quarter-plane $\Gamma_{\varphi 0}$

$$Q_{\phi\,0}(r,0,\beta;N) = \begin{pmatrix} Q_{\phi\,0\rho} \\ Q_{\phi\,0\psi} \\ Q_{\phi\,0\vartheta} \end{pmatrix} = \tag{25}$$
$$= 2m\sum_{k=0}^{n-1}(-1)^k \begin{pmatrix} \left(R_{k0}^{-1} - R_{k0}^{-3}\rho f_{k0} - R_{k0\vartheta}^{-1} + R_{k0\vartheta}^{-3}\rho f_{k0\vartheta}\right)\phi_{k0}\sin\vartheta \\ \left(R_{k0}^{-1} - R_{k0\vartheta}^{-1}\right)s_{k0} - \left(R_{k0}^{-3} - R_{k0\vartheta}^{-3}\right)\rho\phi_{k0}^2\sin\vartheta \\ \left(\left(R_{k0}^{-1} - R_{k0\vartheta}^{-1}\right)\cos\vartheta - \left(R_{k0}^{-3}\eta_{k0} - R_{k0\vartheta}^{-3}\eta_{k0\vartheta}\right)\rho\sin\vartheta\right)\phi_{k0} \end{pmatrix}$$

where

$$\begin{array}{c} R_{k0} = \sqrt{r^2 + \rho^2 - 2r\rho\cos cos\left(\varphi - \omega_{k0}\right)}; \\ R_{k0\vartheta} = \sqrt{r^2 + \rho^2 - 2r\rho\cos cos\left(\varphi - \omega_{k0\vartheta}\right)}; \\ \cos\left(\varphi - \omega_{k0}\right) = \sin\beta\sin\vartheta\cos\left(\psi + 2k\pi/n\right) + \cos\beta\cos\vartheta; \\ \cos\left(\varphi - \omega_{k0\vartheta}\right) = \sin\beta\sin\vartheta\cos\left(\psi + 2k\pi/n\right) - \cos\beta\cos\vartheta; \\ f_{k0} = \rho - rcos\left(\varphi - \omega_{k0}\right);\ f_{k0\vartheta} = \rho - rcos\left(\varphi - \omega_{k0\vartheta}\right); \\ \phi_{k0} = r\sin\beta sin\left(\psi + (2k\pi/n)\right); \\ s_{k0} = r\sin\beta\cos\left(\psi + 2k\pi/n\right) \\ \eta_{k0} = -r\left(\sin\beta\cos\vartheta\cos\left(\psi + 2k\pi/n\right) - \cos\beta\sin\vartheta\right)\ ; \\ \eta_{k0\vartheta} = -r\left(\sin\beta\cos\vartheta\cos\left(\psi + 2k\pi/n\right) + \cos\beta\sin\vartheta\right)\ ; \end{array} \tag{26}$$

c)The matrix $Q_{\varphi\alpha}(r,\alpha,\beta;N)$ on the boundary quarter-plane $\Gamma_{\varphi\alpha}$

$$Q_{\varphi\alpha}(r,\alpha,\beta;N) = \begin{pmatrix} Q_{\varphi\alpha\rho} \\ Q_{\varphi\alpha\psi} \\ Q_{\varphi\alpha\vartheta} \end{pmatrix} = -2m\sum_{k=0}^{n-1}(-1)^k \begin{pmatrix} f_{k\alpha}R_{k\alpha}^{-1} - f_{k\alpha\vartheta}R_{k\alpha\vartheta}^{-1} \\ \varphi_{k\alpha}\left(R_{k\alpha\vartheta}^{-1} - R_{k\alpha}^{-1}\right) \\ \eta_{k\alpha}R_{k\alpha}^{-1} - \eta_{k\alpha\vartheta}R_{k\alpha\vartheta}^{-1} \end{pmatrix}, \tag{27}$$

where

$$\begin{array}{c} R_{k\alpha} = \sqrt{r^2 + \rho^2 - 2r\rho\cos\left(\phi - \omega_{k\alpha}\right)}; R_{k\alpha\vartheta} = \sqrt{r^2 + \rho^2 - 2r\rho\cos\left(\phi - \omega_{k\alpha\vartheta}\right)}; \\ \cos\left(\phi - \omega_{k\alpha}\right) = \sin\beta\sin\vartheta\cos\left(\alpha - \psi - 2k\pi/n\right) + \cos\beta\cos\vartheta; \\ \cos\left(\phi - \omega_{k\alpha\vartheta}\right) = \sin\beta\sin\vartheta\cos\left(\alpha - \psi - 2k\pi/n\right) - \cos\beta\cos\vartheta; \end{array} \tag{28}$$

and

$$\begin{array}{c} f_{k\alpha} = \rho - r\cos\left(\varphi - \omega_{k\alpha}\right); f_{k\alpha\vartheta} = \rho - r\cos\left(\varphi - \omega_{k\alpha\vartheta}\right), \\ \phi_{k\alpha} = r\sin\beta\sin\left(\alpha - \psi - (2k\pi/n)\right); \\ \eta_{k\alpha} = -r\left(\sin\beta\cos\vartheta\cos\left(\alpha - \psi - 2k\pi/n\right) - \cos\beta\sin\vartheta\right); \\ \eta_{k\alpha\vartheta} = -r\left(\sin\beta\cos\vartheta\cos\left(\alpha - \psi - 2k\pi/n\right) + \cos\beta\sin\vartheta\right). \end{array} \tag{29}$$

d) The matrix $Q_{\beta(\pi/2)}(r,\varphi,\pi/2;N)$ on the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}$

$$Q_{\beta\,(\pi/2)}(r,\phi,\pi/2;N)=\begin{pmatrix}Q_{\beta\,(\pi/2)\rho}\\ Q_{\beta\,(\pi/2)\psi}\\ Q_{\beta\,(\pi/2)\vartheta}\end{pmatrix}=-2m\sum_{k=0}^{n-1}(-1)^k\times$$

$$\begin{pmatrix}\left(R^{-1}_{k(\pi/2)}-R^{-3}_{k(\pi/2)}\rho f_{k(\pi/2)}-R^{-1}_{k(\pi/2)\psi}+R^{-3}_{k(\pi/2)\psi}\rho f_{k(\pi/2)\psi}\right)r\cos\vartheta\\ \left(\cos\left(\phi-\psi-2k\pi/n\right)R^{-3}_{k(\pi/2)}-\cos\left(\phi+\psi-2k\pi/n\right)R^{-3}_{k(\pi/2)\psi}\right)r^2\rho\cos\vartheta\\ \left[\left(R^{-1}_{k(\pi/2)\psi}-R^{-1}_{k(\pi/2)}\right)\sin\vartheta+\rho\cos\vartheta\left(R^{-3}_{k(\pi/2)\psi}\eta_{k(\pi/2)\psi}-R^{-3}_{k(\pi/2)}\eta_{k(\pi/2)}\right)\right]r\end{pmatrix}$$

$$\begin{gathered}R_{k(\pi/2)}=\sqrt{r^2+\rho^2-2r\rho\cos\left(\varphi-\omega_{k(\pi/2)}\right)};\\ R_{k(\pi/2)}=\sqrt{r^2+\rho^2-2r\rho\cos\left(\varphi+\omega_{k(\pi/2)}\right)};\\ \cos\left(\varphi-\omega_{k(\pi/2)}\right)=\sin\vartheta\cos\left(\phi-\psi-2k\pi/n\right);\\ \cos\left(\varphi+\omega_{k(\pi/2)}\right)=\sin\vartheta\cos\left(\phi+\psi-2k\pi/n\right);\\ f_{k(\pi/2)}=\rho-r\cos\left(\varphi-\omega_{k(\pi/2)}\right);f_{k(\pi/2)\psi}=\rho-r\cos\left(\varphi+\omega_{k(\pi/2)}\right);\\ \eta_{k(\pi/2)\psi}=-r\left(\sin\beta\cos\vartheta\cos\left(\alpha-\psi-2k\pi/n\right)-\cos\beta\sin\vartheta\right);\\ \eta_{k(\pi/2)}=-r\cos\vartheta\cos\left(\phi-\psi-2k\pi/n\right);\\ \eta_{k(\pi/2)\psi}=-r\cos\vartheta\cos\left(\phi+\psi-2k\pi/n\right).\end{gathered}\tag{30}$$

Proof. First, the well-known Green's function G for Poisson's equation for a spherical semi-wedges we derive, using image method. Next, we derive the volume dilatation $\Theta^{(q)}(M,N)$ (in the subsection 2.1). In subsection 2.2 it is shown how to derive thermoelastic influence functions $U_q(M,N)$. Finally, in subsection 2.3 on the base of the functions $U_q(M,N)$ the Green-type integral formula for stated BVP of thermoelasticity is derived.

To obtain the matrix $U(M,\mathrm{N})$ in Eq. (22) for the BVP in Eqs. (1) and (16) - (20) we use the integral formula in Eq. (7). The functions $G(M,N)$ and $\Theta^{(q)}(M,N)$ in this equation are the Green's functions of mixed problem in heat conduction and respectively the influence functions of a unit concentrated body force $\delta_{ql}\delta(M-N)$ onto volume dilatation in theory of elasticity for the spherical semi-wedges V. So, to get the Green's function $G(M,N)$ we have to solve the BVP, which consists of the heat conduction equation with the homogeneous boundary conditions similar to those in Eq. (19):

$$\begin{gathered}\nabla^2_M G(M,N)=-\delta(M-N);\ M,N\in V,\ M\equiv(r,\varphi,\beta);\ N\equiv(\rho,\psi,\vartheta);\\ G=0;\ \varphi=0;\ 0\le\beta\le\pi/2;\ 0\le r<\infty;\\ \partial G/\partial\varphi=0;\ \varphi=\alpha;\ 0\le\beta\le\pi/2;\ 0\le r<\infty;\\ G=0;\ \beta=\pi/2;\ 0\le\varphi\le\pi;\ 0\le r<\infty.\end{gathered}\tag{31}$$

Using the image method [18, 19] the Green's function for BVP (31) can be derived exactly in elementary functions:

$$G=\frac{1}{4\pi}\sum_{k=0}^{n-1}(-1)^k\left(R_k^{-1}+R_{k\psi}^{-1}-R_{k\vartheta}^{-1}-R_{k\psi\vartheta}^{-1}\right),\tag{32}$$

where R_k, $R_{k\psi}$, $R_{k\vartheta}$ and $R_{k\psi\vartheta}$ are determined by equation (23). □

2.1. **Deriving the Volume Dilatation** $\Theta^{(q)}(M,N)$**.** To get the influence functions $\Theta^{(k)}(x,\xi)$, usually, we have to solve the following BVP, which consists of fundamental Lame's equations in theory of elasticity (9) and homogeneous boundary conditions similar to those in Eqs (16)-(18):

$$\begin{gathered} U_r^{(q)}(r,0,\beta)=\sigma_{\varphi\varphi}^{(q)}(r,0,\beta)=U_\beta^{(q)}(r,0,\beta)=0;\ (r,0,\beta)\in\Gamma_{\varphi 0}; \\ \sigma_{\varphi r}^{(q)}(r,\alpha,\beta)=U_\varphi^{(q)}(r,\alpha,\beta)=\sigma_{\varphi\beta}^{(q)}(r,\alpha,\beta)=0;\ (r,\alpha,\beta)\in\Gamma_{\varphi\alpha}; \\ U_r^{(q)}(r,\varphi,\pi/2)=\sigma_{\beta\varphi}^{(q)}(r,\varphi,\pi/2)=\sigma_{\beta\beta}^{(q)}(r,\varphi,\pi/2)=0;\ (r,\varphi,\beta)\in\Gamma_{\beta(\pi/2)} \end{gathered} \tag{33}$$

and then, on the base of displacements $U_s^{(q)}(M,N);\ s=r,\varphi,\beta;\ q=\rho,\psi,\vartheta$, and the rule in Eq. (8) to compute volume dilatation. In equation (33) the stresses are determined by the Hooke's law:

$$\begin{gathered} \sigma_{sp}^{(q)}=2\mu\varepsilon_{sp}^{(q)}+\delta_{sp}\lambda\Theta^{(q)};\ s,p=r,\varphi,\beta; \\ \varepsilon_{rr}^{(q)}=\tfrac{\partial U_r^{(q)}}{\partial r};\ \varepsilon_{\varphi\varphi}^{(q)}=\tfrac{1}{r\sin\beta}\tfrac{\partial U_\varphi^{(q)}}{\partial\varphi}+\tfrac{U_r^{(q)}}{r}+\operatorname{ctg}\beta\tfrac{U_\beta^{(q)}}{r};\ \varepsilon_{\beta\beta}^{(q)}=\tfrac{1}{r}\tfrac{\partial U_\beta^{(q)}}{\partial\beta}+\tfrac{U_r^{(q)}}{r}; \\ \varepsilon_{r\varphi}^{(q)}=\tfrac{1}{2}\left(\tfrac{1}{r\sin\beta}\tfrac{\partial U_r^{(q)}}{\partial\varphi}-\tfrac{U_\varphi^{(q)}}{r}+\tfrac{\partial U_\varphi^{(q)}}{\partial r}\right); \\ \varepsilon_{r\beta}^{(q)}=\tfrac{1}{2}\left(\tfrac{1}{r}\tfrac{\partial U_r^{(q)}}{\partial\beta}-\tfrac{U_\beta^{(q)}}{r}+\tfrac{\partial U_\beta^{(q)}}{\partial r}\right);\varepsilon_{\varphi\beta}^{(q)}=\tfrac{1}{2}\left(\tfrac{1}{r\sin\beta}\tfrac{\partial U_\beta^{(q)}}{\partial\varphi}-\tfrac{U_\varphi^{(q)}}{r}\operatorname{ctg}\beta+\tfrac{\partial U_\varphi^{(q)}}{r\partial\beta}\right). \end{gathered} \tag{34}$$

As it is shown below, in the case of the boundary conditions in Eqs (33), we can derive the volume dilatation $\Theta^{(q)}(M,N)$ using the equation

$$\begin{gathered} \nabla^2\Theta^{(q)}(M,N)=-(\lambda+2\mu)^{-1}L_M^{(q)}\delta(M-N); \\ M,N\in V,\ M\equiv(r,\phi,\beta);\ N\equiv(\rho,\psi,\vartheta); \\ L_M^{(q)}=\delta_{qr}\frac{\partial}{\partial r}+\frac{\delta_{q\phi}}{r\sin\beta}\frac{\partial}{\partial\phi}+\frac{\delta_{q\beta}}{r}\frac{\partial}{\partial\beta} \end{gathered} \tag{35}$$

and its integral representation via respective Green's function $G_\Theta(M,N)$:

$$\begin{gathered} \Theta^{(q)}(M,N)=-\frac{1}{\lambda+2\mu}L_N^{(q)}G_\Theta(M,N)+ \\ +\int\limits_\Gamma\left[\frac{\partial}{\partial n_{\bar M}}\Theta^{(q)}\left(\bar M,N\right)-\Theta^{(q)}\left(\bar M,N\right)\frac{\partial}{\partial n_{\bar M}}\right]G_\Theta\left(\bar M,M\right)d\Gamma\left(\bar M\right); \\ L_N^{(q)}=\delta_{q\rho}\frac{\partial}{\partial\rho}+\frac{\delta_{q\psi}}{\rho\sin\vartheta}\frac{\partial}{\partial\psi}+\frac{\delta_{q\vartheta}}{\rho}\frac{\partial}{\partial\vartheta}. \end{gathered} \tag{36}$$

It is useful to be mentioned that Eq. (36) is obtained from Eqs. (9), applying rule (8). The Green's function $G_\Theta(M,N)$ in (36) satisfies the following equation:

$$\nabla_M^2G_\Theta(M,N)=-\delta(M-N);\ M,N\in V,\ M\equiv(r,\varphi,\beta);\ N\equiv(\rho,\psi,\vartheta). \tag{37}$$

To show this, first prove that the boundary conditions in Eq. (37) lead to the following values of the volume dilatation:

a. on the boundary plane $\Gamma_{\varphi 0}$ $(0\le r<\infty;\ \ \varphi=0;\ (\pi/2)\le\beta\le\pi)$:

$$\Theta^{(q)}=0 \tag{38}$$

and

b. on the boundary plane $\Gamma_{\varphi\alpha}$ $(0\le r<\infty;\ \ \varphi=\alpha;\ (\pi/2)\le\beta\le\pi)$:

$$\partial\Theta^{(q)}/\partial n_{\varphi\alpha}=0. \tag{39}$$

To prove equality in Eq (36) let us prove that the first boundary conditions in Eq. (33) lead to zero volume dilatation on the boundary plane $\Gamma_{\varphi 0}$, i.e., $\Theta^{(q)}=0$.

So, taking into account the Eq. (8) in the form

$$\begin{aligned}\varepsilon_{\phi\phi}^{(q)} &= \frac{1}{r\sin\beta}\frac{\partial U_{\phi}^{(q)}}{\partial\phi} + \frac{U_r^{(q)}}{r} + \operatorname{ctg}\beta\frac{U_{\beta}^{(q)}}{r} = \Theta^{(q)} - \varepsilon_{rr}^{(q)} - \varepsilon_{\beta\beta}^{(q)} = \\ &= \Theta^{(q)} - \left(\frac{\partial U_r^{(q)}}{\partial r} + \frac{1}{r}\frac{\partial U_{\beta}^{(q)}}{\partial\beta} + \frac{U_r^{(q)}}{r}\right)\end{aligned} \tag{40}$$

and using Hooke's law, as in Eq. (34), written for stresses $\sigma_{\varphi\varphi}^{(q)}$, we obtain

$$\sigma_{\varphi\varphi}^{(q)} = 2\mu\varepsilon_{\varphi\varphi}^{(q)} + \lambda\Theta^{(q)} = (\lambda + 2\mu)\,\Theta^{(q)} - 2\mu\left(\frac{\partial U_r^{(q)}}{\partial r} + \frac{1}{r}\frac{\partial U_{\beta}^{(q)}}{\partial\beta} + \frac{U_r^{(q)}}{r}\right). \tag{41}$$

Also by using (41) with the conditions $\sigma_{\varphi\varphi}^{(q)} = 0$; $U_r^{(q)} = 0$; $\partial U_r^{(q)}/\partial r = 0$; $\partial U_{\beta}^{(q)}/\partial\beta = 0$, which follow from the first boundary conditions (41), we find that on the quarter-plane $\Gamma_{\varphi 0}$ $(0 \le r < \infty;\ \ \varphi = 0;\ (\pi/2) \le \beta \le \pi)$ the volume dilatation is zero. So, the Eq. (38) is true. To prove equality in Eq. (39) let us prove that the second boundary conditions in Eq. (33) lead to zero normal derivative with respect to volume dilatation on the boundary quarter-plane $\Gamma_{\varphi\alpha}$, i.e., $\partial\Theta^{(q)}/\partial n_{\varphi 0} = 0$.

Having substituted the relationships $\sigma_{\varphi r}^{(q)} = 0;\ \Rightarrow\ \partial\sigma_{\varphi r}^{(q)}/\partial r = 0;\ \sigma_{\varphi\beta}^{(q)} = 0;\ \Rightarrow$ $\partial\sigma_{\varphi\beta}^{(q)}/\partial\beta = 0$, which follow from the second boundary conditions in Eq. (33), into the equation of equilibrium in terms of stresses

$$\frac{\partial\sigma_{r\varphi}}{\partial r} + \frac{1}{r\sin\beta}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi} + \frac{1}{r}\frac{\partial\sigma_{\varphi\beta}}{\partial\beta} + \frac{3\sigma_{r\varphi} + 2\sigma_{\varphi\beta}\operatorname{ctg}\beta}{r} = 0 \tag{42}$$

we obtain that on the quarter-plane $\Gamma_{\varphi\alpha}$, the relation $\partial\sigma_{\varphi\varphi}^{(q)}/\partial\varphi = 0$ holds. Using this result in Hooke's law written in the form

$$\frac{\partial\sigma_{\varphi\varphi}^{(q)}}{\partial\varphi} = (\lambda + 2\mu)\,\frac{\partial\Theta^{(q)}}{\partial\varphi} - 2\mu\frac{\partial}{\partial\varphi}\left(\frac{\partial U_r^{(q)}}{\partial r} + \frac{1}{r}\frac{\partial U_{\beta}^{(q)}}{\partial\beta} + \frac{U_r^{(q)}}{r}\right) = 0 \tag{43}$$

and the second boundary conditions in Eq. (33), as well as Hooke's law for the tangential stresses $\sigma_{\varphi r}^{(q)}$ and $\sigma_{\varphi\,\beta}^{(q)}$, we find that the following relations hold:

$$\partial U_r^{(q)}/\partial\varphi = 0;\ \partial^2 U_r^{(q)}/\partial r\partial\varphi = 0;\ \partial^2 U_{\beta}^{(q)}/\partial\beta\partial\varphi = 0. \tag{44}$$

Indeed, Eqs (42) can be obtained from the second boundary conditions (33) on the plane $\Gamma_{\varphi\alpha}$, using the following operations:

$$\begin{aligned}&U_{\phi}^{(q)} = 0\ \Rightarrow \frac{\partial U_{\phi}^{(q)}}{\partial r} = 0,\ \frac{\partial U_{\phi}^{(q)}}{\partial\beta} = 0\,;\\ &\sigma_{\phi\,r}^{(q)} = 0,\ U_{\phi}^{(q)} = 0, \frac{\partial U_{\phi}^{(q)}}{\partial r} = 0\ \Rightarrow\ \mu\left(\frac{1}{r\sin\beta}\frac{\partial U_r^{(q)}}{\partial\phi} - \frac{U_{\phi}^{(q)}}{r} + \frac{\partial U_{\phi}^{(q)}}{\partial r}\right) = 0;\\ &\Rightarrow \frac{\partial U_r^{(q)}}{\partial\phi} = 0,\ \frac{\partial^2 U_r^{(q)}}{\partial r\partial\phi} = 0\,;\\ &\sigma_{\phi\,\beta}^{(q)} = 0,\ U_{\phi}^{(q)} = 0, \frac{\partial U_{\phi}^{(q)}}{\partial\beta} = 0\ \Rightarrow\ \mu\left(\frac{1}{r\sin\beta}\frac{\partial U_{\beta}^{(q)}}{\partial\phi} - \frac{U_{\phi}^{(q)}}{r}ctg\beta + \frac{\partial U_{\phi}^{(q)}}{r\partial\beta}\right) = 0;\\ &\Rightarrow \frac{\partial U_{\beta}^{(q)}}{\partial\phi} = 0,\ \frac{\partial^2 U_{\beta}^{(q)}}{\partial\phi\partial\beta} = 0.\end{aligned} \tag{45}$$

Finally, by substituting the results (44) into (43) we find that on the quarter-plane $\Gamma_{\varphi\alpha}$, there is $\partial\Theta^{(q)}/\partial\varphi = 0$, i.e., $\partial\Theta^{(q)}/\partial n_{\varphi\alpha} = 0$. So, the Eq. (39) is true.

In analogical way, as in eqs (40) and (41), one can be proved that on the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}$ the volume dilatation is zero, $\Theta^{(q)} = 0$.

So, the BVP for $\Theta^{(q)}$ is described by Eqs (35), (37), (39) and (36), where the BVP for Green's function G_Θ for considered spherical semi-wedges can be written in the form:

$$\begin{gathered}\nabla^2_M G_\Theta\left(M,N\right) = -\delta\left(M-N\right);\ M,N\in V,\ M\equiv\left(r,\varphi,\beta\right);\ N\equiv\left(\rho,\psi,\vartheta\right);\\ G_\Theta = 0;\ 0\le r<\infty;\ \varphi=0;\ 0\le\beta\le\pi/2;\\ \partial G_\Theta/\partial\varphi = 0;\ 0\le r<\infty;\ \varphi=\alpha;\ 0\le\beta\le\pi/2;\\ G_\Theta = 0;\ 0\le r<\infty;\ 0\le\varphi\le\alpha;\ 0\le\beta\le\pi/2.\end{gathered}\tag{46}$$

The last BVP coincides with BVP in Eq. (31) so that the Green's function $G_\Theta = G$ is determined by the expressions in Eq. (32).

Finally, if we introduce the expressions (38), (39) and (32) in the representation (36), then we obtain the searched volume dilatation of the elastic BVP in Eqs (9) and (33) for spherical semi-wedges, written in the form:

$$\begin{gathered}\Theta^{(q)} = -\left[4\pi\left(\lambda+2\mu\right)\right]^{-1} L_N^{(q)}\textstyle\sum_{k=0}^{n-1}\left(-1\right)^{(k)}\left(R_k^{-1}+R_{k\psi}^{-1}-R_{k\vartheta}^{-1}-R_{k\psi\vartheta}^{-1}\right);\\ R_k=\sqrt{r^2+\rho^2-2r\rho\cos\left(\phi-\omega_k\right)};\ R_{k\psi}=\sqrt{r^2+\rho^2-2r\rho\cos\left(\phi+\omega_k\right)};\\ R_{k\vartheta}=\sqrt{r^2+\rho^2-2r\rho\cos\left(\phi-\omega_{k\vartheta}\right)};\ R_{k\psi\vartheta}=\sqrt{r^2+\rho^2-2r\rho\cos\left(\phi+\omega_{k\vartheta}\right)};\\ \cos\left(\phi-\omega_k\right)=\sin\beta\sin\vartheta\cos\left(\varphi-\psi-2k\pi/n\right)+\cos\beta\cos\vartheta;\\ \cos\left(\phi+\omega_k\right)=\sin\beta\sin\vartheta\cos\left(\varphi+\psi-2k\pi/n\right)+\cos\beta\cos\vartheta;\\ \cos\left(\phi-\omega_{k\vartheta}\right)=\sin\beta\sin\vartheta\cos\left(\varphi-\psi-2k\pi/n\right)-\cos\beta\cos\vartheta;\\ \cos\left(\phi+\omega_{k\vartheta}\right)=\sin\beta\sin\vartheta\cos\left(\varphi+\psi-2k\pi/n\right)-\cos\beta\cos\vartheta;\\ L_N^{(q)}=\delta_{q\rho}\left(\partial/\partial\rho\right)+\delta_{q\psi}\left(\rho\sin\vartheta\right)^{-1}\left(\partial/\partial\psi\right)+\delta_{q\vartheta}\rho^{-1}\left(\partial/\partial\vartheta\right).\end{gathered}\tag{47}$$

2.2. **Deriving the Thermoelastic Influence Functions $U_q\left(M,N\right)$.** Now we have both functions: $G\left(M,N\right)$ and $\Theta^{(q)}\left(M,N\right)$ needed for deriving the thermoelastic influence functions $U_q\left(M,N\right)$, using the equation (5). So, substituting functions $G\left(M,N\right)$ and $\Theta^{(q)}\left(M,N\right)$ from Eqs (36) and (47) in the equation (6), rewritten for spherical semi-wedges $V\left(0\le r<\infty;\ 0\le\varphi\le\alpha;\ (\pi/2)\le\beta\le\pi\right);\ \alpha=\pi/n;\ n=2,3,4\ldots,$

$$\begin{gathered}U_q\left(M,N\right)=\gamma\int\limits_0^\infty\int\limits_0^\alpha\int\limits_0^{\pi/2}G\left(M;\rho',\psi',\vartheta'\right)\Theta^{(q)}\left(\rho',\psi',\vartheta';N\right)\rho'^2d\rho'd\psi'd\vartheta'=\\ -(\lambda+2\mu)^{-1}\gamma L_N^{(q)}\sum_{k=0}^{n-1}\left(-1\right)^{(k)}\times\int\limits_0^\infty\int\limits_0^\alpha\int\limits_0^{\pi/2}\Big[\left(4\pi\right)^{-1}\left(R_k^{-1}\left(M;\rho',\psi',\vartheta'\right)-\right.\\ \left.R_{k\psi}^{-1}\left(M;\rho',\psi',\vartheta'\right)-R_{k\vartheta}^{-1}\left(M;\rho',\psi',\vartheta'\right)+R_{k\vartheta\psi}^{-1}\left(M;\rho',\psi',\vartheta'\right)\right)\times\\ \left(4\pi\right)^{-1}\left(R_k^{-1}\left(\rho',\psi',\vartheta';N\right)-R_{k\psi}^{-1}\left(\rho',\psi',\vartheta';N\right)-R_{k\vartheta}^{-1}\left(\rho',\psi',\vartheta';N\right)+\right.\\ \left.R_{k\psi\vartheta}^{-1}\left(\rho',\psi',\vartheta';N\right)\right)\Big]\,{\rho'}^2d\rho'd\psi'd\vartheta'\end{gathered}\tag{48}$$

then, calculating the volume integral in the special way (see Appendix A), we obtain the following expression for functions $U_q\left(M,N\right)$:

$$\begin{gathered}U_q\left(M,N\right)=-mL_N^{(q)}\times\sum_{k=0}^{n-1}\left(-1\right)^{(k)}\left[R_k\left(r,\phi,\beta;\rho,\psi,\vartheta\right)\right.\\ \left.-R_{k\psi}\left(r,\phi,\beta;\rho,\psi,\vartheta\right)-R_{k\vartheta}\left(r,\phi,\beta;\rho,\psi,\vartheta\right)+R_{k\psi\vartheta}\left(r,\phi,\beta;\rho,\psi,\vartheta\right)\right].\end{gathered}\tag{49}$$

Calculating the respective derivatives in (49), we can see that the functions $U_q\left(M,N\right)$ coincide with the components of the matrix $U\left(M,\mathrm{N}\right)$ in Eq. (22). From the expressions $U_q\left(M,N\right)$ in Eq. (49) we can see also that the inequality $\left|U_q\left(M,N\right)\right|<\infty$ is valid and the displacements vanish at infinity: $\lim\limits_{\rho\to\infty}U_q\left(M,N\right)\to 0$.

At the next step we have to check the correctness of the functions $U_q\left(M,N\right)$. To this end we note that $U_q=U_q\left(M,N\right)$ satisfies the displacement-temperature Eq. (12) as well as Eq. (11) that can be identified with a heat conduction equation. So, with respect to the coordinates of the point of application $N\equiv(\rho,\psi,\xi)$ they satisfy the BVP of thermoelasticity in Eq. (12) with the following boundary conditions:

$$\begin{array}{c} U_\rho\left(M,\tilde{N}\right)=\sigma_{\psi\psi}\left(M,\tilde{N}\right)=U_\xi\left(M,\tilde{N}\right)=0;\ M\in V,\ \tilde{N}\in\Gamma_{\varphi 0},\\ \sigma_{\psi\rho}\left(M,\tilde{N}\right)=U_\psi\left(M,\tilde{N}\right)=\sigma_{\psi\xi}\left(M,\tilde{N}\right)=0;\ M\in V,\ \tilde{N}\in\Gamma_{\varphi\alpha};\\ U_\rho\left(M,\tilde{N}\right)=\sigma_{\psi\psi}\left(M,\tilde{N}\right)=U_\xi\left(M,\tilde{N}\right)=0;\ M\in V,\ \tilde{N}\in\Gamma_{\beta(\pi/2)}, \end{array} \tag{50}$$

which follows from the integral formula in Eq. (48) and boundary conditions in Eqs (16) - (17). Also, the functions $U_q\left(M,N\right)$ have to satisfy the following fictitious heat conduction BVP with respect to the coordinates of the point of observation $M\equiv(r,\varphi,\beta)$:

$$\begin{array}{c} \nabla_M^2U_q\left(M,N\right)=-\gamma\Theta^{(q)}\left(M,N\right);\ M,N\in V;\\ U_q\left(\tilde{M},N\right)=0;\ \tilde{M}\equiv\left(\tilde{r},\tilde{\varphi}=0,\tilde{\beta}\right)\in\Gamma_{\varphi 0},\\ \frac{\partial U_q\left(\tilde{M},N\right)}{\partial n_{\tilde{\varphi}\alpha}}=0;\ \tilde{M}\equiv\left(\tilde{r},\tilde{\varphi}=\alpha,\tilde{\beta}\right)\in\Gamma_{\varphi\alpha};\\ U_q\left(\tilde{M},N\right)=0;\ \tilde{M}\equiv\left(\tilde{r},\tilde{\varphi},\pi/2\right)\in\Gamma_{\beta(\pi/2)}, \end{array} \tag{51}$$

which follows from the integral formulas in Eq. (48) and boundary conditions in Eq. (31) (see also equation (19)). Note, that in equation (46) the functions $\Theta^{(q)}\left(M',N\right)$ are determined by expression in Eq. (47). But the functions $\Theta\left(M,N\right)$ in Eq. (12) are determined on the basis of displacements $U_q\left(M,N\right)$ in Eq. (49), using the rule similar to that in Eq. (8), in which derivatives must be taken with respect to coordinates of the point $N\equiv(\rho,\psi,\vartheta)$.

2.3. **Deriving the Green-type Integral Formula.** The next step in proving the above theorem is to calculate (on the base of derived main influence functions $U_q\left(M,N\right)$), the other influence functions such as $U_{\varphi\alpha q}\left(\tilde{M},N\right)$, $\partial U_{\varphi 0q}\left(\tilde{M},N\right)/\partial n_{\varphi\,0}$ on the boundary quarter- planes $\Gamma_{\varphi\alpha}$, $\Gamma_{\varphi 0}$ and $\partial U_{\beta(\pi/2)q}\left(\tilde{M},N\right)/\partial n_{\beta(\pi/2)}$ on the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}$. Then we can to obtain finally the Green-type integral formula for the spherical semi-wedges, using Eq. (5). The mentioned above influence functions are determined on the base of the general formulas in Eqs (13) and (14) as following:
a) function of influence of unit surface temperature onto displacements:

$$U_q\left(r,\alpha,\beta;N\right)=-2mL_N^{(q)}\sum_{k=0}^{n-1}\left[R_{k\alpha}\left(r,\alpha,\beta;\rho,\psi,\vartheta\right)-R_{k\alpha\vartheta}\left(r,\alpha,\beta;\rho,\psi,\vartheta\right)\right], \tag{52}$$

where

$$\begin{array}{c} R_{k\alpha}\left(r,\alpha,\beta;\rho,\psi,\vartheta\right)=\sqrt{r^2+\rho^2-2r\rho\cos\left(\phi_\alpha-\omega_{k\alpha}\right)};\\ R_{k\alpha\vartheta}\left(r,\alpha,\beta;\rho,\psi,\vartheta\right)=\sqrt{r^2+\rho^2-2r\rho\cos\left(\phi_\alpha-\omega_{k\vartheta\alpha}\right)};\\ \cos\left(\phi-\omega_{k\alpha}\right)=\sin\beta\sin\vartheta\cos\left(\alpha-\psi-2k\pi/n\right)+\cos\beta\cos\vartheta;\\ \cos\left(\phi-\omega_{k\alpha\vartheta}\right)=\sin\beta\sin\vartheta\cos\left(\alpha-\psi-2k\pi/n\right)-\cos\beta\cos\vartheta. \end{array} \tag{53}$$

on the boundary plane $\Gamma_{\varphi\alpha}$;

b) function of influence of unit surface temperature onto displacements:

$$\begin{aligned}&\partial U_q\left(r,0,\beta;N\right)/\partial n_{\phi 0}=2mL_N^{(q)}\times\sum_{k=0}^{n-1}\left(-1\right)^k r\rho\sin\beta\sin\vartheta\sin\left(\psi+2k\pi/n\right)\\&\left[R_{k0}^{-1}\left(r,0,\beta;\rho,\psi,\vartheta\right)-R_{k0\vartheta}^{-1}\left(r,0,\beta;\rho,\psi,\vartheta\right)\right],\end{aligned}\tag{54}$$

on the boundary plane $\Gamma_{\varphi 0}$.
c) function of influence of unit surface temperature onto displacements:

$$\begin{gathered}\partial U_{\beta(\pi/2)q}\left(r,\phi,\pi/2;N\right)/\partial n_{\beta(\pi/2)}=-2mL_N^{(q)}\times\\\sum_{k=0}^{n-1}\left(-1\right)^k r\rho\cos\vartheta\left[R_{k(\pi/2)}^{-1}\left(r,\phi,\pi/2;\rho,\psi,\vartheta\right)-\ R_{k(\pi/2)\psi}^{-1}\left(r,\phi,\pi/2;\rho,\psi,\vartheta\right)\right],\\R_{k(\pi/2)}=\sqrt{r^2+\rho^2-2r\rho\cos\left(\varphi-\omega_{k(\pi/2)}\right)};\\R_{k(\pi/2)}=\sqrt{r^2+\rho^2-2r\rho\cos\left(\varphi+\omega_{k(\pi/2)}\right)};\\\cos\left(\varphi-\omega_{k(\pi/2)}\right)=\sin\vartheta\cos\left(\phi-\psi-2k\pi/n\right);\\\cos\left(\varphi+\omega_{k(\pi/2)}\right)=\sin\vartheta\cos\left(\phi+\psi-2k\pi/n\right)\end{gathered}\tag{55}$$

on the boundary plane $\Gamma_{\beta(\pi/2)}$.

Let us rewrite the general integral formula in Eq. (5) in the case of BVP in Eqs (1) and (16)-(19) for spherical semi-wedges V:

$$\begin{gathered}u_q\left(N\right)=\frac{1}{a}\int_0^\infty\int_0^\alpha\int_0^{\pi/2}F\left(r,\phi,\beta\right)U_q\left(r,\phi,\beta;N\right)r^2drd\phi\,d\beta+\\\int_0^\infty\int_0^{\pi/2}\frac{\partial T_{\phi\alpha}\left(r,\alpha,\beta\right)}{\partial n_{\phi\alpha}}U_q\left(r,\alpha,\beta;N\right)rdrd\beta\\\int_0^\infty\int_0^{\pi/2}T_{\phi 0}\left(r,0,\beta\right)\frac{\partial U_{\phi q}\left(r,0,\beta;N\right)}{\partial n_{\phi 0}}rdrd\beta-\\\int_0^\infty\int_0^\alpha T_{\beta(\pi/2)}\left(r,\phi,\pi/2\right)\frac{\partial U_{\beta(\pi/2)}\left(r,\phi,\pi/2;N\right)}{\partial n_{\beta(\pi/2)}}rdrd\phi;\ N\equiv\left(\rho,\psi,\vartheta\right);\ q=\rho,\psi,\vartheta.\end{gathered}\tag{56}$$

The formula in Eq. (56) is obtained from the general integral formula in Eq. (5), wherein was taken into account that on the boundary quarter-plane $\Gamma_{\varphi 0}$ and on marginal circular infinite sector $\Gamma_{\beta(\pi/2)}$ the Dirichlet's conditions (temperature) are given and on the boundary quarter-plane $\Gamma_{\varphi\alpha}$ the Neumann's conditions (heat flux) are prescribed (see Eq. (19)). Introducing the influence functions from Eqs (49) and (52)-(54) into the formula in Eq. (56), we obtain the following Green's type integral formula:

$$
\begin{gathered}
u_q(N) = -mL_N^{(q)} \sum_{k=0}^{n-1} (-1)^k \left\{ a^{-1} \left[\int_0^\infty \int_0^\alpha \int_0^{\pi/2} F(r,\phi,\beta) \left(R_k(r,\phi,\beta;\rho,\psi,\vartheta) - \right.\right.\right. \\
\left. R_{k\psi}(r,\phi,\beta;\rho,\psi,\vartheta) - R_{k\vartheta}(r,\phi,\beta;\rho,\psi,\vartheta) + R_{k\psi\vartheta}(r,\phi,\beta;\rho,\psi,\vartheta) \right) r^2 dr d\phi d\beta + \\
\left. +2 \int_0^\infty \int_0^{\pi/2} S_{\phi\alpha}(r,\alpha,\beta) \left(R_{k\alpha}(r,\alpha,\beta;\rho,\psi,\vartheta) - R_{k\alpha\vartheta}(r,\alpha,\beta;\rho,\psi,\vartheta) \right) r dr d\beta \right] - \\
-2 \int_0^\infty \int_0^\pi f_{\phi 0}(r,0,\beta)\, r\rho \sin\beta \sin\vartheta \sin(\psi + (2k\pi/n)) \\
\left(R_{k0}^{-1}(r,0,\beta;\rho,\psi,\vartheta) - R_{k0\vartheta}^{-1}(r,0,\beta;\rho,\psi,\vartheta) \right) r\, dr d\beta \\
-2 \int_0^\infty \int_0^\alpha T_{\beta(\pi/2)}(r,\phi,\pi/2)\, r\rho\cos\vartheta \\
\left. \left(R_{k_{(\pi/2)}}^{-1}(r,\phi,\pi/2;\rho,\psi,\vartheta) - R_{k_{(\pi/2)}\psi}^{-1}(r,\phi,\pi/2;\rho,\psi,\vartheta) \right) r dr d\phi \right\},
\end{gathered} \tag{57}
$$

where the thermoelastic displacements $u_q(\mathrm{N})$ are generated by the inner heat source $F(M)$, heat flux $a\left[\partial T_{\varphi\alpha}(r,\alpha,\beta)/\partial n_{\varphi\alpha}\right] = S_{\varphi\alpha}(r,0,\beta)$, given on the boundary quarter-plane $\Gamma_{\varphi\alpha}$ and the temperatures $T_{\varphi 0}(r,\alpha,\beta)$ and $T_{\beta(\pi/2)}(r,\varphi,\pi/2)$, given on the marginal circular infinite sector $\Gamma_{\beta(\pi/2)}$. Finally, we note that integrals with unbounded intervals exist, which means that the displacements $|u_q(N)| < \infty$, when the following conditions are satisfied:

$$
\begin{gathered}
\int_0^\infty \int_0^\alpha \int_0^{\pi/2} |F(r,\varphi,\beta)|\, r^2 dr d\varphi d\beta < \infty; \ \int_0^\infty \int_0^{\pi/2} |S_{\varphi\alpha}(r,\alpha,\beta)|\, r dr d\beta < \infty; \\
\int_0^\infty \int_0^{\pi/2} |f_{\varphi 0}(r,0,\beta)|\, r dr d\beta < \infty; \ \int_0^\infty \int_0^\alpha \left| f_{\beta(\pi/2)}(r,\varphi,\pi/2) \right| r dr d\varphi < \infty,
\end{gathered} \tag{58}
$$

because, the kernels in Eq. (57) vanish at infinity. The conditions in Eqs (58) are satisfied in the case when the functions $F(r,\varphi,\beta)$,$S_{\varphi\alpha}(r,\alpha,\beta)$,$f_{\varphi 0}(r,0,\beta)$ and $f_{\beta(\pi/2)}(r,\varphi,\pi/2)$ are given on the bounded domains. The investigations have shown that the displacements, described by Green-type integral formula (57), satisfy the BVP in Eqs (1) and (16)-(17) rewritten for the point N. If in Eqs (49) and (52)-(54) we calculate the derivatives, and, if we present the obtained results in matrix form, then we can be sure that the influence matrices $U(M,\mathrm{N})$,$Q_{\varphi\alpha}(r,\alpha,\beta;N) = U_{\varphi\alpha}(r,\alpha,\beta;N)$, $Q_{\varphi 0}(r,0,\beta;N) = \partial U_{\varphi 0}(r,0,\beta;N)/\partial n_{\varphi 0}$, $Q_{\beta(\pi/2)}(r,\varphi,\pi/2;N) = \partial U_{\beta(\pi/2)}(r,\varphi,\pi/2;N)/\partial n_{\beta(\pi/2)}$ and Green-type integral formula in Eq. (57) coincide with the results in Eqs (22) - (29) and (21), respectively. So, now we are sure that all items as well as the theorem are proved.

As a corollaries of the proved theorem can serve some particular cases of angle α. So, from Eqs (21)-(29) at $\alpha = \pi/2$ we obtain the Green-type integral formula and the Green's functions for octant $V(0 \le r < \infty; 0 \le \varphi \le \pi/2; 0 \le \beta \le \pi/2)$ ($\alpha = \pi/2$; $n = 2$; $k = 0,\ 1$). As an example of application of the derived Green's function can serve the BVP for determination of the thermoelastic displacements in a ground foundation, having the form of a octant; on its vertical boundary quarter-plane $\Gamma_{\varphi\alpha}$ is established massive rigid retaining wall at a contact without friction (normal displacement and tangential stresses are zero). On the others boundaries $\Gamma_{\varphi 0}$ and $\Gamma_{\beta(\pi/2)}$ is installed an inextensible thin plate fully contacted with the octant (normal stress and tangential displacement are zero).

Finally, it is very important to note that analogical theorems may be proved for 32 BVPs of thermoelasticity for spherical semi-wedges (On the boundaries the following homogeneous mechanical boundary conditions in any combinations may be given: a. normal displacements and tangential stresses; and b. normal stresses and tangential displacements. Also, on the boundaries the following non-homogeneous thermal boundary conditions in any combinations may be given: c. temperature; and d. heat flux. In addition, taking also in consideration the particular cases: octant, wedge, quarter-space and half-space that follows from the equation for the considered above spherical semi-wedges it is possible to derive new exact Green's type integral formulas and the thermoelastic Green's functions in elementary functions for about 80 BVPs of thermoelastostatics. Thus, one of the main conclusions that follows from the proved above theorem is that the proposed method to derive Green's type integral formulas and the thermoelastic Green's functions in thermoelastostatics is applicable for many BVPs.

3. The Advantages Of The Obtained Results And The Possibilities To Derive New Green'S Functions And Integral Formulae

First, it is important to note that the Green-type integral formula in Eq. (21) and Green's functions in Eqs (22)–(34) for general BVP in Eqs (1) and (16)-(20), have been obtained for the first time. The advantages of the Green's functions in Eqs (22)-(34) and the Green-type integral formula in Eq. (21) are as follows: 1. When compared with the thermoelastic potential method proposed by Timoshenko and Goodier, other thermoelastic potentials methods, and many other traditional methods to solve BVP of thermoelasticity, in our method: (a) we have not solved BVP in heat conduction to determine the preliminary temperature field (first-stage of solution) and subsequently solve Lame's equations at the known temperature field (second stage of solution). In the obtained Green's type integral formula, the solution is presented directly via the given (known) heat actions (heat source and boundary temperature and heat flux). This formula gives the final solution in terms of integrals; hence, we did not have to solve the preliminary BVP in heat conduction or any other BVP; (b) it is not necessary for us to solve additional BVP of elasticity. Particularly, in the singular integral equations method (SIEM), i.e., one of the methods of thermoelastic potentials, to solve the additional BVP, we had to solve systems of boundary SIE. The Green-type integral formula presented in Eq. (21) gives the complete solution to the considered BVP in Eqs (1), (15)-(19), and hence, it is not necessary to solve additional BVP; 2. The Green's functions given in Eqs (21)-(34) and the Green-type integral formula presented in Eq. (21) obtained in elementary functions are of significant importance in the applications because of its short computation and high accuracy; 3. The Green-type integral formula presented in Eq. (21) is completely ready and convenient for immediate numerical implementation; 4. On the basis of this formula, it is possible to elaborate a boundary element in BEM [20]; and 5. Green's functions given in Eqs (21)-(34) allow to write closed solutions to many particular BVPs with heat actions (heat source, boundary temperature and heat flux) including discontinuities and dislocations in the form of line integrals or in elementary functions. The above-mentioned advantages, usefulness, and importance of the new exact Green's functions and Green-type integral formula for thermoelastic spherical semi-wedges obtained in closed form explain why the specialists highly appreciate the new closed-form Green's functions and solution in integrals of any BVP. Hence, we tried to analyze the possibilities to derive the Green-type integral formula in thermoelasticity not only for spherical semi-wedges, but also for the different cylindrical and other orthogonal canonical domains [21, 22, 23, 24]. As observed earlier, the main difficulty in deriving the thermoelastic Green's functions U_q

is the determination of the influence functions $\Theta^{(q)}$ for BVP in elasticity and the Green's functions G for BVP in heat conduction.

4. Conclusions

1. The Green-type integral formula in Eq. (21) and Green's functions in Eqs (22)-(29) for general BVP in Eqs (1) and (16)-(20) for thermoelastic spherical semi-wedges have been obtained for the first time. They are useful and completely ready to be efficiently applied for computing the thermoelastic displacements $u_q(N)$ in spherical semi-wedges. The main advantage of the obtained integral formula is that the sought thermoelastic displacements within spherical semi-wedges are expressed directly via given inside heat source, boundary heat flux, temperature and known kernels. So it is not necessary to determine the intermediate inner temperature field or to solve, as in traditional methods, additional BVP; 2. The most difficult problem in the proposed technique of deriving the influence functions $U_q(M, N)$ and writing the Green's type integral formulas is the derivation of the Green's functions $G(M, N)$ in heat conduction and the functions of influence for volume dilatation $\Theta^{(q)}(M, N)$ in elasticity. It is very important to note that analogous theorems may be proved for many BVPs of thermoelasticity for spherical semi-wedges (on the boundaries the following homogeneous mechanical boundary conditions in any combinations may be given: (a). normal displacements and tangential stresses; (b). normal stresses and tangential displacements). Thus, taking also in consideration the particular cases: octant, wedges, quarter-space and half-space that follow from the equation for the considered above spherical semi-wedges and different combinations of possible inhomogeneous boundary conditions in the heat conduction problem, it is possible to derive new exact Green's type integral formulas and the thermoelastic Green's functions in elementary functions for considerable more much BVPs of thermoelastostatics. Thus, one of the main conclusions that follows from the proved above theorem is that the proposed method to derive Green's type integral formulas and the thermoelastic Green's functions in thermoelastostatics is applicable to many BVPs. The results presented in this paper can provide great possibilities for the researchers to derive many new thermoelastic influence functions and new Green's type integral formulas, not only for spherical semi-wedges, but also for many other canonical spherical domains. These new Green's type integral formulas are very useful to solve effectively not only deterministic BVPs of thermoelasticity, but also the stochastic ones [25]; 3. The equations and general formulas given in this paper for the determination of the influence functions $U_q(M, N)$ in Eq. (7), as well as the Green-type integral formula in Eq. (5), are valid not only for thermoelasticity, but also for other physical phenomena such as poroelasticity, described by the same BVPs as that described for thermoelasticity [26]; and 4. The approach presented in this paper for spherical semi-wedges can be extended to any other spherical canonical domains. This extension can be carried out when the lists of both the functions G and $\Theta^{(q)}$ are completed.

References

[1] Boley, B.A and Weiner, I.H., *Theory of Thermal Stresses*, Wiley, New York (1960).
[2] Kovalenko, A.D., *Fundamentals of Thermoelasticity*, Naukova Dumka, Kiev (1970) (in Russian).
[3] Melan, E. and Parkus, H., *Wärmespannungen infolge Stationarer Temperaturfelder*, Springer, Vienna (1953).
[4] Maysel, V.M., *The Temperature Problem of the Theory of Elasticity*, Publisher AN SSSR, Kiev (1951) (in Russian).
[5] Nowacki, W., *Thermoelasticity*, Pergamon Press and Polish Scientific Publishers, Oxford, Warszawa (1962).
[6] Nowacki, W., *The Theory of Elasticity*, Mir, Moscow (1975) (in Russian).

[7] Hetnarski, R.B. and Eslami, M.R., *Thermal Stresses - Advanced Theory and Applications*, XXXII+ 559, pages, Springer (2009).

[8] Seremet, V. D., *The modification of Maysel's formula in the stationary thermoelasticity*, Bulletin of Academy of Science of Republic of Moldova, Mathematics, 3, pp. 19 – 22, Chisinau, Republic of Moldova (1997).

[9] Seremet, V., *Some New Influence Functions and Integral Solutions in Theory of Thermal Stresses*, Proc. IV-th International Congress on Thermal Stresses, Osaka Prefecture University, Japan, p. 423 , (2001).

[10] Seremet, V., *New Results in 3-D Thermoelasticity*, Proc. 14th U.S. National Congress of Theoretical and Applied Mechanics, Blacksburg, Virginia, p.29, (2002).

[11] Şeremet, V.D., *Handbook on Green's Functions and Matrices*, WIT Press, Southampton and Boston (2003).

[12] Şeremet, V., Vlad, I. and Şeremet, A., *New Integral Formulae in Thermoelasticity*, Proc.16th ASCE Engineering Mechanics Conference (EM 2003), Washington University, Seattle, p. 82, (2003).

[13] Sheremet, V., *The Integral Equations and Green's Matrices of the Influence Elements Method in the Mechanics of Solids*, Dr. Habilitat Thesis, Technical University of Moldova, Chisinau, (1995) (in Romanian).

[14] Sheremet, V., *Generalization of Green's Formulae in Thermoelasticity*, Collection: Multi-scale Green's Functions for Nanostructures, National Science Digital Library, NSF, website: http://209.85.135.104/search?q=cache:-_vI47Sx7XYJ:matdl.org/repository/view/matdl:571, pp. 1-4, (2003).

[15] Sheremet, V., *New Formulae for Dynamical Thermal Stresses*, J. Thermal Stresses, 25, 2, pp.123-153, (2002).

[16] Sheremet, V., Precupan, D., Vlad, I. and Sheremet, A., *The Constructing of Green's Matrices in Cylindrical Coordinates*, Proc. The 17 th ASCE Engineering Mechanics Conference (EM 2004), University of Delaware, Newark, p. 87 (2004).

[17] Seremet, V., Bonnet, G. and Speianu, T., *New Results in Construction of the Green's Matrices in Spherical Coordinates*, Proc. University of Minnesota, Minneapolis, p. 240, (2008).

[18] Budac, V.M., Samarskii, A.A. and Tihonov, A.N., *Set of Problems on Mathematical Physics*, Gostehnizdat, Moscow, (1980) (in Russian).

[19] Kartashov, A.D., *Analytic Methods in the Theory of Heat Conductibility of Rigid Bodies*, High School, Moscow, (1980) (in Russian).

[20] Brebbia, C.A., *The Boundary Element Method for Engineers*, Wiley, New York (1978).

[21] Seremet, V., *Influence Elements Method*, 2003. Edit. Center, Agrar. Univ. of Moldova., Chisinau (2003).

[22] Seremet, V. and Bonnet, G., *Encyclopedia of Domain Green's Functions (Thermo-magneto-electrostatics of solids in rectangular and polar coordinates)*, Edit. Center Agr. Univ. of Moldova, Chisinau, http://greenfunction.md , (2008).

[23] Seremet, V., Vlad I. and Seremet, A., *New Influence Functions for Thermoelastic Sperical Shells*, Proc. V-th International Congress on Thermal Stresses (ICTS 2003), Virginia Tech., Blacksburg, vol.1, p. MA-5-1-1 (2003).

[24] Seremet, V., Bonnet G. and Speianu, T., *Influence Functions and Integral Formulae for Spherical Thermoelastic Bodies*, Proc. The XXII International Congress of Theoretical and Applied Mechanics, ICTAM 2008, Adelaide University, Australia, p. 226 (2008).

[25] Sabelfeld, K., *Expansion of random boundary excitations for elliptic PDEs*, J. Monte Carlo Methods and Appl., Vol. 13, pp. 403-451 (2007).

[26] Norris, A., *On the correspondence between poroelasticity and thermoelasticity*, J. Appl. Phys., Vol. 71, No. 3, pp. 1138-1141 (1992).

Appendix A. Computing Of The Improper Integral (47)

To compute the integral

$$U_q(M,N)=\gamma\int_0^\infty\int_0^\alpha\int_0^{\pi/2}G(M;\rho',\psi',\vartheta')\,\Theta^{(q)}(\rho',\psi',\vartheta';N)\,\rho'^2d\rho'd\psi'd\vartheta', \tag{59}$$

where

$$\begin{gathered}
\Theta^{(q)}=-[4\pi(\lambda+2\mu)]^{-1}L_N^{(q)}\sum_{k=0}^{n-1}(-1)^{(k)}\left(R_k^{-1}-R_{k\psi}^{-1}-R_{k\vartheta}^{-1}+R_{k\psi\vartheta}^{-1}\right);\\
G=\frac{1}{4\pi}\sum_{k=0}^{n-1}(-1)^{(k)}\left(R_k^{-1}-R_{k\psi}^{-1}-R_{k\vartheta}^{-1}+R_{k\psi\vartheta}^{-1}\right)\\
R_k=\sqrt{r^2+\rho^2-2r\rho\cos(\varphi-\omega_k)};\ R_{k\psi}=\sqrt{r^2+\rho^2-2r\rho\cos(\varphi+\omega_k)};\\
R_{k\vartheta}=\sqrt{r^2+\rho^2-2r\rho\cos(\varphi-\omega_{k\vartheta})};\ R_{k\psi\vartheta}=\sqrt{r^2+\rho^2-2r\rho\cos(\varphi+\omega_{k\vartheta})};\\
\cos(\varphi-\omega_k)=\sin\beta\sin\vartheta\cos(\phi-\psi-2k\pi/n)+\cos\beta\cos\vartheta;\\
\cos(\varphi+\omega_k)=\sin\beta\sin\vartheta\cos(\phi+\psi-2k\pi/n)+\cos\beta\cos\vartheta;\\
\cos(\varphi-\omega_{k\vartheta})=\sin\beta\sin\vartheta\cos(\phi-\psi-2k\pi/n)-\cos\beta\cos\vartheta;\\
\cos(\varphi+\omega_{k\vartheta})=\sin\beta\sin\vartheta\cos(\phi+\psi-2k\pi/n)-\cos\beta\cos\vartheta;\\
L_N^{(q)}=\delta_{q\rho}(\partial/\partial\rho)+\delta_{q\psi}(\rho\sin\vartheta)^{-1}(\partial/\partial\psi)+\delta_{q\vartheta}\rho^{-1}(\partial/\partial\vartheta)\ .
\end{gathered} \tag{60}$$

First we need to substitute (60) into (59) and obtain the following improper integral

$$\begin{gathered}
U_q(M,N)=-2mL_N^{(q)}\\
\int_0^\infty\int_0^\alpha\int_0^\pi\sum_{k=0}^{n-1}(-1)^{(k)}\Big(R_k^{-1}(M;\rho',\psi',\vartheta')-R_{k\psi}^{-1}(M;\rho',\psi',\vartheta')-\\
-R_k^{-1}(M;\rho',\psi',\vartheta')+R_{k\psi}^{-1}(M;\rho',\psi',\vartheta')\Big)\times\\
(4\pi)^{-1}\sum_{k=0}^{n-1}(-1)^{(k)}\Big(R_k^{-1}(\rho',\psi',\vartheta';N)-R_{k\psi}^{-1}(\rho'^2,\psi',\vartheta';N)-\\
-R_k^{-1}(\rho',\psi',\vartheta';N)+R_{k\psi}^{-1}(\rho'^2,\psi',\vartheta';N)\Big)\rho'^2d\rho'd\psi'd\vartheta',
\end{gathered} \tag{61}$$

The improper integral in Eq. (48) was taken using:
a) the following equalities on the boundary quarter-planes of the spherical semi-wedges

$$\begin{gathered}
\frac{\partial}{\partial\psi'}\left[\frac{1}{4\pi}\sum_{k=0}^{n-1}(-1)^{(k)}\left(R_k(M;\rho',\psi',\vartheta')-R_{k\psi}(M;\rho',\psi',\vartheta')-\right.\right.\\
\left.\left.-R_{k\vartheta}(M;\rho',\psi',\vartheta')+R_{k\psi\vartheta}(M;\rho',\psi',\vartheta')\right)\right]|_{\psi'=\alpha}=0;\\
\left[\frac{1}{4\pi}\sum_{k=0}^{n-1}(-1)^{(k)}\left(R_k(M;\rho',\psi',\vartheta')-R_{k\psi}(M;\rho',\psi',\vartheta')-\right.\right.\\
\left.\left.-R_{k\vartheta}(M;\rho',\psi',\vartheta')+R_{k\psi\vartheta}(M;\rho',\psi',\vartheta')\right)\right]|_{\psi'=0}=0;
\end{gathered}$$

$$\frac{\partial}{\partial \psi'}\left[\frac{1}{4\pi}\sum_{k=0}^{n-1}(-1)^{(k)}\left(R_k^{-1}(\rho',\psi',\vartheta';N)-R_{k\psi}^{-1}(\rho',\psi',\vartheta';N)-\right.\right.$$
$$\left.\left.-R_{k\vartheta}^{-1}(\rho',\psi',\vartheta';N)+R_{k\psi\vartheta}^{-1}(\rho',\psi',\vartheta';N)\right)\right]|_{\psi'=\alpha}\;=0; \tag{62}$$

$$\left[\frac{1}{4\pi}\sum_{k=0}^{n-1}(-1)^{(k)}\left(R_k^{-1}(\rho',\psi',\vartheta';N)-R_{k\psi}^{-1}(\rho',\psi',\vartheta';N)-\right.\right.$$
$$\left.\left.-R_{k\vartheta}^{-1}(\rho',\psi',\vartheta';N)+R_{k\psi\vartheta}^{-1}(\rho',\psi',\vartheta';N)\right)\right]|_{\psi'=0}\;=0;$$
$$\left[\frac{1}{4\pi}\sum_{k=0}^{n-1}(-1)^{(k)}\left(R_k^{-1}(\rho',\psi',\vartheta';N)-R_{k\psi}^{-1}(\rho',\psi',\vartheta';N)-\right.\right.$$
$$\left.\left.-R_{k\vartheta}^{-1}(\rho',\psi',\vartheta';N)+R_{k\psi\vartheta}^{-1}(\rho',\psi',\vartheta';N)\right)\right]|_{\vartheta'=0}\;=0.$$

b) the relations

$$\begin{aligned}
&\nabla^2\left[(R_k(M;\rho',\psi',\vartheta')-R_{k\psi}(M;\rho',\psi',\vartheta'))\right]=\\
&=2\left[R_k^{-1}(M;\rho',\psi',\vartheta')-R_{k\psi}^{-1}(M;\rho',\psi',\vartheta')\right];\\
&\nabla^2\left[(R_{k\vartheta}(M;\rho',\psi',\vartheta')-R_{k\psi\vartheta}(M;\rho',\psi',\vartheta'))\right]=\\
&=2\left[R_{k\vartheta}^{-1}(M;\rho',\psi',\vartheta')-R_{k\psi\vartheta}^{-1}(M;\rho',\psi',\vartheta')\right]
\end{aligned} \tag{63}$$

and

$$\nabla_N^2 R_k^{-1}(M,N)=\begin{cases}-\delta(M-N), & k=0\\ 0,\quad k\neq 0, & k=1,2,3,\ldots,n-1\end{cases}$$
$$\nabla_M^2 R_k^{-1}(M,N)=\nabla_M^2 R_{k\psi}^{-1}(M,N)=\nabla_N^2 R_{k\psi}^{-1}(M,N)=\nabla_N^2 R_{k\vartheta}^{-1}(M,N)$$
$$=\nabla_M^2 R_{k\vartheta}^{-1}(M,N)=\nabla_M^2 R_{k\psi\vartheta}^{-1}(M,N)=\nabla_N^2 R_{k\psi\vartheta}^{-1}(M,N)=0; \tag{64}$$

c) the following characteristic of Dirac's function

$$\int_V f(M)\,\delta(M-N)\,dV(M)=f(N) \tag{65}$$

d) the Green's formula within the spherical semi-wedges

$$\int_V\left(g\nabla^2 f-f\nabla^2 g\right)dV=\int_\Gamma\left[g\,(\partial f/\partial n)-f\,(\partial g/\partial n)\right]d\Gamma, \tag{66}$$

where

$$\begin{aligned}
f=&\frac{1}{4\pi}\sum_{k=0}^{n-1}(-1)^{(k)}\left(R_k^{-1}\left(M;\rho',\psi',\vartheta'\right)-R_{k\psi}^{-1}\left(M;\rho',\psi',\vartheta'\right)-\right.\\
&\left.-R_{k\vartheta}^{-1}\left(M;\rho',\psi',\vartheta'\right)+R_{k\psi\vartheta}^{-1}\left(M;\rho',\psi',\vartheta'\right)\right);\\
g=&\frac{1}{4\pi}\sum_{k=0}^{n-1}(-1)^{(k)}\left(R_k\left(\rho',\psi',\vartheta';N\right)-R_{k\psi}\left(\rho',\psi',\vartheta';N\right)-\right.\\
&\left.-R_{k\vartheta}\left(\rho',\psi',\vartheta';N\right)+R_{k\psi\vartheta}\left(\rho',\psi',\vartheta';N\right)\right).
\end{aligned} \tag{67}$$

So, applying Eqs. (66) in formula (66) we obtain:

$$\begin{gathered}
U_q(M,N) = -2mL_N^{(q)} \times \int_0^\infty \int_0^\alpha \int_0^\pi \nabla^2 \left[\frac{1}{8\pi} \sum_{k=0}^{n-1} (-1)^{(k)} \left(R_k(M;\rho',\psi',\vartheta') - \right.\right. \\
\left. -R_{k\psi}(M;\rho',\psi',\vartheta') - R_k(M;\rho',\psi',\vartheta') + R_{k\psi}(M;\rho',\psi',\vartheta') \right) \Big] \times \\
\sum_{k=0}^{n-1} (-1)^{(k)} \Big(R_k^{-1}(\rho',\psi',\vartheta';N) - R_{k\psi}^{-1}(\rho',\psi',\vartheta';N) - \\
-R_{k\vartheta}^{-1}(\rho',\psi',\vartheta';N) + R_{k\psi\vartheta}^{-1}(\rho',\psi',\vartheta';N) \Big) \rho'^2 d\rho' d\psi' d\vartheta' = \\
= -2mL_N^{(q)} \int_0^\infty \int_0^\alpha \int_0^\pi \frac{1}{8\pi} \sum_{k=0}^{n-1} (-1)^{(k)} \left(R_k(M;\rho',\psi',\vartheta') - \right. \\
\left. -R_{k\psi}(M;\rho',\psi',\vartheta') - R_k(M;\rho',\psi',\vartheta') + R_{k\psi}(M;\rho',\psi',\vartheta') \right) \times \\
\nabla^2 \Big[\Big(R_k^{-1}(\rho',\psi',\vartheta';N) - R_{k\psi}^{-1}(\rho',\psi',\vartheta';N) - \\
-R_{k\vartheta}^{-1}(\rho',\psi',\vartheta';N) + R_{k\psi\vartheta}^{-1}(\rho',\psi',\vartheta';N) \Big) \Big] \rho'^2 d\rho' d\psi' d\vartheta' = \\
= -mL_N^{(q)} \frac{1}{8\pi} \sum_{k=0}^{n-1} (-1)^{(k)} \times \int_0^\infty \int_0^\alpha \int_0^\pi \left(R_k(M;\rho',\psi',\vartheta') - R_{k\psi}(M;\rho',\psi',\vartheta') - \right. \\
\left. -R_k(M;\rho',\psi',\vartheta') + R_{k\psi}(M;\rho',\psi',\vartheta') \right) \delta(N'-N) \rho'^2 d\rho' d\psi' d\vartheta',
\end{gathered} \tag{68}$$

where boundary integrals over $\Gamma_{\varphi 0}$, $\Gamma_{\varphi\alpha}$ and $\Gamma_{\beta(\pi/2)}$ due to the relations (62) are equal to zero.

Finally, using the characteristic of Dirac's function from (65) in volume integral we obtain the following main thermoelastic Green's functions in terms of elementary functions:

$$\begin{gathered}
U_q(M,N) = -mL_N^{(q)} \sum_{k=0}^{n-1} (-1)^{(k)} \left(R_k(M;\rho,\psi,\vartheta) - \right. \\
\left. -R_{k\psi}(M;\rho,\psi,\vartheta) - R_{k\vartheta}(M;\rho,\psi,\vartheta) + R_{k\psi\vartheta}(M;\rho,\psi,\vartheta) \right)
\end{gathered} \tag{69}$$

which coincides with the final expressions in Eq. (49).

NOMENCLATURE

α_t is the coefficient of the linear thermal expansion
λ, μ are Lame's constants of elasticity
$\gamma = \alpha_t\,(2\mu + 3\lambda)$ is the thermoelastic constant
α is the coefficient of convective heat conductivity, a is the thermal diffusivity
δ_{ql} is the Kronecker's symbol
V is the body volume
Γ is the surface of the body V
(r, φ, β) are the sherical coordinates
$M\,(r, \varphi, \beta)$, $M \in V$ is an inner point of the body V
$F\,(M)$ is the inner heat source
$T\,(M)$ is the inner temperature
ε_{sp}; $s, p = r, \varphi, z$ are the strains
Θ is the volume dilatation, $\Theta = \frac{\partial U_r}{\partial r} + \frac{1}{r}\left(2U_r + \frac{1}{\sin\beta}\cdot\frac{\partial U_\varphi}{\partial \varphi} + \frac{\partial U_\beta}{\partial \beta} + \operatorname{ctg}\beta\, U_\beta\right)$ due to temperature G
θ is the volume dilatation,$\theta = \frac{\partial u_r}{\partial r} + \frac{1}{r}\left(2u_r + \frac{1}{\sin\beta}\cdot\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_\beta}{\partial \beta} + \operatorname{ctg}\beta u_\beta\right)$ due to temperature T
$\tilde{M} \equiv \left(\tilde{r}, \tilde{\varphi}, \tilde{\beta}\right)$ is a point of the surface $\Gamma \equiv \Gamma_D + \Gamma_N + \Gamma_M$
Γ_D, Γ_N and Γ_M denote the surfaces on which the boundary conditions of Dirichlet's, Neumann's, or mixed type are prescribed
$T\left(\tilde{M}\right); \tilde{M} \in \Gamma_D$ is the temperature prescribed on the surface Γ_D
$a\left[\partial T\left(\tilde{M}\right)/\partial n_{\tilde{M}}\right]\,\tilde{M}\ \in \Gamma_N$ is the heat flux prescribed on the surface Γ_N
$\left(\beta\left(\partial/\partial n_{\tilde{M}}\right) + a\right)\,T\left(\tilde{M}\right)$; $\tilde{M} \in \Gamma_M$ is the law of the heat exchange of the body with exterior medium prescribed on the surface Γ_M
$N\,(\rho, \psi, \vartheta)$ $N \in V$ is an inner point of application of the unit point source (heat source, body force etc) (ρ, ψ, ϑ) are the spherical coordinates of the point of application of the source
$\delta\,(M - N)$ is the Dirac's delta function
$G\,(M, N)$ is the Green's function for a boundary value problem in heat conduction
$\Theta^{(q)}\,(M, N)$ is the influence function represents a volume dilatation in an inner point M of elastostatics problem corresponding to a unit concentrated body force, applied in an inner point N in the direction of the spherical axis $(q \equiv \rho, \psi, \vartheta)$, $\Theta^{(q)} = \frac{\partial U_r^{(q)}}{\partial r} + \frac{1}{r}\left(2U_r^{(q)} + \frac{1}{\sin\beta}\cdot\frac{\partial U_\varphi^{(q)}}{\partial \varphi} + \frac{\partial U_\beta^{(q)}}{\partial \beta} + \operatorname{ctg}\beta U_\beta^{(q)}\right)$
$G_\Theta\,(M, N)$ is the Green's function in an inner point M for a boundary value of elastostatics problem for dilatation
$U_S^{(q)}\,(M, N)$ are the displacements in an inner point of observation M in the direction of the axis $(s \equiv r, \varphi, \beta)$ corresponding to an inner unit point body force applied in an inner point N in the direction of the spherical axis $(q \equiv \rho, \psi, \vartheta)$ (components of the elastostatics Green's tensor)
$U_q\,(M, N)$ are the thermoelastic displacements in an inner point of observation M in the direction of the spherical axis $(q \equiv \rho, \psi, \vartheta)$ corresponding to an inner unit point heat source applied in an inner point N
$U_q\left(\tilde{M}, N\right)$ are the influence functions corresponding to a unit point heat flux on the surface Γ_N and representing the static thermoelastic displacements

$\partial U_q\left(\tilde{M}, N\right)/\partial n_{\tilde{M}}$ are the influence functions corresponding to a unit point temperature on the surface Γ_D and representing the static thermoelastic displacements
$U_q\left(\tilde{M}, N\right)$ are the influence functions corresponding to a unit point heat exchange of the body through the surface Γ_M and representing the static thermoelastic displacements
$L_N^{(q)} = \delta_{q\rho}\left(\partial/\partial\rho\right) + \delta_{q\psi}\left(\rho\sin\vartheta\right)^{-1}\left(\partial/\partial\psi\right) + \delta_{q\vartheta}\rho^{-1}\left(\partial/\partial\vartheta\right)$ is a differential operator in spherical coordinates of the point N
$L_M^{(q)} = \delta_{qr}\left(\partial/\partial r\right) + \delta_{q\varphi}\left(r\sin\beta\right)^{-1}\left(\partial/\partial\varphi\right) + \delta_{q\vartheta}r^{-1}\left(\partial/\partial\beta\right)$ is a differential operator in spherical coordinates of the point M
$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\beta}\frac{\partial}{\partial\beta}\left(\sin\beta\frac{\partial}{\partial\beta}\right) + \frac{1}{r^2\sin\beta}\frac{\partial^2}{\partial\varphi^2}$ is the placeLaplace differential operator with respect to the spherical coordinates of the inner point M

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