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GREEN'S TENSOR FOR AN ELASTIC CIRCLE AND ITS APPLICATION IN MICROMECHANICS OF DEFECTS IN SOLIDS

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Abstract. The Green's tensor for the first (displacement given) boundary value problem of elasticity for a circular domain is computed under a closed form expression. The method of solution uses the "incompressible influence element" for which the Green's tensor is given by representation using Green's function for Poisson's equation. Using such a representation, is shown that the main problem is to find the dilatation along the boundary induced by the displacements Green's function. The volume dilatation is than obtained by solving an integral equation along the circular boundary. Explicit expressions are obtained for the Green's displacements tensor and for the traction along the circular boundary, allowing expressing the solution for any kind of "displacement" boundary condition and body forces. On the basis of the constructed Green's tensor is given the integral formula which presents a generalization of the well known Poisson's integral formula from the theory of harmonic potentials onto the theory of elasticity. An example of application of Green's tensor in micromechanics of defects in solids as radial Volterra's slip dislocation in an elastic circle is presented. These results were obtained in explicit form and for the first time. Applied here the "incompressible influence element method" (IIEM) can be used to derive the Green's tensor for a wide classis of different boundary value problems for canonical domains of many systems of coordinates. So, IIEM will increase considerable the possibilities to solve new complicate boundary value problems in bounded and "unbounded" solids, acted by different inner actions: body forces, temperature dislocations, Volterra's dislocations, eigenstrains, inclusions etc and any boundary displacements.

1. Introduction

The method of Green's functions gives the solutions of boundary value problem under the form of an integral on the boundary involving the known boundary conditions. In addition, it leads to explicit solutions involving volume forces or point forces applied within the domain. The method has been applied to many boundary problems for Poisson's equation. The case of 2D or 3D elasticity does not contain as many solutions of the Green's tensor, because it is more difficult to apply the method of separation of variables to the equations of elasticity, even in the case of an isotropic medium, as explained for example by (Morse and Feshbach, 1954 [3]). A great number of solutions for the Green's tensor were obtained by (Seremet, 2003 [8, 9]) in the case of mixed boundary conditions (for example one component of displacements and one or two components of the traction vector along the boundary). The Green's tensors for a problem related to Dirichlet boundary conditions (displacements fixed on the boundary) or to Neuman boundary conditions (tension vector fixed on the boundary) were obtained for infinite domains only. No solutions were obtained up to now for the Green's tensor in the inner points (so called first Green's function) of the finite domains.

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A solution was given for any Neuman boundary conditions by (Baker and al., 1993 [1]) within a plane rectangular domain under the form of Fourier series (Lame problem), but the solution does not use the Green's tensor formalism and does not include volume forces. For first boundary value problem for an elastic circle the components of the Green's functions for tractions in the points of the circular boundary were obtained by (Cahniashvili N. S, 1960, 1963 see Kupradze V.D., 1965 [2]) using method of elastic potentials (MEP). Also, for second boundary value problem for elastic circle (Basheleishvili M.O., see Kupradze V.D, 1965 [2]) were obtained the value of the components of Green's functions for displacements in the points of the circular boundary using the seam method. It is necessary to mention that these results do not permit to solve the boundary value problems (including in form of integrals over domain) when in the inner points are given some actions, such as masses forces, temperature fields, dislocations, inclusions and others. To do this it is necessary to have the Green's tensor in the inner points of the domain. To construct this Green's tensor using the obtained by Kupradze V.D and his elves solutions in the form of integrals over the boundary it is very difficult because we need to compute a lot of complicated convolutions between the products of singular functions on the boundary.

To construct the Green's tensor in the inner points of the domain we propose to use the incompressible influence elements method (IIEM) elaborated by (Seremet V.D. 1995, 1997, 2003, 2004 [6, 7, 8, 9, 10]) which is essentially more efficient than method of elastic potentials. The efficiently consist in:

(1) We need to solve only one regular boundary integral equation with respect to the dilatation on the boundary: - in IIEM, and a system of two singular boundary integral equations with respect to densities on the boundary:-in MEP;
(2) The number and complicity of boundary integrals needed to be computed in IIEM are considerable less then in MEP.
(3) Green's tensor obtained by IIEM is presented in a compact form: in the form of some linear differential operators from two elementary functions, that is very suitable for applications.
    In the MEP Green's tensor can be obtained in a bulk form and has very long expressions via a lot of elementary functions.
(4) Beside this IIEM can be successfully applied to derive the Green's tensor not only for the first basic problem for an elastic circle, but also for the wide class of boundary value problems (including locally mixed boundary conditions) of elasticity for many others domains of polar and cylindrical system of coordinates.

This paper presents the Green's tensor related to Dirichlet boundary conditions (fixed displacements) for a circular elastic domain. A fully explicit expression of the solution is given and an example of application in micromechanics of defects in solids is presented.

The Green's tensor for elasticity is closely related to the Green's function for Poisson's equation within the same domain. In a first step, the Green's function for Poisson's equation within a circular domain is recalled. Next, a representation of the elasticity Green's tensor is given, involving the Green's function for Poisson's equation and the unknown dilatation induced by the elasticity Green's tensor along the boundary. It is then possible to compute the unknown dilatation along the boundary of the circle and to give the solution for the Green's tensor in a closed form, which is presented as some linear differential operators from two elementary function. An example of applications of the Green's tensor for determination of the displacements in an elastic circle, generated by Volterra's radial slip dislocation is finally given. These results were obtained in closed form.

2. Green's function for Poisson's equation within a circle

Let us consider the domain interior to a circle $V\{0 \le r \le r_0,\, 0 \le \varphi \le 2\pi\}$.

The solution U of the following boundary value problem is now considered: $U(M) = g(M)$ for any point M along the boundary of the circle and $\Delta(U) + f = 0$ within the circle, where f and g have suitable regularity properties.

The solution of such a boundary value problem is given by:

$$U(r,\varphi) = \int_0^{2\pi}\int_0^{r_0} f(\rho,\psi)\, G(r,\varphi,\rho,\psi)\,\rho d\rho d\psi - \int_0^{2\pi} g(\varphi')\,\frac{\partial G(r_0,\varphi';r,\varphi)\, r_0 d\varphi'}{\partial n_{r_0}} \tag{1}$$

where G is the first Green's function for Poisson's equation within a circular domain.

The first Green's function for Poisson's equation within a circular domain is a classical result of the theory of potential ((Polojii, 1964 [5]), (Vladimirov, 1974 [11])), which is recalled thereafter:

A closed form expression of $G(M,N)$ is given by:

$$\begin{aligned} G(r,\varphi;\rho,\psi) &= \frac{1}{2\pi}\ln\frac{\bar{R}}{R};\\ \bar{R} &= \sqrt{r_0^2 - 2r\rho\cos\tilde{\varphi} + (r\rho/r_0)^2};\\ R &= \sqrt{r^2 - 2r\rho\cos(\varphi-\psi) + \rho^2}. \end{aligned} \tag{2}$$

The normal derivative is then given by:

$$\frac{\partial G(r_0,\varphi';r,\varphi)}{\partial n_{r_0}} = \frac{1}{2\pi r_0}\cdot\frac{\left(r^2 - r_0^2\right)}{r_0^2 - 2rr_0\cos(\varphi'-\varphi) + r^2} \tag{3}$$

The aim of the following is to find the expression of the Green's tensor for plane elasticity within a domain with the circular boundary and to show how it can be used for solving boundary value problems.

3. Green's tensor for the problem related to an elastic circular domain

The Green's tensor for the elastic circle is sought by using polar coordinates r, φ. It is the solution of the following set of equations:

$$\begin{aligned} &\mu\left(\Delta U_r^{(q)} - \tfrac{1}{r^2}U_r^{(q)} - \tfrac{2}{r^2}\tfrac{\partial U_\varphi^{(q)}}{\partial_\varphi}\right) + (\lambda+\mu)\tfrac{\partial\Theta^{(q)}}{\partial r} + \delta_{rq}\delta(M-N) = 0;\\ &\mu\left(\Delta U_\varphi^{(q)} - \tfrac{1}{r^2}U_\varphi^{(q)} + \tfrac{2}{r^2}\tfrac{\partial U_r^{(q)}}{\partial_\varphi}\right) + (\lambda+\mu)\tfrac{\partial\Theta^{(q)}}{r\partial\varphi} + \delta_{\varphi q}\delta(M-N) = 0, \end{aligned} \tag{4}$$

and

$$U_r^{(q)} = U_\varphi^{(q)} = 0 \tag{5}$$

along the boundary of the circle: $r = r_0$, where $N(q = (\rho,\psi))$ is the point where a unit forces is applied and $M(s = (r,\varphi))$ is the point where the displacements is given; δ_{pq} is the Kronecker symbol; λ and μ are the Lame elasticity coefficients; $\Theta^{(q)}$ is the volume dilatation computed from the displacement vector U having the components $\left(U_r^{(q)}, U_\varphi^{(q)}\right)$ (see figure 1).

The Green's tensor has the following symmetry property (the Maxwell's theorem of reciprocity of displacements):

$$U_s^{(q)}(M,N) = U_q^{(s)}(N,M). \tag{6}$$

a) Integral representation of the Green's tensor for the elastic circle.

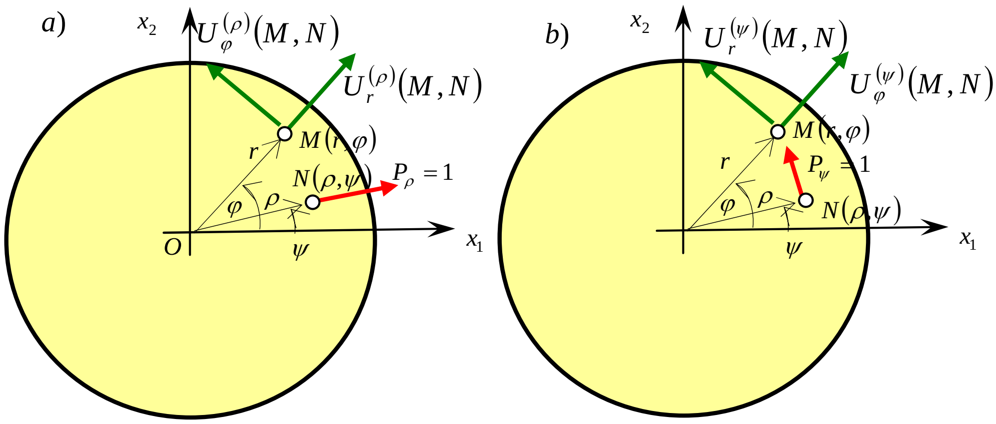

FIGURE 1. a) The displacements $U_S^{(\rho)}(M,N)$ in the point M toward the direction s, $(s=r,\varphi)$ of the Green's tensor, generated by concentrated unit force $P_\rho = 1$, applied in the point N toward the direction ρ;
b) The displacements $U_S^{(\psi)}(M,N)$ in the point M toward the direction s, $(s=r,\varphi)$ of the Green's tensor, generated by concentrated unit force $P_\psi = 1$, applied in the point N toward the direction ψ.

The solution of the previous boundary value problem could be obtained using the incompressible influence element method (IIEM) described in polar or cylindrical coordinates ((Seremet, 2003 [8, 9]), (Seremet, 1997 [7])). Using such solution, the Green's tensor can be written under the form of an integral representation in polar coordinates:

$$\begin{array}{c} U_r^{(q)} = \tilde{U}_r^{(q)} - \frac{\lambda+\mu}{2\mu}\int_0^{2\pi}\Theta^{(q)}\left[r_0\cos\bar{\varphi} - r\right]\frac{\partial G}{\partial n_r} r_0 d\varphi'; \\ U_\varphi^{(q)} = \tilde{U}_\varphi^{(q)} - \frac{\lambda+\mu}{2\mu}\int_0^{2\pi}\Theta^{(q)} r_0 \sin\bar{\varphi}\frac{\partial G}{\partial n_r} r_0 d\varphi'; \quad \bar{\varphi} = \varphi' - \varphi \end{array} \tag{7}$$

In these expressions the displacement components are functions of polar coordinates: $U_r^{(q)} \equiv U_r^{(q)}(r,\varphi;\rho,\psi)$, $U_\varphi^{(q)} \equiv U_\varphi^{(q)}(r,\varphi;\rho,\psi)$; the normal derivative of G: $\partial G/\partial n_\Gamma \equiv \partial G(r_0,\varphi';r,\varphi)/\partial n_\Gamma$ is determined by expressions (3).
The displacements $\tilde{U}_S^{(q)} \equiv \tilde{U}_r^{(q)}(r,\varphi;\rho,\psi)$ are determined as

$$\begin{array}{c} \tilde{U}_r^{(q)} = A\left[(B+1)\left(\delta_{q\rho}\cos\tilde{\varphi} + \delta_{q\psi}\sin\tilde{\varphi}\right) + L^{(q)}\left(r - \rho\cos\tilde{\varphi}\right)\right] G; \\ \tilde{U}_\varphi^{(q)} = A\left[(B+1)\left(\delta_{q\psi}\cos\tilde{\varphi} - \delta_{q\rho}\sin\tilde{\varphi}\right) + L^{(q)}\rho\sin\tilde{\varphi}\right] G, \ \tilde{\varphi} = \varphi - \psi, \end{array} \tag{8}$$

where G is the first Green's function for Dirichlet's problem for Poisson's equation defined in section 2; $A = (\lambda+\mu)/2\mu(\lambda+2\mu)$; $B = (\lambda+3\mu)/\lambda+\mu$; $L^{(q)}$ is the differential operator defined as $L^{(q)} = \delta_{q\rho}\partial/\partial\rho + \delta_{q\psi}\partial/\rho\partial\psi$ and $\delta_{q\rho}$, $\delta_{q\psi}$ are the Kronecker's symbols $\delta_{q\rho} = 1$, $q=\rho$; $\delta_{q\rho} = 0$, $q \neq \rho$; $\delta_{q\psi} = 1$, $q = \psi$; $\delta_{q\rho} = 0$, $q \neq \psi$.

The representations (10) are one of the particular case of the more general integral representations for the Green's tensor of first 3D boundary value problem of elasticity in cylindrical coordinates for the domain V with the boundary Γ, obtained by (Seremet 1995 [6]) and are presented also in the works (Seremet, 1997) [7], (Seremet, 2003b) [9], (Seremet, 2004 [10]).

Using the expressions

$$[r_0\cos\bar{\varphi} - r]\frac{\partial G}{\partial r} = \frac{r_0^2 - r^2}{2\pi r_0}\frac{\partial \ln R}{\partial r}; \ \sin\bar{\varphi}\frac{\partial G}{\partial r} = \frac{r_0^2 - r^2}{2\pi r_0^2 r}\frac{\partial \ln R}{\partial \varphi}; \ \bar{\varphi} = \varphi' - \varphi \tag{9}$$

the representations (10) of the components of the Green's tensor can be written finally as:

$$\begin{aligned} U_r^{(q)} &= \tilde{U}_r^{(q)} + \tfrac{\lambda+\mu}{2\mu}\left(r^2 - r_0^2\right)\tfrac{1}{2\pi}\int_0^{2\pi}\Theta^{(q)}\tfrac{\partial}{\partial r}\ln R d\varphi'; \\ U_\varphi^{(q)} &= \tilde{U}_\varphi^{(q)} + \tfrac{\lambda+\mu}{2\mu}\left(r^2 - r_0^2\right)\tfrac{1}{2\pi\, r}\int_0^{2\pi}\Theta^{(q)}\tfrac{\partial}{\partial \varphi}\ln R d\varphi'. \end{aligned} \tag{10}$$

It is easy already to be seen that the new obtained representations (10) when $r = r_0$ automatically satisfy the boundary conditions (5), because the displacements $\tilde{U}_S^{(q)} = \tilde{U}_S^{(q)}(r, \varphi; \rho, \psi)$ satisfy them due to Green's function G for Poisson's equation; $\Theta^{(q)}$ is given by using derivatives of $U_r^{(q)}$, $U_\varphi^{(q)}$ and the dilatation formula:

$$\Theta^{(q)} = \frac{\partial U_r^{(q)}}{\partial r} + \frac{U_r^{(q)}}{r} + \frac{1}{r}\frac{\partial U_\varphi^{(q)}}{\partial \varphi}. \tag{11}$$

In eqn (10) all is known except the volume dilatation $\Theta^{(q)} \equiv \Theta^{(q)}(r_0, \varphi'; \rho, \psi)$ along the boundary Γ of the circle. It will be shown thereafter that the previous lead to an integral equation where the unknown is the value of $\Theta^{(q)}$ along the boundary Γ. The main problem is to obtain an explicit expression of the function $\Theta^{(q)}$ on the boundary.

b) The volume dilatation on the contour of the circle

The main problem is to get the dilatation computed from the Green's tensor along the boundary of the circle within (10). Let us compute the volume dilatation $\Theta^{(q)} \equiv \Theta^{(q)}(r_0, \varphi; \rho, \psi)$ along the boundary of the circle, using the definition of $\Theta^{(Q)}$. Using the representations (7) or (10) for the displacements $U_r^{(q)}$, $U_\varphi^{(q)}$, the following equation for the volume dilatation is obtained:

$$\Theta^{(q)}(r, \varphi; \rho, \psi) + \frac{\lambda+\mu}{2\mu}\int_0^{2\pi}\Theta^{(q)}(r_0, \varphi'; \rho, \psi)\, L(r_0, \varphi'; r, \varphi)\, r_0 d\varphi' = \tilde{\Theta}^{(q)}(r, \varphi; \rho, \psi) \tag{12}$$

with the kernel

$$L(r_0, \varphi', r, \varphi) = \left[\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)[r_0\cos\bar{\varphi} - r] + \frac{1}{r}\frac{\partial}{\partial \varphi} r_0 \sin\bar{\varphi}\right] - \frac{\partial G(r_0, \varphi', r, \varphi)}{\partial n_{r_0}} - \frac{1}{2\pi r_0} \tag{13}$$

The kernel (13) is obtained using in the eqn (7) the analytical expression for the derivative in respect with exterior normal on counter of the circle in eqn (3) and the rule (11).

Introducing the value (13) of L into eqn (12), using the representation

$$\begin{aligned} &-\int_0^{2\pi}\Theta^{(q)}(r_0, \varphi'; \rho, \psi)\frac{\partial G(r_0, \varphi'; r, \varphi)}{\partial n_{r_0}} r_0 d\varphi' = \\ &= \Theta^{(q)}(r, \varphi; \rho, \psi) + \frac{1}{\lambda + 2\mu} L^{(q)} G(r, \varphi; \rho, \psi) \end{aligned} \tag{14}$$

and passing to the limit from the point $M(r, \varphi)$, $M \in V$ to the point $M'(r_0, \varphi')$, $M' \in \Gamma$, the integral equation in $\Theta^{(q)}$ on the boundary is obtained:

$$\Theta^{(q)}(r_0, \varphi'; \rho, \psi) - \frac{1}{2\pi B}\int_0^{2\pi}\Theta^{(q)}(r_0, \varphi'; \rho, \psi)\, d\varphi' = \frac{2\mu}{\lambda + 3\mu}\tilde{\Theta}^{(q)}(r_0, \varphi'; \rho, \psi)\,. \tag{15}$$

In (15) the term $\tilde{\Theta}^{(q)}(r_0, \varphi'; \rho, \psi)$ can be computed on the basis of the displacements (8) and the formula (11). Using the relations:

$$\begin{aligned} &\sin\tilde{\varphi}'\, \tfrac{\partial G}{\partial r}\Big|_{r=r_0} = \tfrac{r_0^2-\rho^2}{2\pi r_0^2 \rho}\,\tfrac{\partial \ln R}{\partial \psi}\Big|_{r=r_0}; \;\; \tfrac{\partial G}{\partial r}\Big|_{r=r_0} = -\tfrac{1}{2\pi r_0} + \tfrac{2\rho}{2\pi r_0}\tfrac{\partial \ln R}{\partial \rho}\Big|_{r=r_0}; \\ &\cos\tilde{\varphi}'\tfrac{\partial G}{\partial r}\Big|_{r=r_0} = \tfrac{r_0^2+\rho^2}{2\pi r_0^2}\,\tfrac{\partial \ln R}{\partial \rho}\Big|_{r=r_0} - \tfrac{\rho}{2\pi r_0^2};\;\; \tilde{\varphi}' = \varphi' - \psi \end{aligned} \tag{16}$$

we will obtain the dilatation $\tilde{\Theta}^{(q)}$ on the boundary, as:

$$\begin{gathered}\tilde{\Theta}^{(q)}\Big|_{r=r_0}\frac{\partial\tilde{U}_r^{(q)}}{\partial r}\,|_{r=r_0}=A\tfrac{\delta_{q\rho}}{2\pi}\left(1-B\right)\tfrac{\rho}{r_0^2}+A\left\{\tfrac{\delta_{q\rho}}{2\pi}\left[C_2\left(\rho\right)\tfrac{\partial}{\partial\rho}-\right.\right.\\ \left.\left.-\tfrac{C_1(\rho)}{\rho}\tfrac{\partial^2}{\partial\psi^2}\right]+\tfrac{\delta_{q\psi}}{2\pi}C_1\left(\rho\right)\left[\tfrac{(B+1)}{\rho}+\tfrac{\partial}{\partial\rho}\right]\tfrac{\partial}{\partial\psi}\right\}\ln R\,|_{r=r_0}\,,\end{gathered}\tag{17}$$

in which were used the following notations and relation:

$$\begin{gathered}C_1\left(\rho\right)=\left(1-\tfrac{\rho^2}{r_0^2}\right);\;C_2\left(\rho\right)=B\bar{C}_1\left(\rho\right)+C_1\left(\rho\right);\;\bar{C}_1\left(\rho\right)=\left(1+\tfrac{\rho^2}{r_0^2}\right);\\ \rho\tfrac{\partial^2}{\partial\rho^2}\ln R=-\left(\tfrac{\partial}{\partial\rho}+\tfrac{\partial^2}{\rho\partial\psi^2}\right)\ln R.\end{gathered}\tag{18}$$

The solution of the integral equation (15) is:

$$\Theta^{(q)}=\frac{2\mu}{\lambda+3\mu}\left[\tilde{\Theta}^{(q)}+\frac{1}{2\pi\left(1-B\right)}\int_0^{2\pi}\tilde{\Theta}^{(q)}d\varphi'\right]\,.\tag{19}$$

Substituting in (19) $\tilde{\Theta}^{(q)}$ defined by (17), (18) and calculating the integrals

$$\int_0^{2\pi}\frac{\partial}{\partial r}G\left(r_0,\varphi';r,\varphi\right)d\varphi'=-r_0^{-1};\int_a^{2\pi}\ln R\left(r_0,\varphi';r,\varphi\right)d\varphi'=2\pi\ln r_0,\tag{20}$$

we will obtain $\int_0^{2\pi}\tilde{\Theta}^{(q)}d\varphi'=-A\delta_{q\rho}\left(1-B\right)\rho/r_0^2$ and then the final expression for dilatation along the boundary:

$$\begin{gathered}\Theta^{(q)}\big|_{r=r_0}=-AB\tfrac{2\mu}{\lambda+3\mu}\tfrac{\rho}{r_0^2}\tfrac{\delta_{q\rho}}{2\pi}+A\tfrac{2\mu}{\lambda+3\mu}\left\{\tfrac{\delta_{q\rho}}{2\pi}\left[C_2\left(\rho\right)\tfrac{\partial}{\partial\rho}-\right.\right.\\ \left.\left.-\tfrac{C_1(\rho)}{\rho}\tfrac{\partial^2}{\partial\psi^2}\right]+\tfrac{\delta_{q\psi}}{2\pi}C_1\left(\rho\right)\left[\tfrac{(B+1)}{\rho}+\tfrac{\partial}{\partial\rho}\right]\tfrac{\partial}{\partial\psi}\right\}\ln R\,|_{r=r_0}\,.\end{gathered}\tag{21}$$

c) Final closed form expressions of the Green's tensor

To obtain the analytical expressions for the Green's tensor, the expression for the volume dilatation $\Theta^{(q)}\left(r_0,\varphi';\rho,\psi\right)$ (21) must be introduced in (10) and must be compute the integrals:

$$\begin{gathered}\int_a^{2\pi}\frac{\partial}{\partial r}G\left(r,\varphi';\rho,\psi\right)|_{r=r_0}\ln R\left(r_0,\varphi';r,\varphi\right)d\varphi'=-r_0^{-1}\ln\bar{R};\\ \frac{\partial}{\partial r}\int_a^{2\pi}\ln R\left(r_0,\varphi';\rho,\psi\right)\ln R\left(r_0,\varphi';r,\varphi\right)d\varphi'=\frac{\pi}{r}\left(\ln r_0-\ln\bar{R}\right)\,;\\ \frac{\partial}{r\partial\varphi}\int_a^{2\pi}\ln R\left(r_0,\varphi';\rho,\psi\right)\ln R\left(r_0,\varphi';r,\varphi\right)d\varphi'=-\frac{\pi}{r}\operatorname{arctg}\bar{F};\\ \frac{\partial}{r\partial\varphi}\int_a^{2\pi}\frac{\delta_{q\rho}\rho}{2\pi r_0^2}\ln R\left(r_0,\varphi';r,\varphi\right)d\varphi'=0;\quad\frac{\partial}{\partial r}\int_a^{2\pi}\frac{\delta_{q\rho}\rho}{2\pi r_0^2}\ln R\left(r_0,\varphi';r,\varphi\right)d\varphi'=0.\end{gathered}\tag{22}$$

Finally, applying the integrals (22) in (10) and using some of the relations

$$\begin{gathered}\frac{\partial^2}{\partial r\partial\varphi}\operatorname{arctg}\bar{F}=-\rho\frac{\partial^2}{\partial r\partial\rho}\ln\bar{R};\frac{\partial}{\partial r}\ln\bar{R}=\frac{\rho}{r}\frac{\partial}{\partial\rho}\ln\bar{R};\frac{\partial}{\partial\rho}\ln\bar{R}=\frac{r}{\rho}\frac{\partial}{\partial r}\ln\bar{R};\\ \frac{\partial}{\partial\varphi}\operatorname{arctg}\bar{F}=-r\frac{\partial}{\partial r}\ln\bar{R};\frac{\partial}{\partial r}\operatorname{arctg}\bar{F}=\frac{\partial}{r\partial\varphi}\ln\bar{R};\frac{\partial}{\partial\rho}\operatorname{arctg}\bar{F}=\frac{\partial}{\rho\partial\varphi}\ln\bar{R};\\ \frac{\partial}{\partial\varphi}\operatorname{arctg}\bar{F}=-\rho\frac{\partial}{\partial\rho}\ln\bar{R};\bar{F}=\frac{r\rho\sin\tilde{\varphi}}{r_0^2-r\rho\cos\tilde{\varphi}};\quad\bar{R}=\sqrt{r_0^2-2r\rho\cos\tilde{\varphi}+\left(\frac{r\rho}{r_0}\right)^2},\end{gathered}\tag{23}$$

we will obtain the final expressions in a closed form for Green's tensor given as displacement components as:

$$U_r^{(\rho)}=\tilde{U}_r^{(\rho)}+\frac{AC}{2\pi\rho}C_1\left(r\right)\left[C_2\left(\rho\right)+C_1\left(\rho\right)\rho\frac{\partial}{\partial\rho}\right]\frac{\partial}{\partial r}\ln\bar{R};\tag{24}$$

$$U_r^{(\psi)} = \tilde{U}_r^{(\psi)} - \frac{AC}{2\pi\rho} C_1(r) C_1(\rho) \left[\frac{(B+1)}{r} + \frac{\partial}{\partial r} \right] \frac{\partial}{\partial \varphi} \ln \bar{R}; \tag{25}$$

$$U_\varphi^{(\rho)} = \tilde{U}_\varphi^{(\rho)} + \frac{AC}{2\pi\rho} C_1(r) \left[C_2(\rho) \frac{1}{r} + C_1(\rho) \frac{\partial}{\partial r} \right] \frac{\partial}{\partial \varphi} \ln \bar{R}; \tag{26}$$

$$U_\varphi^{(\psi)} = \tilde{U}_\varphi^{(\psi)} + \frac{AC}{2\pi\rho} C_1(r) C_1(\rho) \left[\rho \frac{\partial}{\partial \rho} + (B+1) \right] \frac{\partial}{\partial r} \ln \bar{R}, \tag{27}$$

where

$$C_1(r) = \left(1 - \frac{r^2}{r_0^2} \right); \; C = \frac{B^{-1} r_0^2}{2} \tag{28}$$

and the displacements $\tilde{U}_S^{(q)} = \tilde{U}_S^{(q)}(r, \varphi; \rho, \psi)$ are determined by eqns (8).

Using the relations (23) the expressions (25)-(28) could be written in different commode for applications forms. It is not so difficult to convince that the obtained expressions satisfy to equilibrium equations (5)(in special the singular part of the Green's tensor, which contains singular function $\ln R$, satisfy non homogeneous equilibrium eqns (5), which contains Dirac's function, but remaining it's part is regular and satisfy homogeneous ones), boundary conditions (6) (due to Dirichlet's Green's functions from the displacements $\tilde{U}_s^{(q)}$ and due to function $C_1(r)$ of remaining parts of eqns (24)-(28)) and the Maxwell's theorem of reciprocity of displacements (6). We must notice that to check Maxwell's theorem of reciprocity it is necessary to do certain mathematical transformations, using some of the given relations (23).

So, the expressions (24)-(28) represents the components of Green's tensor for displacements in the first basic boundary value problem (4)-(5) of theory of elasticity for a circle, obtained here in the closed form and for first time.

4. Green's integral formula for the solution of the first boundary value problem of elasticity for a circle

The solution of the first boundary value problem of elasticity is classically expressed from the previously obtained Green's tensor by:

$$\begin{aligned} U_q(\rho, \psi) = & \int_0^{2\pi} \int_0^{r_0} f_S(r, \varphi) U_S^{(q)}(r, \varphi; \rho, \psi) \, r \, dr \, d\varphi - \\ & - \int_0^{2\pi} g_S(\varphi') P_S^{(q)}(r_0, \varphi'; \rho, \psi) \, r_0 d\varphi', \end{aligned} \tag{29}$$

where $s = r, \varphi;\ q = \rho, \psi$ - are polar coordinates, and the index s is the summating index: g_s are the components of the given displacement vectors along the boundary and f_s are the volume density of forces applied within the circle. In this relation, the forces $P_S^{(q)}(r_0, \varphi'; \rho, \psi)$,$s = r, \varphi;\ q = \rho, \psi$ represents the traction vector components on the circular contour, computed from the previously obtained displacement components of the Green's tensor (24)-(28). They can be obtained from the respective expressions in the displacements (24)-(28), using the classical Hooke's law, the eqns for traction via stresses and some of the relations (23) and relation (18). The respective expressions for tractions $P_r^{(\rho)}$ and $P_r^{(\psi)}$ could be obtained using formula (21) for $\Theta^{(q)}|_{r=r_0}$, so that $P_r^{(q)}|_{r=r_0} = (\lambda + 2\mu)\Theta^{(q)}|_{r=r_0}$. The obtained tractions $P_r^{(\rho)}$ and $P_r^{(\psi)}$ from the displacements (24)-(28) or from the last formula via dilatation will coincide if will be taken into account some of the relations (23) and relation (18). The final expressions $P_S^{(q)}(r_0, \varphi'; \rho, \psi)$ in a closed form are given as:

$$P_r^{(\rho)} = -\frac{B^{-1}}{2\pi}\left\{\frac{B\rho}{r_0^2} + \left[C_1(\rho)\frac{\partial^2}{\rho\,\partial\psi^2} - C_2(\rho)\frac{\partial}{\partial\rho}\right]\ln R\right\} \quad (30)$$

$$P_r^{(\psi)} = \frac{B^{-1}}{2\pi}C_1(\rho)\left[\frac{(B+1)}{\rho} + \frac{\partial}{\partial\rho}\right]\frac{\partial}{\partial\psi}\ln R; \quad (31)$$

$$P_\varphi^{(\rho)} = \frac{A\mu}{2\pi}C_1(\rho)\left[\frac{\left(B^{-1}-B\right)}{\rho} + \left(1+B^{-1}\right)\frac{\partial}{\partial\rho}\ln R\right]\frac{\partial}{\partial\psi}\ln R; \quad (32)$$

$$\begin{gathered}P_\varphi^{(\psi)} = \frac{A\mu}{2\pi}\left\{-(B+1)\frac{\rho}{r_0^2} + \left[(B+1)\left(1+\frac{\rho^2}{r_0^2}\right)\frac{\partial}{\partial\rho} + \right.\right. \\ \left.\left. +C_1(\rho)\left(B^{-1}+1\right)\left(\frac{\partial^2}{\rho\partial\psi^2} - \frac{\partial}{\partial\rho}\right)\right]\ln R\right\},\end{gathered} \quad (33)$$

where

$$R = \sqrt{r_0^2 - 2r_0\rho\cos\bar{\varphi}' + \rho^2},\ \bar{\varphi}' = \varphi' - \psi.$$

To be mentioned that integral formulas (29)-(33) at absence of the volume forces represents the generalization of the Poisson's formula (1) from the theory of harmonic potentials for the circle onto theory of elasticity.

Sometimes we need to have the values of the traction components P_ρ, P_ψ on the circular contour generated only by the displacements $g_S(\varphi')$. To do this we must use the formula (29) at absence of the volume forces, formulas (30)-(33) for the tractions, Hooke's low, and to calculate obtained expressions for $P_\rho(\rho,\psi)$, $P_\psi(\rho,\psi)$ when $\rho = r_0$. The final results will be written as:

$$\begin{pmatrix} P_\rho(r_0,\psi) \\ P_\psi(r_0,\psi)\end{pmatrix} = -\int_0^{2\pi}\begin{pmatrix} K_{r\rho}K_{\varphi\rho} \\ K_{r\psi}K_{\varphi\psi}\end{pmatrix}\begin{pmatrix} g_r(\varphi') \\ g_\varphi(\varphi')\end{pmatrix} r_0 d\varphi', \quad (34)$$

where the kernels $K_{Sq} = K_{Sq}(\bar{\varphi})$, $\quad \bar{\varphi} = \varphi' - \psi$ are determinate by the following formulas:

$$K_{r\rho} = \frac{B^{-1}(\lambda+2\mu)}{2\pi r_0^2}\left[\frac{(B+1) - B\cos\bar{\varphi}}{(1-\cos\bar{\varphi})^2}\cos\bar{\varphi} - 1\right]; \quad (35)$$

$$K_{\varphi\rho} = \frac{A\mu}{2\pi r_0^2}\frac{(\lambda+2\mu)\left(B^{-1}-B\right)}{1-\cos\bar{\varphi}}\sin\bar{\varphi};\ K_{r\psi} = \frac{\left(1+B^{-1}\right)\mu}{2\pi r_0^2}\frac{\sin\bar{\varphi}}{1-\cos\bar{\varphi}}; \quad (36)$$

$$K_{\varphi\psi} = \frac{A\mu^2}{2\pi r_0^2}\left[B^{-1} + 1 + \frac{B - B^{-1} - (B+1)\cos\bar{\varphi}}{(1-\cos\bar{\varphi})^2}\cos\bar{\varphi}\right]. \quad (37)$$

5. An example of application in micromechanics of defects in solids

In the following boundary value problem we show one of the applications of the constructed displacements Green's tensor (24)-(28) in micromechanics of defects in solids.

Let us consider the following boundary value problem:

$$\begin{gathered}\mu\left(\Delta U_r - \tfrac{1}{r^2}U_r - \tfrac{2}{r^2}\tfrac{\partial U_\varphi}{\partial\varphi}\right) + (\lambda+\mu)\tfrac{\partial\Theta}{\partial r} = 0; \\ \mu\left(\Delta U_\varphi - \tfrac{1}{r^2}U_\varphi + \tfrac{2}{r^2}\tfrac{\partial U_r}{\partial\varphi}\right) + (\lambda+\mu)\tfrac{\partial\Theta}{r\partial\varphi} = 0\end{gathered} \quad (38)$$

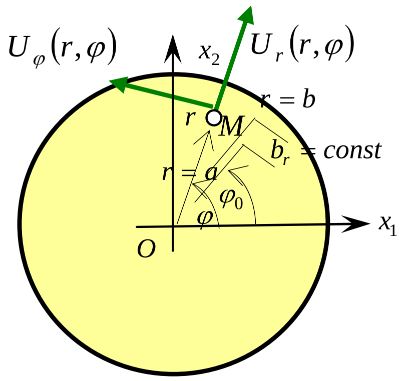

FIGURE 2. The displacements $U_r\left(r,\varphi\right)$, $U_\varphi\left(r,\varphi\right)$ in the point $M\left(r,\varphi\right)$, created by slip Volterra's dislocations $b_r = \text{const}$, applied on the segment of straight line, $\varphi=\varphi_0$, $a \le r \le b$

in respect to the displacements in the inner points (r,φ) of the circle, which are generated by the radial Volterra's slip dislocations $b_r = \text{const}$, which are given on the segment $r \in [a,b]$, $\varphi=\varphi_0$, as (figure 2):

$$\begin{gathered}\lim_{\varphi=\varphi_0}\Delta U_r=\lim_{\varphi=\varphi_0}\left[U_r\left(r,\varphi=\varphi_0+0\right)-U_r\left(r,\varphi=\varphi_0-0\right)\right]=b_r=\text{const}; r\in[a,b]\\ \lim_{\varphi=\varphi_0}\Delta U_r=\lim_{\varphi=\varphi_0}\left[U_r\left(r,\varphi=\varphi_0+0\right)-U_r\left(r,\varphi=\varphi_0-0\right)\right]=0; r\in[0,\ a)\bigcup\left(b,r_0\right]\\ \lim_{\varphi=\varphi_0}\Delta U_\varphi=\lim_{\varphi=\varphi_0}\left[U_\varphi\left(r,\varphi=\varphi_0+0\right)-U_\varphi\left(r,\varphi=\varphi_0-0\right)\right]=0; 0\le r\le r_0\end{gathered} \tag{39}$$

with the fixed boundary:

$$U_r\left(r=r_0,\varphi\right)=U_\varphi\left(r=r_0,\varphi\right)=0. \tag{40}$$

To obtain the solution of this problem we will use the following integral formula

$$U_q\left(\rho,\psi\right)=-b_r\int_a^b\sigma_{r\varphi}^{(q)}\left(r,\rho;\varphi=\varphi_0,\psi\right)dr, \tag{41}$$

where the stresses $\sigma_{r\varphi}^{(q)}\left(r,\rho;\varphi=\varphi_0,\psi\right)$ are determined as:

$$\begin{aligned}&\sigma_{r\varphi}^{(q)}\left(r,\rho;\varphi=\varphi_0,\psi\right)=\sigma_{r\varphi}^{(q)}\left(r,\rho;\varphi,\psi\right)|_{\varphi=\varphi_0}=\\&=\mu\left[r^{-1}\left(U_{r,\varphi}^{(q)}-U_\varphi^{(q)}\right)+U_{\varphi,r}^{(q)}\right]|_{\varphi=\varphi_0}.\end{aligned} \tag{42}$$

Introducing the formula (42) in (41) we will obtain:

$$\begin{aligned}&U_q\left(\rho,\psi\right)=-b_r\mu\left\{\left[U_\varphi^{(q)}\left(b,\rho;\varphi_0-\psi\right)-U_\varphi^{(q)}\left(a,\rho;\varphi_0-\psi\right)\right]+\right.\\&\left.+\int_a^b r^{-1}\left(U_{r,\varphi}^{(q)}-U_\varphi^{(q)}\right)|_{\varphi=\varphi_0}\,dr\right\}.\end{aligned} \tag{43}$$

The expressions for $r^{-1}\left(U_{r,\varphi}^{(q)}-U_\varphi^{(q)}\right)|_{\varphi=\varphi_0}$ in the integral (43) were calculated using the Green's tensor components (24)-(28) and some of the relations (23). So, for them we obtain the following final expressions in commode for integration forms (the integrals will be taken on the variable r, so will be preferable to have derivatives with respect to r):

$$r^{-1}\left(U_{r,\varphi}^{(\rho)}-U_{\varphi}^{(\rho)}\right)|_{\varphi=\varphi_0} = \tfrac{A}{2\pi}\left\{\left[(B+1)\cos\bar{\varphi}_0+\tfrac{\partial}{\partial\rho}\left(r-\rho\cos\bar{\varphi}_0\right)\right]\times \tfrac{\partial}{r\partial\varphi}\ln\tfrac{\bar{R}}{R}+\tfrac{C}{\rho}C_1(r)\left[C_2(\rho)\tfrac{\partial^2}{\partial r^2}\operatorname{arctg}\bar{F}+ C_1(\rho)\tfrac{\partial^3}{\partial\varphi\partial r^2}\ln\bar{R}\right]\right\}|_{\varphi=\varphi_0}\ ; \tag{44}$$

and

$$r^{-1}\left(U_{r,\varphi}^{(\psi)}-U_{\varphi}^{(\psi)}\right)|_{\varphi=\varphi_0} = \tfrac{A}{2\pi}\left\{\left[(B+1)\sin\bar{\varphi}_0+\tfrac{\partial}{\rho\partial\psi}\left(r-\rho\cos\bar{\varphi}_0\right)\right]\times \tfrac{\partial}{r\partial\varphi}\ln\tfrac{\bar{R}}{R}+\tfrac{C}{\rho}C_1(r)\,C_1(\rho)\left[(B+1)\tfrac{\partial^2}{\partial r^2}\ln\bar{F}- \tfrac{\partial^3}{\partial\varphi\partial r^2}\operatorname{arctg}\bar{F}\right]\right\}|_{\varphi=\varphi_0}\ ;\bar{\varphi}_0=\varphi_0-\psi. \tag{45}$$

To obtain the commode for integration expressions (44), (45) were used such formulas as:

$$\tfrac{1}{r}\left(\tfrac{\partial}{\partial r}-\tfrac{1}{r}\right)\tfrac{\partial}{\partial\varphi}\ln\bar{R}=\tfrac{\partial^2}{\partial r^2}\operatorname{arctg}\bar{F};\ \tfrac{1}{r}\left(\rho\tfrac{\partial}{\partial\rho}-1\right)\tfrac{\partial^2}{\partial r\partial\varphi}\ln\bar{R}=\tfrac{\partial^3}{\partial\varphi\partial r^2}\ln\bar{R};$$
$$-\tfrac{1}{r}\left(\tfrac{\partial}{\partial r}+\tfrac{1}{r}\tfrac{\partial^2}{\partial\varphi^2}\right)\ln\bar{R}=\tfrac{\partial^2}{\partial r^2}\ln\bar{R};\ \tfrac{1}{r}\tfrac{\partial}{\partial r}\left(\rho\tfrac{\partial}{\partial\rho}+\tfrac{\partial^2}{\partial\varphi^2}\right)\ln\bar{R}=\tfrac{\partial^3}{\partial\varphi\partial r^2}\operatorname{arctg}\bar{F},$$

which follows from respective relations (23).

If we will introduce (26), (44) and (45) in (43) and if we will compute appears integrals (see appendix 1), we will obtain the following expressions for displacements $U_q(\rho,\psi)$ of boundary value problem (38)-(40) containing different derivatives:

$$\begin{aligned}\mathrm{U}_\rho(\rho,\bar{\varphi}_0)=-\frac{Ab_r\mu}{2\pi}&\left\{\left[(1-B)\sin\bar{\varphi}_0\ln\frac{\bar{R}}{R}+(B+1)\cos\bar{\varphi}_0\left(\operatorname{arctg}\bar{\Pi}-\operatorname{arctg}\Pi\right)\right]+\right.\\ &\left[\frac{c}{\rho}\left(C_2(\rho)\,C_1(r)\,\frac{1}{r}+\frac{2C_1(\rho)}{\rho}\cos\bar{\varphi}_0\right)\cdot\frac{\partial}{\partial\varphi}+\left(r_0^2+\rho^2\right)\sin\bar{\varphi}_0\frac{\partial}{\rho\partial\rho}+\right.\\ &\left.+\frac{2c}{\rho}C_1(\rho)\,C_1(r)\,\frac{\partial^2}{\partial\varphi\partial r}\right]\ln\bar{R}-2\sin\bar{\varphi}_0\rho\frac{\partial}{\partial\rho}\ln R+\\ &\left.\left[\frac{\left(r_0^2-\rho^2\right)}{\rho}\cos\bar{\varphi}_0\frac{\partial}{\partial\rho}-\frac{2c}{\rho^2}C_1(\rho)\sin\bar{\varphi}_0\frac{\partial}{\partial\varphi}\right]\operatorname{arctg}\bar{\Pi}+\frac{c}{\rho}C_2(\rho)\,C_1(r)\,\frac{\partial}{\partial r}\operatorname{arctg}\bar{F}\right\}|_{r=a}^{r=b}\end{aligned} \tag{46}$$

– for radial displacement, and

$$\begin{aligned}\mathrm{U}_\psi(\rho,\bar{\phi}_0)=-\frac{Ab_r\mu}{2\pi}&\left\{\left[\left(B-\frac{r_0^2}{\rho^2}\right)\cos\bar{\phi}_0+\ \left(\frac{r_0^2}{\rho^2}-1\right)\right]\ln\bar{R}+(1-B)\cos\bar{\phi}_0\ln R\right.\\ &+\left[\left(B-1+\frac{2r_0^2}{\rho^2}\right)\operatorname{arctg}\bar{\Pi}-(B+1)\operatorname{arctg}\Pi\right]\sin\bar{\phi}_0+\frac{r}{\rho}\left(\frac{B+1}{B}\right)\left(1-\frac{\rho^2}{r_0^2}\right)+\\ &\left[\frac{r_0^2}{B\rho}C_1(\rho)\,C_1(r)\left((B+1)\frac{\partial}{\partial r}-\frac{1}{\rho}\sin\bar{\phi}_0\frac{\partial}{\partial\phi}\right)+\left(1+\frac{r_0^2}{\rho^2}\right)\sin\bar{\phi}_0\frac{\partial}{\partial\psi}\right]\ln\bar{R}-\\ &2\sin\bar{\phi}_0\frac{\partial}{\partial\psi}\ln R+\frac{B+1}{B}\cos\bar{\phi}_0\left(\frac{r_0^2}{\rho^2}-1\right)\frac{\partial}{\partial\psi}\operatorname{arctg}\bar{\Pi}-\\ &\left.-\frac{c}{\rho}C_1(\rho)\,C_1(r)\,\frac{\partial^2}{\partial r\partial\phi}\operatorname{arctg}\bar{F}\right\}|_{r=a}^{r=b}.\end{aligned} \tag{47}$$

– for tangential displacement, where

$$\bar{\Pi}=\frac{r\rho-r_0^2\cos\bar{\varphi}_0}{r_0^2\sin\bar{\varphi}_0};\ \Pi=\frac{r-\rho\cos\bar{\varphi}_0}{\rho\sin\bar{\varphi}_0};\ \bar{F}=\frac{r\rho\sin\bar{\varphi}_0}{r_0^2-r\rho\cos\bar{\varphi}_0}. \tag{48}$$

Calculating the derivatives in the given above displacements (46)-(48) we will obtain the final expressions of the boundary value problem (38)-(40) in a closed form as:

$$\begin{gathered}\mathrm{U}_\rho\left(\rho,\bar{\phi}_0\right)=-\frac{Ab_r\mu}{2\pi}\left\{\left[(1-B)\sin\bar{\phi}_0\ln\frac{\bar{R}}{R}+(B+1)\cos\bar{\phi}_0\left(\operatorname{arctg}\bar{\Pi}-\operatorname{arctg}\Pi\right)\right]\right.\\ +\frac{2\rho\sin\bar{\phi}_0\left(r\cos\bar{\phi}_0-\rho\right)}{R^2}+\\ \left[\frac{\left(r_0^2-\rho^2\right)}{B}\cdot\left(\frac{2r\cos\bar{\phi}_0}{\rho}+2C_1(r)-\left(\frac{r_0}{\rho}\right)^2\right)+r_0^2+\rho^2-2r\rho\cos\bar{\phi}_0\right]\frac{\sin\bar{\phi}_0}{\bar{R}^2}+\\ \left.+\frac{\left(r_0^2-\rho^2\right)C_1(r)}{B}\left[\left(\frac{r\rho}{r_0}\right)^2-r\rho\cos\bar{\phi}_0\right]\frac{\sin\bar{\phi}_0}{\bar{R}^4}\right\}\Big|_{r=a}^{r=b},\end{gathered}\tag{49}$$

for radial ($q=\rho$) displacements $U_\rho(\rho,\psi)$, and

$$\begin{gathered}\mathrm{U}_\psi\left(\rho,\bar{\phi}_0\right)=-\frac{Ab_r\mu}{2\pi}\left\{\left[\left(B-\frac{r_0^2}{\rho^2}\right)\cos\bar{\phi}_0+\left(\frac{r_0^2}{\rho^2}-1\right)\right]\ln\bar{R}+(1-B)\cos\bar{\phi}_0\ln R\right.\\ +\left[\left(B-1+\frac{2r_0^2}{\rho^2}\right)arctg\bar{\Pi}-(B+1)\,arctg\Pi\right]\sin\bar{\phi}_0+\frac{r}{\rho}\left(\frac{B+1}{B}\right)\left(1-\frac{\rho^2}{r_0^2}\right)+\\ \left[\frac{r_0^2}{B\rho}C_1(\rho)\,C_1(r)\left((B+1)\left(\frac{r\rho^2}{r_0^2}-\rho\cos\bar{\phi}_0\right)-r\sin^2\bar{\phi}_0+\frac{1}{2}\left(\frac{2r\rho^2}{r_0^2}-\rho\cos\bar{\phi}_0\right)\right)\right.\\ \left.-\left(1+\frac{r_0^2}{\rho^2}\right)r\rho\sin^2\bar{\phi}_0+\frac{B+1}{B}\cos\bar{\phi}_0\left(r\rho\cos\bar{\phi}_0-r_0^2\right)\left(\frac{r_0^2}{\rho^2}-1\right)\right]\frac{1}{\bar{R}^2}+\\ \left.\frac{2r\rho\sin^2\bar{\phi}_0}{R^2}-\frac{r_0^2}{B}\frac{r\rho\left(\cos\bar{\phi}_0-r\rho\,r_0^{-2}\right)^2}{\bar{R}^4}\right\}\Big|_{r=a}^{r=b}.\end{gathered}\tag{50}$$

for tangential ($q=\psi$) displacements $U_\psi(\rho,\psi)$.

It is not extremely difficult to check that displacements satisfy eqns (38)-(40). When will be checked equilibrium equations (38) will be useful to remember that the functions: $\ln\bar{R}$, $\operatorname{arctg}\bar{\Pi}$, $\operatorname{arctg}\bar{F}$ and others ones are harmonically functions.

The boundary conditions (40) on the fixed circumference are satisfied by both displacements U_ρ and U_ψ, when $\rho=r_0$. To check this confirmation is useful to take into account that the differences: $\bar{R}-R$; $\bar{\Pi}-\Pi$; $\ln\bar{R}-\ln R$; $\operatorname{arctg}\bar{\Pi}-\operatorname{arctg}\Pi$ and the function $C_1(\rho)$ are equal to zero, when $\rho=r_0$. The conditions of discontinuity of the displacements U_ρ when $\varphi=\varphi_0$, $a\le\rho\le b$ due to the Volterra's radial dislocations $b_r=$ const definite by eqn (39), are satisfied by term $\frac{Ab_r\mu}{2\pi}(B+1)\cos\bar{\varphi}_0\operatorname{arctg}\Pi$ of eqns (46) and (49), because

$$\begin{gathered}\frac{Ab_r\mu}{2\pi}(B+1)\lim_{\psi=\phi_0}\left(\cos\bar{\phi}_0\operatorname{arctg}\Pi\big|_{\psi=\phi_0+0}-\cos\bar{\phi}_0\operatorname{arctg}\Pi\big|_{\psi=\phi_0-0}\right)\Big|_{r=a}^{r=b}=\\ =\begin{cases}b_r,\ \text{when}\ a\leqslant\rho\leqslant b\\ 0,\ \text{when}\ \rho\in[0,\ a)\cup(b,\ r_0]\end{cases}\end{gathered}\tag{51}$$

The analogical limits (39) from the others remaining terms of displacements U_ρ and U_ψ are equal to zero (continuity conditions for displacements U_ψ and remaining terms for U_ρ).

The obtained by present in literature solutions of boundary value problems in closed form in mechanics of defects with Volterra's dislocations are referred only to unbounded solids (see as examples solutions for the space and half space with Volterra's dislocations which are given in the monograph by T. Mura, 1991 [4]). The obtained in this paper in elementary functions solution presents as we know the first result in micromechanics of defects obtained for the bounded solids acted by Volterra's slip dislocations. Before has been never obtained any such kind of solution for bounded domain.

6. Conclusions

1. Proposed by Seremet (1995, 1997, 2003, 2004 [6, 7, 8, 9, 10]) IIEM applied here to derive the Green's tensor for the circle is essentially more efficient then method of elastic potentials (MEP). The efficiently consist in:

A. It is necessary to solve only one regular boundary integral equation with respect to the dilatation on the boundary: - in IIEM, and a system of two singular boundary integral equations with respect to densities on the boundary:-in MEP;

B. The number and complicity of boundary integrals needed to be computed in IIEM are considerable less then in MEP.

C. Green's tensor obtained by IIEM is presented in a compact form: in the form of some linear differential operators from two elementary functions, that is very suitable for applications.
In the MEP Green's tensor can be obtained in a bulk form and has very long expressions via a lot of elementary functions.

D. Beside this IIEM can be successfully applied to derive the Green's tensor not only for the first basic problem for an elastic circle, but also for the wide classes of boundary value problems. The MEP is more suitable to obtain numerical solutions of the problems.

2. Presented partially here IIEM can be successfully applied to derive the Green's tensor for the wide class of boundary value problems (including locally mixed boundary conditions) of elasticity for many others canonical domains of polar system of coordinates (such as half and quarter of the circle, circular sector, circular layer etc). To carry out this we need to apply some others, then (7) integral representations for Green's tensor related to considered boundary value problem. All these integral representations can be easy obtained from the more general integral representations for Green's tensor in cylindrical coordinates, proposed by (Seremet 1995 [6]), (Seremet 1997 [7]), (Seremet 2003b [9]) and (Sheremet 2004 [10]). The Green's functions for Poisson's equation for many canonical domains of polar system of coordinates, which contains in these integral representations, are already known or can be obtained in a closed form.

3. The presented here Green's tensor and the solution of the boundary value problem with dislocation are obtained for the first time. The obtained in closed form solution for the problem with dislocation presents the first problem in micromechanics of defects for which is obtained in general for bounded domain. Before has not been obtained any such kind of solution for bounded domain.

4. As IIEM permit to derive Green's tensor for a lot of bounded elastic bodes of Cartesian, polar, cylindrical, spherical and others systems of orthogonal coordinates it became possible to solve many new boundary value problems (not only for the first basic problem for elastic circle) of micromechanics of defects as Volterra's dislocations, inclusions, eigenstrains, temperature dislocations etc.

Appendix A. Computing of some integrals

$$\int_a^b \frac{\partial}{r\partial\varphi} \ln \bar{R}\,|_{\varphi=\varphi_0}\, dr = \operatorname{arctg} \bar{\Pi}; \int_a^b \frac{\partial}{r\partial\varphi} \ln R\,|_{\varphi=\varphi_0}\, dr = \operatorname{arctg} \Pi$$

$$\int_a^b C_1(r)\frac{\partial^2}{\partial r^2}\operatorname{arctg}\bar{F}\,|_{\varphi=\varphi_0}\,dr = C_1(r)\frac{\partial}{\partial r}\operatorname{arctg}\bar{F}\,|_{\varphi=\varphi_0}\,|_{r=a}^{r=b} + \frac{2}{r_0^2}\int_a^b r\frac{\partial}{\partial r}\operatorname{arctg}\bar{F}\,|_{\varphi=\varphi_0}\,dr$$

$$\int_a^b r\frac{\partial}{\partial r}\operatorname{arctg}\bar{F}\,|_{\varphi=\varphi_0}\,dr = \int_a^b \frac{\partial}{\partial \varphi}\ln\bar{R}\,|_{\varphi=\varphi_0}\,dr = \frac{r_0^2}{2\rho}\sin\bar{\varphi}_0\left[\ln\bar{R}^2 + 2\operatorname{ctg}\bar{\varphi}_0\operatorname{arctg}\,\bar{\Pi}\right]\,|_{r=a}^{r=b}$$

$$\int_a^b C_1(r)\frac{\partial^2}{\partial r^2}\ln\bar{R}\,|_{\varphi=\varphi_0}\,dr = C_1(r)\frac{\partial}{\partial r}\ln\bar{R}\,|_{\varphi=\varphi_0}\,|_{r=a}^{r=b} + \frac{2}{r_0^2}\int_a^b r\frac{\partial}{\partial r}\ln\bar{R}\,|_{\varphi=\varphi_0}\,dr$$

$$\int_a^b r\frac{\partial}{\partial r}\ln\bar{R}\,|_{\varphi=\varphi_0}\,dr = r\,|_a^b + \frac{r_0^2}{2\rho}\cos\bar{\varphi}_0\left[\ln\bar{R}^2 - 2tg\bar{\varphi}_0\operatorname{arctg}\bar{\Pi}\right]\,|_{r=a}^{r=b}$$

where

$$\bar{\Pi} = \frac{r\rho - r_0^2\cos\bar{\phi}_0}{r_0^2\sin\bar{\phi}_0};\ \Pi = \frac{r-\rho\cos\bar{\phi}_0}{\rho\sin\bar{\phi}_0};\ \bar{F} = \frac{r\rho\sin\bar{\phi}_0}{r_0^2 - r\rho\cos\bar{\phi}_0};$$

$$\bar{R} = \sqrt{r_0^2 - 2r\rho\cos\bar{\phi}_0 + (r\rho/r_0)^2};\ R = \sqrt{r^2 - 2r\rho\cos\bar{\phi}_0 + \rho^2};\ \bar{\phi}_0 = \phi_0 - \psi.$$

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