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ON OPTIMIZING LINEAR MULTIPLE REGRESSION MODELS USING STEPWISE REGRESSION

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ABSTRACT. This article is focused on the problem of optimizing linear multiple regression models using the stepwise regression method. Thus, the technique for constructing the computation matrix is initially presented, then, the optimization algorithm is described, and finally, the method for determining the elements that are not contained in the optimal regression model is presented. The last part of this paper contains a case study where the optimization algorithm is applied on a dataset containing authentic territorial statistical data.

1. Introduction

The stepwise regression method[1] is a combination between the methods of backward elimination and forward selection. The idea behind the method is to enlarge the regression model as long as there are still some factorial variables outside the model that have a significant influence on the target variable, while at the same time ensuring that the model does not contain variables that, because of cumulated influences from other factorial variables (possibly recently inserted in the model), have an insignificant influence on the target variable.

The stepwise regression method is very advantageous because, compared with other methods, it generally provides a faster way of computing the optimal regression model as, at each step, it analyzes all the variables, included or not included in the model, and this ensures the fact that no important variable is left outside the model and no insignificant variable will be contained in the final model.

2. The stages of optimization when using stepwise regression

The optimization process of linear multiple regression models using the method of stepwise regression can be divided in three main stages:

(1) the construction of the computation matrix;
(2) the run of the optimization algorithm;
(3) the calculation of the optimal model and of the elements left outside of this model.

2.1. **The construction of the computation matrix.** **Step 1.** The selection means and standard deviations of the factorial variables $X_1, X_2, \ldots, X_p$ and of the target variable Y are computed using the formulae:

$$\bar{x}_i = \frac{1}{n}\sum_{k=1}^{n} x_{ki} \ ; \ \forall\ i = \overline{1,p}; s_{X_i} = \sqrt{\frac{\sum_{k=1}^{n}(x_{ki}-\bar{x}_i)^2}{n-1}} \ ; \ \forall\ i = \overline{1,p}$$

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[1] The algorithm of the method has been described, in an initial draft by Efroymson M.A. in 1960.

$$\bar{y} = \frac{1}{n}\sum_{k=1}^{n} y_k \, ; s_Y = \sqrt{\frac{\sum_{k=1}^{n}(y_k - \bar{y})^2}{n-1}} \, .$$

Step 2. The correlation coefficients between the factorial variables $X_1, X_2, \ldots, X_p$, are computed using the formula:

$$r_{ij} = r_{X_i/X_j} = \frac{\sum_{k=1}^{n}(x_{ki} - \bar{x}_i)(x_{kj} - \bar{x}_j)}{(n-1)\cdot s_{X_i}\cdot s_{X_j}} \; ; \; \forall\, i = \overline{1,p} \; , \; \forall\, j = \overline{1,p}$$

and thus the correlation matrix $R = \begin{pmatrix} r_{11} & r_{12} & \ldots & r_{1p} \\ r_{21} & r_{22} & \ldots & r_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ r_{p1} & r_{p2} & \ldots & r_{pp} \end{pmatrix}$ is obtained.

Step 3. The correlation coefficients between the target variable Y and the factorial variables$X_1, X_2, \ldots, X_p$, are computed using the formula:

$$t_j = r_{Y/X_j} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})(x_{ij} - \bar{x}_j)}{(n-1)\cdot s_Y \cdot s_{X_j}} \; , \; \forall\, j = \overline{1,p}$$

and the vector $T_{1\times p}^T = (\, t_1, \; t_2 \, , \; \ldots \; , \, t_p \,)$of the correlations between the target variable and the factorial variables is obtained.

Step 4. Matrix$A$ defined as: $A = \begin{pmatrix} R_{p\times p} & T_{p\times 1} & I_{p\times p} \\ T_{1\times p}^T & s_{1\times 1} & O_{1\times p}^T \\ -I_{p\times p} & O_{p\times 1} & \Theta_{p\times p} \end{pmatrix}$ is constructed: R and T are the matrix and the vector previously defined, $I_{p\times p}$ is the unit matrix of order p, $s_{1\times 1}$ is $r_{Y/Y} = 1$,$O_{p\times 1}$ is the null vector with p elements, and $\Theta_{p\times p}$ is the null matrix of order p. As such, matrix A is:

$$A = \begin{pmatrix} r_{11} & r_{12} & \ldots & r_{1p} & t_1 & 1 & 0 & \ldots & 0 \\ r_{21} & r_{22} & \ldots & r_{2p} & t_2 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ r_{p1} & r_{p2} & \ldots & r_{pp} & t_p & 0 & 0 & \ldots & 1 \\ t_1 & t_2 & \ldots & t_p & 1 & 0 & 0 & \ldots & 0 \\ -1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} .$$

2.2. **The optimization algorithm.** The optimization algorithm employed by the *stepwise regression* method operates on the matrix A and is based on two inter-twined stages: *the insertion of a factorial variable in the model* and, respectively, *the elimination of a factorial variable from the model.* The initial premise is that the model doesn't contain any factorial variables. We will mark by J_q the set containing the indices of the factorial variables included in the model; initially $q = 0$ and$J_q = J_0 = \Phi$ (the empty set) and in fact q represents the number of degrees of freedom for SP_{reg} corresponding to the current model.

2.2.1. *The insertion of a factorial variable in the model.* In order to insert a new factorial variable in the model, one must follow these steps: the selection of the variable to be tested for insertion, the statistical testing with regards to the admittance criterion and, eventually, the extension of the regression model through the updating of the matrix A and of the set containing the indices included in the model.

1. ***The selection of the variable to be tested for insertion***. Using matrix A we compute the next statistics for all the variables that are not included in the model:

$W_i = \frac{a_{i,p+1} \cdot a_{p+1,i}}{a_{ii}}$, $\forall\, i \notin J_q$ **,** and afterwards we .select the maximum value. Let there be k such as $W_{\max} = W_k = \max\limits_{i \notin J_q} \{W_i\}$. In this case, variable X_k that corresponds to W_k can be included in the model if it satisfies the admittance criterion.

2. ***The testing of the admittance criterion***. The $F^*_{X_k}$ statistic for the partial F test that is used to test the admittance criterion, is given by:

$$F^*_{X_k} = \frac{n_g \cdot W_{\max}}{a_{p+1,p+1} - W_{\max}} = \frac{n_g \cdot W_k}{a_{p+1,p+1} - W_k},$$

where n_g is the number of degrees of freedom for SP_{rez} after the insertion of the new variable, i.e. $n_g = (n-1)-(q+1) = n-q-2$. If $F^*_{X_k} > F_{\alpha;1,n_g}$, where α is the significance level, then the variable X_k satisfies the admittance criterion and will be inserted in the model.

3. ***The extension of the model through the insertion of variable*** X_k**.** The extension of the model requires updating matrix A, through a swiveling procedure that we shall name **swivelinp** and updating the set containing the indices included in the model.

a) Through the procedure **swivelinp**, a new configuration of the matrix $A = (a_{ij})_{2p+1,2p+1}$ will be obtained by following the next steps:

P1. $piv \leftarrow a_{kk}$,

P2. $a_{kj} \leftarrow \frac{a_{kj}}{piv}$; $\forall\, j = \overline{1, 2p+1}$,

P3. For $\forall\, i = \overline{1, 2p+1}$ do:

i) $b \leftarrow a_{ik}$;

ii) $a_{ij} \leftarrow a_{ij} - a_{kj} \cdot b$; $\forall\, j = \overline{1, 2p+1}$, if $i \neq k$.

P4. $a_{p+1+k,k} \leftarrow a_{kk}$.

b) The value of q is increased by one unit ($q \leftarrow q+1$) and the new set containing the indices included in the model will be $J_q = J_{q-1} \bigcup \{k\}$.

2.2.2. *The elimination of a factorial variable from the model.* The elimination of a factorial variable from the model consists of these steps: the selection of the variable to be tested for elimination, the statistical testing with regards to the elimination criterion and, eventually, the reduction of the regression model through the updating of the matrix A and of the set containing the indices included in the model.

1. ***The selection of the variable to be tested for removal***. Using the elements of matrix A we compute for all the variables contained in the model the next statistics[2]:

$$F^*_{X_i} = \frac{a_i^2}{s^2(a_i)} = \frac{a^2_{i,p+1} \cdot \frac{s_Y^2}{s^2_{X_i}}}{a_{i,p+1+i} \cdot \frac{s_u^2}{s^2_{X_i}} \cdot \frac{1}{n-1}} =$$

$$= \frac{(n-1) \cdot a^2_{i,p+1} \cdot s_Y^2}{a_{i,p+1+i} \cdot s_u^2} = \frac{(n-1) \cdot a^2_{i,p+1} \cdot \frac{c}{n-1}}{a_{i,p+1+i} \cdot \frac{a_{p+1,p+1} \cdot c}{n_g}}, \forall i \in J_q$$

i.e.

$$F^*_{X_i} = \frac{n_g \cdot a^2_{i,p+1}}{a_{p+1,p+1} \cdot a^*_{i,p+1+i}} = \frac{(n-q-1) \cdot a^2_{i,p+1}}{a_{p+1,p+1} \cdot a_{i,i+p+1}} \ , \ \forall\, i \in J_q,$$

and then we determine k such that $F^*_{X_k} = \min\limits_{i \in J_q} \left\{F^*_{X_i}\right\}$. In this case, variable X_k corresponding to $F^*_{X_k}$ can be removed from the regression model if it satisfies the elimination criterion.

[2] See formulae (1) and (2) from §2.3 for c, s_Y^2, s_u^2.

2. ***The testing of the elimination criterion***. The $F^*_{X_k}$ statistic has a F distribution with 1 and $(n-q-1)$ degrees of freedom. The X_k variable satisfies the elimination criterion if $F^*_{X_k} < F_{\alpha;1,n-q-1}$, in which case it will be removed from the model, thus helping to obtain a more reduced regression model.

3. ***The reduction of the model through the elimination of variable*** X_k**.** The reduction of the model requires updating matrix A, through a swiveling procedure that we shall name **swivelout** and updating the set containing the indices included in the model..

a) Through the procedure **swivelout**, a new configuration of the matrix $A = (a_{ij})_{2p+1,2p+1}$ will be obtained by following the next steps:

P1. $piv \leftarrow a_{p+1+k,p+1+k}$,

P2. $a_{p+1+k,j} \leftarrow \frac{a_{p+1+k,j}}{piv}$; $\forall\, j = \overline{1, 2p+1}$,

P3. For $\forall\, i = \overline{1, 2p+1}$ do:

i) $b \leftarrow a_{ik}$,

ii) $a_{ij} \leftarrow a_{ij} - a_{p+1+k,j} \cdot b$; $\forall\, j = \overline{1, 2p+1}$, if $i \neq p+1+k$,

P4. $a_{kj} \leftrightarrow a_{p+1+k,j}$; $\forall\, j = \overline{1, 2p+1}$, i.e. swap lines k and $(p+1+k)$.

P5. $a_{p+1+k,k} \leftarrow -1$.

Observation. The transformation defined by the above steps is in fact the inverse transformation of the one performed when variable X_k has been inserted in the model, meaning that, if before inserting X_k, matrix A was in the state A', and after the insertion it changed its state to A^*, by removing X_k, matrix A will return to state A'. This aspect is very important because it offers the possibility to exclude a variable that has been previously included in the model and to re-include in the model a variable that has been previously excluded.

b) The value of q is decreased by one unit $(q \leftarrow q-1)$, and the new set containing the indices included in the model will be $J_q = J_{q+1} \setminus \{k\}$.

The above stages are repeated successively until no more variables can be inserted or removed from the model[3]**.**

2.3. **Determining the optimal model and the additional elements of the regression.** At any given stage of the algorithm, matrix A is structured in a similar manner as its initial form, meaning:

$$A = \begin{pmatrix} B_{p\times p} & C_{p\times 1} & F_{p\times p} \\ D^T_{1\times p} & f_{1\times 1} & -G^T_{1\times p} \\ Z_{p\times p} & V_{p\times 1} & P_{p\times p} \end{pmatrix}$$

, and each of its corresponding blocks has a particular meaning after performing a swiveling procedure. Thus:

(1) the vectors $V$and G^T contained the normalized regression coefficients, and the zero values correspond to the variables that are not included in the model at that stage;

(2) the matrix P from which we eliminate the null lines and columns is the inverse of the correlation matrix of the variables that are included in the model at that stage;

(3) the non-zero elements of vector V identified by $a_{i+p+1,p+1}$, $\forall\, i \in J_q$ can also be found in vector C where they are identified by $a_{i,p+1}$, $i \in J_q$ and as such $a_{i+p+1,p+1} = a_{i,p+1} = -a_{p+1,i+p+1}$, $\forall\, i \in J_q$;

(4) the non-zero elements of matrix P identified by $a_{i+p+1,j+p+1}$, $\forall\, i,j \in J_q$ can also be found in matrix F where they are identified by $a_{i,j+p+1}$, $\forall\, i,j \in J_q$ and as such $a_{i+p+1,j+p+1} = a_{i,j+p+1}$, $\forall\, i,j \in J_q$.

[3] If in the end $J_q = \Phi$, we shall choose as the best model $Y = \bar{y}$.

From matrix A, structured in the above manner, one can obtain all the necessary information after inserting or removing variable X_k in/from the model. The data are in a normalized form and in order to obtain the real values, whenever the case, one must multiply them by with the mean adjusted sum of squares of the target variable Y, i.e. multiply by

$$c = \sum_{i=1}^{n} y_i^2 - \frac{1}{n} \cdot \left(\sum_{i=1}^{n} y_i \right)^2 . \tag{1}$$

The following relations are true:

$$\begin{cases} SP_{reg} = (1 - a_{p+1,p+1}) \cdot c \\ SP_{rez} = a_{p+1,p+1} \cdot c \\ SP_{total} = SP_{reg} + SP_{rez} = c \\ s_Y = \sqrt{\frac{SP_{total}}{n-1}} = \sqrt{\frac{c}{n-1}} \end{cases} ; \begin{cases} n_g = n - q - 1 \\ R^2 = 1 - \frac{SP_{rez}}{SP_{total}} = 1 - a_{p+1,p+1} \\ s_u^2 = \frac{SP_{rez}}{n_g} = \frac{a_{p+1,p+1} \cdot c}{n-q-1} \\ F^* = \frac{SP_{reg}}{SP_{rez}} \cdot \frac{n_g}{q} = \frac{SP_{reg}}{SP_{rez}} \cdot \frac{n-q-1}{q} \end{cases} \tag{2}$$

The coefficients of the variables X_i, included in the model, are determined by:

$$a_i = a_{i,p+1} \cdot \frac{s_Y}{s_{X_i}}, \ \forall\, i \in J_q, \tag{3}$$

and the constant term of the regression equation is always obtained by:

$$a_o = \bar{y} - \sum_{i \in J_q} a_i \bar{x}_i \, . \tag{4}$$

The variances and standard errors of the estimated coefficients are obtained using the formulae:

$$s^2(a_i) = a_{i,i+p+1} \cdot \frac{s_u^2}{s_{X_i}^2} \cdot \frac{1}{n-1} = a_{i+p+1,i+p+1} \cdot \frac{s_u^2}{s_{X_i}^2} \cdot \frac{1}{n-1} \ , \ \forall\, i \in J_q, \tag{5}$$

and

$$e_i = \sqrt{s^2(a_i)} = \frac{s_u}{s_{X_i}} \sqrt{\frac{a_{i+p+1,i+p+1}}{n-1}} \ , \ \forall\, i \in J_q. \tag{6}$$

3. Case study

For the 1992 territorial statistical data presented in Table 1, optimization through the method of stepwise regression should be carried out like this:

I. The construction of the computation matrix

(1) the selection means and the standard deviations of the factorial variables and of the target variable are

Variables	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Y
$\bar{x}_i$ / $\bar{y}$	60.526	74.2168	8.7320	10.8683	123.5766	37.7168	34.9298	21
s_{X_i} / s_Y	1.7187	4.5123	3.0427	3.2402	14.7783	11.6094	8.6205	11.9791

[4] X_1 - the percentage of work resources in the total population; X_2 - the employment rate of the work force; X_3 - unemployment rate; X_4 - women unemployment rate; X_5 - the economic dependency rate; X_6 - rate of the population working in the primary sector; X_7 - rate of the population working in the secondary sector.

[5] Y - the final rank obtained in the hierarchy constructed through the rankings method

TABLE 1

County	Factorial variables[4]							Target variable[5]
	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Y
BC	60.8437	67.0411	9.6012	11.3000	145.1562	30.5473	42.3217	20
BT	56.9972	75.1458	11.1163	11.5000	133.4758	54.3347	24.1935	37
IS	60.4372	70.1680	10.9603	13.1000	135.8068	37.8559	32.7426	26
NT	60.4230	74.3354	12.6978	15.9000	122.6396	41.8231	34.9674	36
SV	58.9291	75.3046	10.3879	11.3000	125.3455	45.4255	31.0940	28
VS	57.4288	70.7452	14.6943	12.8000	146.1347	51.1936	28.3289	39
BR	61.5851	71.1993	12.5748	15.9000	128.0597	35.7474	36.0950	32
BZ	59.5332	75.8151	13.0855	17.7000	121.5569	45.7192	30.4366	38
CT	64.0522	74.1970	6.2838	10.2000	110.4163	26.5364	28.8304	10
GL	62.1418	72.3371	9.4262	11.8000	122.4618	31.0153	39.3950	19
TL	61.7358	65.5128	15.6453	18.9000	147.2502	43.4703	25.6621	41
VR	59.2106	73.8973	10.8603	11.8000	128.5451	51.3310	24.9421	35
AG	62.6051	77.3546	6.6381	9.2000	106.4924	28.6626	45.7447	9
CL	59.6949	72.1272	7.0922	8.9000	132.2543	53.8619	24.4019	24
DB	60.0095	76.2012	5.7790	7.5000	118.6846	39.5785	39.4223	13
GR	57.7943	66.6193	8.3627	9.6000	159.7258	55.0833	21.7500	34
IL	60.3659	71.0646	8.4261	10.1000	133.1068	54.6148	18.7643	33
PH	62.1804	74.7457	5.5119	8.3000	115.1594	20.6221	50.3061	4
TR	58.0980	75.1957	6.8879	9.1000	128.8998	55.0095	25.1890	25
DJ	60.4357	74.7748	10.8187	13.7000	121.2846	43.6337	28.0233	31
GJ	60.4024	87.3938	2.7753	4.0000	89.4371	31.8334	43.8966	3
MH	59.5274	71.0990	7.0195	9.2000	136.2758	45.0604	28.4293	22
OT	60.3826	71.3459	8.8233	10.2000	132.1238	49.1375	28.4387	27
VL	60.6897	77.6483	8.0604	10.0000	112.2040	40.7927	33.6878	21
AR	60.3993	73.4241	6.0217	8.1000	125.4910	37.4247	30.5697	14
CS	61.8061	69.1025	11.3812	16.6000	134.1397	35.1621	37.6559	29
HD	64.0218	74.3413	7.2251	12.5000	110.1078	20.9178	51.6252	7
TM	61.9034	79.0547	4.5816	6.6000	104.3421	30.8397	35.6298	5
BH	60.6634	81.9367	7.3569	8.8000	101.1846	38.0277	33.8374	15
BN	59.3775	71.1609	16.1124	16.7000	136.6667	43.8672	30.0144	40
CJ	62.3940	83.1893	7.9957	10.3000	92.6591	26.9556	39.8784	11
MM	61.9077	73.2347	7.1642	8.8000	120.5660	40.9202	33.5098	17
SM	61.2279	74.2622	8.7774	10.1000	119.9293	41.9178	31.8904	23
SJ	59.3594	72.1737	10.7331	13.4000	133.4164	43.4821	32.1085	30
AB	60.7320	81.8756	5.6912	7.3000	101.1072	34.1131	41.3743	8
BV	62.8708	74.9942	5.0563	7.4000	112.0914	14.9293	56.7248	2
CV	60.4291	70.0712	7.8431	10.9000	136.1641	35.2467	37.5629	18
HG	60.5208	76.3084	8.7466	10.1000	116.5323	33.1886	38.5856	16
MS	60.8399	73.1939	7.2175	8.9000	124.5620	31.7396	39.2424	12
SB	61.5384	71.5393	7.7656	10.3000	127.1481	20.5154	48.9136	6
B-IF	56.0992	81.7558	4.8132	6.8000	118.0344	4.2516	45.9356	1

Data source: Data processed by the authors based on data from Territorial Statistics, 2003, National Institute of Statistics

(2) the correlation matrix between the factorial variables $X_1, X_2, \ldots, X_7$ is

$$R = \begin{pmatrix} 1.0000 & 0.0182 & -0.2209 & 0.0158 & -0.4512 & -0.4654 & 0.4723 \\ 0.0182 & 1.0000 & -0.5523 & -0.5566 & -0.8966 & -0.3551 & 0.3535 \\ -0.2209 & -0.5523 & 1.0000 & 0.9194 & 0.5855 & 0.4638 & -0.4432 \\ 0.0158 & -0.5566 & 0.9194 & 1.0000 & 0.4832 & 0.2923 & -0.3000 \\ -0.4512 & -0.8966 & 0.5855 & 0.4832 & 1.0000 & 0.5163 & -0.5201 \\ -0.4654 & -0.3551 & 0.4638 & 0.2923 & 0.5163 & 1.0000 & -0.8837 \\ 0.4723 & 0.3535 & -0.4432 & -0.3000 & -0.5201 & -0.8837 & 1.0000 \end{pmatrix}$$

(3) the correlation vector between the target variable Y and the factorial variables $X_1, X_2, \ldots, X_7$ is

$$T = \begin{pmatrix} -0.4353 & -0.5652 & 0.8729 & 0.7572 & 0.6907 & 0.7995 & -0.7631 \end{pmatrix};$$

(4) the initial computation matrix is

$$A = \begin{pmatrix} R & T & I_7 \\ T^T & 1 & O^T \\ -I_7 & O & \Theta \end{pmatrix} =$$

$$\begin{pmatrix} 1.0000 & 0.0182 & -0.2209 & 0.0158 & -0.4512 & -0.4654 & 0.4723 & -0.4353 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0182 & 1.0000 & -0.5523 & -0.5566 & -0.8966 & -0.3551 & 0.3535 & -0.5652 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.2209 & -0.5523 & 1.0000 & 0.9194 & 0.5855 & 0.4638 & -0.4432 & 0.8729 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.0158 & -0.5566 & 0.9194 & 1.0000 & 0.4832 & 0.2923 & -0.3000 & 0.7572 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.4512 & -0.8966 & 0.5855 & 0.4832 & 1.0000 & 0.5163 & -0.5201 & 0.6907 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.4654 & -0.3551 & 0.4638 & 0.2923 & 0.5163 & 1.0000 & -0.8837 & 0.7995 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0.4723 & 0.3535 & -0.4432 & -0.3000 & -0.5201 & -0.8837 & 1.0000 & -0.7631 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -0.4353 & -0.5652 & 0.8729 & 0.7572 & 0.6907 & 0.7995 & -0.7631 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

II. The optimization algorithm

q is initialized with the value 0 and the set containing the indices included in the model J_q with the empty set, i.e. $q = 0$ and $J_0 = \Phi$, and $p = 7$.

The value of $c = \sum_{i=1}^{n} y_i^2 - \frac{1}{n} \cdot \left(\sum_{i=1}^{n} y_i\right)^2 = 5740$ is also determined.

Step 1.

a) The $W_i = \frac{a_{i,p+1} \cdot a_{p+1,i}}{a_{ii}}$ statistics of the variables that are not included in the model are

i	1	2	3	4	5	6	7
W_i	0.1895	0.3195	0.7620	0.5734	0.4770	0.6393	0.5823

$W_{\max} = \max\limits_{i \notin J_q} \{W_i\} = W_3$, thus $k = 3$ and the statistic $F^*_{X_3} = \frac{n-q-2 \cdot W_3}{a_{p+1,p+1} - W_3} = \frac{39 \cdot 0.7620}{1-0.7620} = 124.8667$. Because $F_{0.05;\,1,39} = 4.0913$ variable $X_k = X_3$ will be included in the model as $F^*_{X_3} > F_{0.05;\,1,39}$ (the admittance criterion is satisfied) and as such, matrix A will be updated through the swiveling procedure **swivelinp**. The set containing the indices included in the model will also be updated: q is incremented by one ($q = 1$) and the new set will be $J_q = J_{q-1} \bigcup \{k\}$, i.e. $J_1 = J_0 \bigcup \{3\} = \Phi \bigcup \{3\} = \{3\}$.

b) The computation of the $F^*_{X_i} = \frac{n_g \cdot a^2_{i,p+1}}{a_{p+1,p+1} \cdot a_{i,p+1+i}} = \frac{(n-q-1) \cdot a^2_{i,7+1}}{a_{7+1,7+1} \cdot a_{i,7+1+i}}$ statistics of the factorial variable included in the model is reduced, at this moment, to determining $F^*_{X_3}$. After performing the computations $F^*_{X_3} = 124.8667$ and because $F^*_{X_k} = \min_{i \in J_q} \left\{ F^*_{X_i} \right\} = F^*_{X_3} = 124.8667$ and $F_{0.05;\,1,39} = 4.0913$ variable $X_k = X_3$ cannot be removed from the model as $F^*_{X_3} > F_{0.05;\,1,39}$ (it could be removed only if $F^*_{X_3} < F_{0.05;\,1,39}$).

For the current regression model: $SP_{reg} = 4373.8900$, $SP_{rez} = 1366.1100$, $R^2 = 0.7620$.

Step 2.

a) The W_i statistics of the variables that are not included in the model are:

i	1	2	4	5	6	7
W_i	0.0618	0.0099	0.0133	0.0491	0.1985	0.1762

$W_{\max} = \max_{i \notin J_q} \{W_i\} = W_6$, thus $k = 6$ and $F^*_{X_6} = 190.6872$. Because $F_{0.05;\,1,38} = 4.0982$ variable $X_k = X_6$ will be included in the model as $F^*_{X_6} > F_{0.05;\,1,38}$ (the admittance criterion is satisfied) and as such, matrix A will be updated through the swiveling procedure **swivelinp**. The set containing the indices included in the model will also be updated: q is incremented by one ($q = 2$) and the new set will be $J_q = J_{q-1} \bigcup \{k\}$, i.e. $J_2 = J_1 \bigcup \{6\} = \{3\} \bigcup \{6\} = \{3,\, 6\}$.

b) The $F^*_{X_i}$ statistics of the variables that are included in the model are:

i	3	6
$F^*_{X_i}$	308.6323	190.6840

Because $F^*_{X_k} = \min_{i \in J_q} \left\{ F^*_{X_i} \right\} = F^*_{X_6} = 190.6872$ and $F_{0.05;\,1,38} = 4.0982$ variable $X_k = X_6$ cannot be removed from the model as $F^*_{X_6} > F_{0.05;\,1,38}$ (it could be removed only if $F^*_{X_6} < F_{0.05;\,1,38}$).

For the current regression model: $SP_{reg} = 5512.9992$, $SP_{rez} = 227.0009$, $R^2 = 0.9605$.

Step 3.

a) The W_i statistics of the variables that are not included in the model are:

i	1	2	4	5	7
W_i	0.0046	0.0016	0.0037	0.0055	0.0057

$W_{\max} = \max_{i \notin J_q} \{W_i\} = W_7$, thus $k = 7$ and $F^*_{X_7} = 6.2445$. Because $F_{0.05;\,1,37} = 4.1055$ variable $X_k = X_7$ will be included in the model as $F^*_{X_7} > F_{0.05;\,1,37}$ (the admittance criterion is satisfied) and as such, matrix A will be updated through the swiveling procedure **swivelinp**. The set containing the indices included in the model will also be updated: q is incremented by one ($q = 3$) and the new set will be $J_q = J_{q-1} \bigcup \{k\}$, i.e. $J_3 = \{3, 6, 7\}$.

b) The $F^*_{X_i}$ statistics of the variables that are included in the model are:

i	3	6	7
$F^*_{X_i}$	341.4931	30.6205	6.2445

Because $F^*_{X_k} = \min_{i \in J_q} \left\{ F^*_{X_i} \right\} = F^*_{X_7} = 6.2445$ and $F_{0.05;\,1,37} = 4.1055$ variable $X_k = X_7$ cannot be removed from the model as$F^*_{X_7} > F_{0.05;\,1,37}$ (it could be removed only if $F^*_{X_7} < F_{0.05;\,1,37}$).

For the current regression model: $SP_{reg} = 5545.7780$, $SP_{rez} = 194.2220$, $R^2 = 0.9662$.

Step 4.

a) The W_i statistics of the variables that are not included in the model are:

i	1	2	4	5
W_i	0.0033	0.0013	0.0033	0.0041

$W_{\max} = \max_{i \notin J_q} \{W_i\} = W_5$, thus $k = 5$ and $F^*_{X_5} = 5.0223$. Because $F_{0.05;\,1,36} = 4.1132$ variable $X_k = X_5$ will be included in the model as $F^*_{X_5} > F_{0.05;\,1,36}$ (the admittance criterion is satisfied) and as such, matrix A will be updated through the swiveling procedure **swivelinp**. The set containing the indices included in the model will also be updated: q is incremented by one ($q = 4$) and the new set will be $J_q = J_{q-1} \bigcup \{k\}$, i.e. $J_4 = \{3, 5, 6, 7\}$.
b) The $F^*_{X_i}$ statistics of the variables that are included in the model are:

i	3	5	6	7
$F^*_{X_i}$	267.7462	5.0223	31.9533	5.2711

Because $F^*_{X_k} = \min_{i \in J_q} \left\{F^*_{X_i}\right\} = F^*_{X_5} = 5.0223$ and $F_{0.05;\,1,36} = 4.1132$ variable $X_k = X_5$ cannot be removed from the model as $F^*_{X_5} > F_{0.05;\,1,36}$ (it could be removed only if $F^*_{X_5} < F_{0.05;\,1,36}$).
For the current regression model: $SP_{reg} = 5569.5562,\ SP_{rez} = 170.4438\,,\ R^2 = 0.9703.$
Step 5.
a) The W_i statistics of the variables that are not included in the model are:

i	1	2	4
W_i	0.0016	0.0015	0.0037

$W_{\max} = \max_{i \notin J_q} \{W_i\} = W_4$, thus $k = 4$ and $F^*_{X_4} = 5.0075$. Because $F_{0.05;\,1,35} = 4.1213$ variable $X_k = X_4$ will be included in the model as $F^*_{X_4} > F_{0.05;\,1,35}$ (the admittance criterion is satisfied) and as such, matrix A will be updated through the swiveling procedure **swivelinp**. The set containing the indices included in the model will also be updated: q is incremented by one ($q = 5$) and the new set will be $J_q = J_{q-1} \bigcup \{k\}$, i.e. $J_5 = \{3, 4, 5, 6, 7\}$.
b) The $F^*_{X_i}$ statistics of the variables that are included in the model are:

i	3	4	5	6	7
$F^*_{X_i}$	26.1473	5.0075	6.1618	40.0379	5.2406

Because $F^*_{X_k} = \min_{i \in J_q} \left\{F^*_{X_i}\right\} = F^*_{X_4} = 5.0075$ and $F_{0.05;\,1,35} = 4.1213$ variable $X_k = X_4$ cannot be removed from the model as $F^*_{X_4} > F_{0.05;\,1,35}$ (it could be removed only if $F^*_{X_4} < F_{0.05;\,1,35}$).
For the current regression model: $SP_{reg} = 5590.8896,\ SP_{rez} = 149.1104\,,\ R^2 = 0.9740.$
Step 6.
a) The W_i statistics of the variables that are not included in the model are:

i	1	2
W_i	0.0068	0.0069

$W_{\max} = \max_{i \notin J_q} \{W_i\} = W_2$ thus $k = 2$ and $F^*_{X_2} = 12.3712$. Because $F_{0.05;\,1,34} = 4.1300$ variable $X_k = X_2$ will be included in the model as $F^*_{X_2} > F_{0.05;\,1,34}$ (the admittance criterion is satisfied) and as such, matrix A will be updated through the swiveling procedure **swivelinp**. The set containing the indices included in the model will also be updated: q is incremented by one ($q = 6$) and the new set will be $J_q = J_{q-1} \bigcup \{2\}$, i.e.$J_6 = \{2, 3, 4, 5, 6, 7\}$.
For the current regression model: $SP_{reg} = 5630.6710,\ SP_{rez} = 109.3290\,,\ R^2 = 0.9810.$
b) The $F^*_{X_i}$ statistics of the variables that are included in the model are:

i	2	3	4	5	6	7
$F^*_{X_i}$	12.3715	14.1995	16.2943	19.5086	55.7960	3.9930

Because $F^*_{X_k} = \min_{i \in J_q} \left\{F^*_{X_i}\right\} = F^*_{X_7} = 3.9930$ and $F_{0.05;\,1,34} = 4.1300$ variable $X_k = X_7$ will be removed from the model as $F^*_{X_7} < F_{0.05;\,1,34}$ (the elimination criterion is satisfied) and as such, matrix A will be updated through the swiveling procedure **swivelout**. The set containing the indices included in the model will also be updated: q is decremented by one unit ($q = 5$) and the new set will be $J_q = J_{q+1} \setminus \{k\}$, i.e. $J_5 = J_6 \setminus \{7\} = \{2, 3, 4, 5, 6\}$. *For the current regression model:* $SP_{reg} = 5617.8312$, $SP_{rez} = 122.1688$, $R^2 = 0.9787$.

Step 7.

a) The W_i statistics of the variables that are not included in the model are:

i	1	7
W_i	0.0001	0.0022

$W_{\max} = \max\limits_{i \notin J_q} \{W_i\} = W_7$, thus $k = 7$ and $F^*_{X_7} = 3.9930$. Because $F_{0.05;\,1,34} = 4.1300$ variable $X_k = X_7$ cannot be included in the model as $F^*_{X_7} < F_{0.05;\,1,34}$ (the admittance criterion is not satisfied).

b) The $F^*_{X_i}$ statistics of the variables that are included in the model are:

i	2	3	4	5	6
$F^*_{X_i}$	14.1147	11.6709	17.4393	22.6568	229.3420

Because $F^*_{X_k} = \min\limits_{i \in J_q} \left\{F^*_{X_i}\right\} = F^*_{X_3} = 11.6709$, and $F_{0.05;\,1,35} = 4.1213$ variable $X_k = X_3$ cannot be removed from the model as $F^*_{X_3} > F_{0.05;\,1,35}$ (it could be removed only if $F^*_{X_3} < F_{0.05;\,1,35}$). At this stage, no more factorial variables can be included in the model and also no more factorial variables can be removed from the model. Thus, the optimal regression model is the one that contains the variables X_2, X_3,X_4,X_5 and X_6.

III. Determining the coefficients of the optimal model and the additional elements of the regression

Matrix A obtained after the last update is:

$$A = \begin{pmatrix} B & C & F \\ D & f & -G \\ Z & V & P \end{pmatrix} =$$

0.0346	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0013	-0.0013	1.0000	1.9587	0.0033	0.0059	2.1881	0.0280	0.0000
-1.9587	1.0000	0.0000	0.0000	0.0000	0.0000	–0.2179	0.2572	0.0000	7.7058	–3.9771	4.5718	7.2983	–0.5235	0.0000
–0.0033	0.0000	1.0000	0.0000	0.0000	0.0000	0.1803	0.2844	0.0000	–3.9771	11.3948	-10.0269	–4.6998	–1.3401	0.0000
–0.0059	0.0000	0.0000	1.0000	0.0000	0.0000	–0.2017	0.3307	0.0000	4.5718	–10.0269	10.3141	4.5028	0.9347	0.0000
–2.1881	0.0000	0.0000	0.0000	1.0000	0.0000	–0.2927	0.3450	0.0000	7.2983	–4.6998	4.5028	8.6377	–1.0043	0.0000
–0.0280	0.0000	0.0000	0.0000	0.0000	1.0000	–0.8346	0.4842	0.0000	–0.5235	–1.3401	0.9347	–1.0043	1.6810	0.0000
–0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.2067	–0.0215	0.0000	0.2179	–0.1803	0.2017	0.2927	0.8346	1..0000
–0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	–0.0215	0.0213	0.0000	–0.2572	–0.2844	-0.3307	–0.3450	–0.4842	0.0000
–1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
–1.9587	1.0000	0.0000	0.0000	0.0000	0.0000	–0.2179	0.2572	0.0000	7.7058	–3.9771	4.5718	7.2983	-0.5235	0.0000
–0.0033	0.0000	1.0000	0.0000	0.0000	0.0000	0.1803	0.2844	0.0000	–3.9771	11.3948	-10.0269	–4.6998	-1.3401	0.0000
–0.0059	0.0000	0.0000	1.0000	0.0000	0.0000	–0.2017	0.3307	0.0000	4.5718	-10.0269	10.3141	4.5028	0.9347	0.0000
–2.1881	0.0000	0.0000	0.0000	1.0000	0.0000	–0.2927	0.3450	0.0000	7.2983	-4.6998	4.5028	8.6377	-1.0043	0.0000
–0.0280	0.0000	0.0000	0.0000	0.0000	1.0000	–0.8346	0.4842	0.0000	–0.5235	-1.3401	0.9347	–1.0043	1.6810	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	–1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

and contains all the information needed in order to determine the coefficients of the regression equation as well as all the information needed to determine the additional elements of the regression.

The vectors V and G^T contain the normalized regression coefficients, and the coefficients equal to zero correspond to the variables that are not included in the regression model. In order to determine the real values of the coefficients of the variables from the optimal regression model, one can use the following formulae:

$a_i = a_{p+1+i,p+1} \frac{S_Y}{S_{X_i}}$, $\forall\, i \in J_q$ or $a_i = -a_{p+1,p+1+i} \frac{S_Y}{S_{X_i}}$, $\forall\, i \in J_q$

and as $J_q = J_5 = \{2, 3, 4, 5, 6\}$, and $p = 7$ we have that

$$\begin{cases} a_2 = a_{10,8}\frac{s_Y}{s_{X_2}} = 0.2572\frac{11.9791}{4.5123} = 0.6828 \\ a_3 = a_{11,8}\frac{s_Y}{s_{X_3}} = 0.2844\frac{11.9791}{3.0427} = 1.1196 \\ a_4 = a_{12,8}\frac{s_Y}{s_{X_4}} = 0.3307\frac{11.9791}{3.2402} = 1.2226 \end{cases} ; \begin{cases} a_5 = a_{13,8}\frac{s_Y}{s_{X_5}} = 0.3450\frac{11.9791}{14.7783} = 0.2796 \\ a_6 = a_{14,8}\frac{s_Y}{s_{X_6}} = 0.4842\frac{11.9791}{11.6094} = 0.4996 \end{cases}$$

and the free term of the regression equation is $a_0 = \bar{y} - \sum_{i \in J_q} a_i \bar{x}_i = \bar{y} - (a_2\bar{x}_2 + a_3\bar{x}_3 + a_4\bar{x}_4 + a_5\bar{x}_5 + a_6\bar{x}_6)$, i.e.

$$a_0 = 21 - (0.6828 \times 74.2168 + 1.1196 \times 8.7320 +$$
$$+ 1.2226 \times 10.8683 + 0.2796 \times 123.5766 + 0.4996 \times 37.7168) =$$
$$= 21 - 127.1344 = -106.1344$$

The equation of the optimal regression model is:

$$Y = -106.1344 + 0.6828X_2 + 1.1196X_3 + 1.2226X_4 + 0.2796X_5 + 0.4996X_6$$

The additional elements of the regression are:

(1) The mean squared deviation of the residues:

$$s_u = \sqrt{s_u^2} = \sqrt{\frac{a_{7+1,7+1} \cdot c}{n-q-1}} = \sqrt{\frac{0.0213 \times 5740}{41-5-1}} = 1.8690;$$

(2) The standard errors of the coefficients:

$$\begin{cases} e_2 = \frac{s_u}{S_{X_2}}\sqrt{\frac{a_{2+7+1,2+7+1}}{n-1}} = \frac{s_u}{S_{X_2}}\sqrt{\frac{a_{10,10}}{n-1}} = \frac{1.8690}{4.5123}\sqrt{\frac{7.7058}{40}} = 0.4142 \times 0.4389 = 0.1817 \\ e_3 = \frac{s_u}{S_{X_3}}\sqrt{\frac{a_{3+7+1,3+7+1}}{n-1}} = \frac{s_u}{S_{X_3}}\sqrt{\frac{a_{11,11}}{n-1}} = \frac{1.8690}{3.0427}\sqrt{\frac{11.3948}{40}} = 0.6143 \times 0.5337 = 0.3279 \\ e_4 = \frac{s_u}{S_{X_4}}\sqrt{\frac{a_{4+7+1,4+7+1}}{n-1}} = \frac{s_u}{S_{X_4}}\sqrt{\frac{a_{12,12}}{n-1}} = \frac{1.8690}{3.2402}\sqrt{\frac{10.3141}{40}} = 0.5768 \times 0.5078 = 0.2929 \\ e_5 = \frac{s_u}{S_{X_5}}\sqrt{\frac{a_{5+7+1,5+7+1}}{n-1}} = \frac{s_u}{S_{X_5}}\sqrt{\frac{a_{13,13}}{n-1}} = \frac{1.8690}{14.7783}\sqrt{\frac{8.6377}{40}} = 0.1265 \times 0.4647 = 0.0588 \\ e_6 = \frac{s_u}{S_{X_6}}\sqrt{\frac{a_{6+7+1,6+7+1}}{n-1}} = \frac{s_u}{S_{X_6}}\sqrt{\frac{a_{14,14}}{n-1}} = \frac{1.8690}{11.6094}\sqrt{\frac{1.6810}{40}} = 0.1610 \times 0.2050 = 0.0330 \end{cases}$$

(3) The multiple determination coefficient:

$$R^2 = 1 - \frac{SP_{rez}}{SP_{total}} = 1 - \frac{122.2620}{122.2620 + 5617.7380} = 1 - \frac{122.2620}{5740} = 1 - 0.0213 = 0.9787;$$

(4) The F^* statistic:$F^* = \frac{SP_{reg}}{SP_{rez}} \cdot \frac{n-q-1}{q} = \frac{5617.7380}{122.2620} \times \frac{35}{5} = 321.6384$;

(5) The number of degrees of freedom: $n_g = n - q - 1 = 41 - 5 - 1 = 35$;

(6) The sum of the regression squares: $SP_{reg} = (1 - 0.0213) \times 5740 = 5617.7380$;

(7) The sum of the residues squares: $SP_{rez} = a_{8,8} \cdot c = 0.0213 \times 5740 = 122.2620$.

Conclusion. As $F^* > 4 \cdot F_{0.05;\,5,35}$, i.e. $321.6384 > 4 \times 2.4838 = 9.9352$, the chosen model is valid from a statistical point of view.

Comment. Variable X_7 was introduced in the model at step *3*, but then, after the introduction of other, more important, variables, it was removed from the model at *step 6*. The multiple determination coefficient of the model that contains the variables X_2,X_3,X_4,X_5,X_6 and X_7 is $R^2 = 0.9810$ and shows that 98,10% of the total variation of variable Y is explained by the variables included in the model. The multiple determination coefficient of the chosen optimal model that contains only the variables X_2, X_3,X_4,X_5 andX_6, is $R^2 = 0.9787$ and shows that 97,87% din of the total variation of variable Y is explained by the variables included in the model. Because the differences between the two multiple determination coefficients is insignificant and the values of the correlation coefficient$R = \sqrt{R^2}$ are 0.9904 and 0.9892 (i.e. almost equal) it is preferable to implement the model that contains the least variables because obtaining and processing information for more variables is more costly and time consuming.

4. Final conclusions

(1) The optimization of multiple regression models through the method of stepwise regression has the advantage that at every step all the factorials variables are analyzed and thus it is ensured that no important variable is left outside the model and that no insignificant variable is included in the final model.
(2) The fact that the data are balanced, as the method used the correlation matrix of the factorial variables and the correlation vector between the target variable and the factorial variables, both of which have values in the interval $[-1, 1]$, is another advantage because it eliminates the shortcoming of having different orders of size for variables and as such, rounding and parameter estimation errors are very small.
(3) Another advantage of the stepwise selection method is the fact that this approach diminishes the influence of the multicollinearity phenomenon. In our case, this phenomenon can be considered "omnipresent" because of the multiple interdependencies and triggering effects existent in the economic environment. The correlation coefficient R^2 of the optimal model is insignificantly smaller than the one of the complete model but the quality of the estimated parameters is not negatively affected, although, casually, it might even be positively influenced.
(4) The efficiency of a regression optimization method can be evaluated using two criteria: the status of the variables included in the analysis and the number of solved regressions in the worst case scenario in order to obtain the optimal regression model. With regards to optimizing regression models through the method of stepwise selection, the first criterion is satisfied because all the factorial variables are considered for inclusion in the model. On the second criterion, the maximal number of regressions that must be solved is $2p$ and it is a lot smaller than that required by the method of all possible regressions that requires $2^p - 1$ regressions or than that required by the method of forward selection that requires at most $\frac{p(p+1)}{2} - 1$ regressions.
(5) The presented case study has been solved solely by using a software application created by the authors.

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