TJMM
**4** (2012), No. 1, 37-46

# ON POLYNOMIAL INSTABILITY OF VARIATIONAL DIFFERENCE EQUATIONS IN BANACH SPACES

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Abstract. The object of this paper is to study two concepts of polynomial instability for variational nonautonomous difference equations in Banach spaces. Obtained characterizations for these concepts are generalizations of the classical results due to E.Barbashin ([1]), R.Datko ([4]) and A. Lyapunov ([6]) for variational difference equations.

## 1. Introduction

Systems studied in this work generates the discrete evolution cocycles. The general concept of evolution cocycle was introduced by M.Megan and C.Stoica in [9]. It generalizes the classical notions of $C_0$-semigroups, evolution operators and linear skew-product semiflows.

The concept of polynomial asymptotic behavior has been considered in the works of L.Barreira and C.Valls [2] and [3] for evolution operators. Remarkable results were obtained by M.Megan, T.Ceaușu and L.M.Rămneanțu in [8] for polynomial instability of evolution operators and for polynomial dichotomy in [11]. Also, characterizations of the exponential instability for discrete variational systems were given in [12].

My results are an extension of uniform polynomial instability and (nonuniform) polynomial instability studied in [8], for variational nonautonomous difference equations in Banach spaces, using the technique applied in case of exponential instability. The results obtained generalize the stability theorems due to L.Barreira and C.Valls, R.Datko, E.A.Barbashin, A.Lyapunov and M.Megan for case of polynomial instability for variational nonautonomous difference equations in Banach spaces.

Let $\mathbb{N}$ be the set of all positive integer and let $\Delta$, $\Delta_c$ respectively $T$, $T_c$ be the sets defined by

$$\Delta = \left\{(m,n) \in \mathbb{N}^2, \text{ with}\, m \geq n\right\},$$
$$\Delta_c = \left\{(t,s) \in \mathbb{R}_+^2, \text{ with}\, t \geq s\right\},$$

respectively

$$T = \left\{(m,n,p) \in \mathbb{N}^3, \text{ with}\, m \geq n \geq p\right\},$$
$$T_c = \left\{(t,s,r) \in \mathbb{R}_+^3, \text{ with}\, t \geq s \geq r\right\}.$$

Let $(X,d)$ be a metric space and $V$ a real or complex Banach space. The norm on $V$ and $\mathcal{B}(V)$ (the Banach algebra of all bounded linear operators on $V$) will be denoted by $\|\cdot\|$.

We recall that a mapping $\varphi : \Delta \times X \to X$ is called a *discrete evolution semiflow* on X if the following conditions hold:

$s_1$) $\varphi(n,n,x) = x$, for all $(n,x) \in \mathbb{N} \times X$;

$s_2$) $\varphi(m,n,\varphi(n,p,x)) = \varphi(m,p,x)$, for all $(m,n,p,x) \in T \times X$.

2010 *Mathematics Subject Classification.* 34D05,34E05.

*Key words and phrases.* variational difference equations, polynomial instability, discrete evolution cocycle.

**Example 1.** *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a bounded function and for* $s \in \mathbb{R}_+$ *we denote* $f_s(t) = f(t+s)$ *for all* $t \in \mathbb{R}_+$*. Then* $X = \{f_s, s \in \mathbb{R}_+\}$ *is a metric space with the metric* $d(x_1, x_2) = \sup_{t\in\mathbb{R}_+} |x_1(t) - x_2(t)|$.

*The mapping* $\varphi : \Delta \times X \to X$ *defined by* $\varphi(m, n, x) = x_{m-n}$ *is a discrete evolution semiflow.*

Let $\varphi : \Delta \times X \to X$ be a discrete evolution semiflow on $X$ and let $A : X \to \mathcal{B}(V)$.

Given a sequence $(A_m)_{m\in\mathbb{N}}$ with $A_m : X \to \mathcal{B}(V)$ and a discrete evolution semiflow $\varphi : \Delta \times X \to X$, we consider the problem of existence of a sequence $(v_m)_{m\in\mathbb{N}}$ with $v_m : \mathbb{N} \times X \to X$ such that

$$v_{m+1}(n, x) = A_m(\varphi(m, n, x))v_m(n, x)$$

for all $(m, n, x) \in \Delta \times X$. We shall denote this problem with $(A, \varphi)$ and we say that $(A, \varphi)$ is a *variational (nonautonomous) discrete-time system.*

For $(m, n) \in \Delta$ we define the application $\Phi_m^n : X \to \mathcal{B}(V)$ by

$$\Phi_m^n(x)v = \begin{cases} A_{m-1}(\varphi(m-1, n, x)) \dots A_{n+1}(\varphi(n+1, n, x)) A_n(x)v, & \text{if } m > n \\ v, & \text{if } m = n. \end{cases}$$

**Remark 1.** *From the definitions of* $v_m$ *and* $\Phi_m^n$ *it follows that:*
$c_1$) $\Phi_m^m(x)v = v$*, for all* $(m, x, v) \in \mathbb{N} \times X \times V$*;*
$c_2$) $\Phi_m^p(x) = \Phi_m^n(\varphi(n, p, x)) \Phi_n^p(x)$*, for all* $(m, n, p, x) \in T \times X$*;*
$c_3$) $v_m(n, x) = \Phi_m^n(x)v_n(n, x)$*, for all* $(m, n, x) \in \Delta \times X$*.*

The properties $(c_1)$ and $(c_2)$ shows that the mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$ defined by

$$\Phi(m, n, x)v = \Phi_m^n(x)v$$

for all $(m, n, x, v) \in \Delta \times X \times V$ is a discrete evolution cocycle over discrete evolution semiflow $\varphi$.

## 2. Uniform polynomial instability

Let $(A, \varphi)$ be a discrete variational system associated to the discrete evolution semiflow $\varphi : \Delta \times X \to X$ and to the sequence of mappings $A = (A_m)$, where $A_m : X \to \mathcal{B}(V)$, for all $m \in \mathbb{N}$.

**Definition 1.** *The system* $(A, \varphi)$ *is said to be* ***uniformly polynomially instable*** *(and denote u.p.is.) if there are the constants* $N \geq 1$ *and* $\alpha > 1$ *such that:*

$$\left(\frac{m+1}{n+1}\right)^{\alpha} \|v\| \leq N \|\Phi_m^n(x)v\|$$

*for all* $(m, n, x, v) \in \Delta \times X \times V$*.*

**Remark 2.** *It is easy to see that* $(A, \varphi)$ *is uniformly polynomially instable if there are the constants* $N \geq 1$ *and* $\alpha > 1$ *with:*

$$\left(\frac{m+1}{n+1}\right)^{\alpha} \|\Phi_n^p(x)v\| \leq N \|\Phi_m^p(x)v\|$$

*for all* $(m, n, p, x, v) \in T \times X \times V$*.*

**Example 2.** *Let* $\mathcal{C} = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ *be the metric space of all continuous functions* $x : \mathbb{R}_+ \to \mathbb{R}$*, with the topology of uniform convergence on compact subsets of* $\mathbb{R}_+$*.* $\mathcal{C}$ *is metrizable relative to the metric given in Example 1*

*Let $f : \mathbb{R}_+ \to (0,\infty)$ be a decreasing function with the property that there exists $\lim_{t\to\infty} f(t) = l > 0$. We denote by $X$ the closure in $\mathcal{C}$ of the set $\{f_t, t \in \mathbb{R}_+\}$, where $f_t(s) = f(t+s)$, for all $s \in \mathbb{R}_+$. The mapping $\varphi : \Delta \times X \to X$ defined by $\varphi(m,n,x) = x_{m-n}$ is a discrete evolution semiflow.*

*Let us consider the Banach space $V = \mathbb{R}$ and we define the sequence of mappings $A_m : X \to \mathcal{B}(V)$ by*

$$A_m(x)v = \left(\frac{m+2}{m+1}\right)^2 \frac{x(\tau)}{x(\tau+1)}v.$$

*Then*

$$\Phi_m^n(x)v = \left(\frac{m+1}{n+1}\right)^2 \frac{x(\tau)}{x(m-n+\tau)}v$$

*so*

$$|\Phi_m^n(x)v| = \left(\frac{m+1}{n+1}\right)^2 \frac{x(\tau)}{x(m-n+\tau)}|v| \geq$$

$$\geq \frac{1}{N}\left(\frac{m+1}{n+1}\right)^2 |v|$$

*for all $(m,n,x,v) \in \Delta \times X \times \mathbb{R}$,where $N = \frac{x(0)}{l}$.*

The Datko-type characterization for uniform polynomial instability property is presented in

**Theorem 1.** *The system $(A,\varphi)$ is uniformly polynomially instable if and only if there are $D \geq 1$, $d > 0$, $M \geq 1$ and $\omega > 0$ such that:*

*(i)* $\sum_{k=n}^{m} \frac{(m+1)^d}{(k+1)^{d+1}} \|\Phi_k^n(x)v\| \leq D\|\Phi_m^n(x)v\|$ *for all $(m,n,x,v) \in \Delta \times X \times V$;*

*(ii)* $\|\Phi_n^p(x)v\| \leq M\left(\frac{m+1}{n+1}\right)^\omega \|\Phi_m^p(x)v\|$ *for all $(m,n,x,v) \in \Delta \times X \times V$.*

*Proof.* *Necessity* According with the Remark 2 there are $N \geq 1$ and $\alpha > 1$ such that

$$\|\Phi_k^n(x)v\| \leq N\left(\frac{k+1}{m+1}\right)^\alpha \|\Phi_m^n(x)v\|$$

for all $(m,k,n,x,v) \in T \times X \times V$. Using the above relation we obtain

$$\sum_{k=n}^{m} \frac{(m+1)^d}{(k+1)^{d+1}} \|\Phi_k^n(x)v\| \leq N\|\Phi_m^n(x)v\| \sum_{k=n}^{m} \frac{(m+1)^d}{(k+1)^{d+1}} \left(\frac{k+1}{m+1}\right)^\alpha =$$

$$= \frac{N}{m+1}\|\Phi_m^n(x)v\| \sum_{k=n}^{m} \left(\frac{k+1}{m+1}\right)^{\alpha-d-1} \leq$$

$$\leq N\frac{m-n+1}{m+1}\|\Phi_m^n(x)v\| \leq N\|\Phi_m^n(x)v\|$$

for all $(m,n,x,v) \in \Delta \times X \times V$.

*Sufficiency.* If $m > 2n$, then using the assumption it results the inequalities:

$$(n+1)\|v\| = \sum_{k=n}^{2n} \|v\| \leq M \sum_{k=n}^{2n} \left(\frac{k+1}{n+1}\right)^\omega \|\Phi_k^n(x)v\| =$$

$$= M \sum_{k=n}^{2n} \frac{(m+1)^d}{(k+1)^{d+1}} \left(\frac{k+1}{m+1}\right)^d \left(\frac{k+1}{n+1}\right)^\omega (k+1)\|\Phi_k^n(x)v\| \leq$$

$$\leq MD\left(\frac{2n+1}{m+1}\right)^{d}\left(\frac{2n+1}{n+1}\right)^{\omega}(2n+1)\left\|\Phi_{m}^{n}(x)v\right\| \leq$$

$$\leq MD2^{d+\omega}(2n+1)\left(\frac{n+1}{m+1}\right)^{d}\left\|\Phi_{m}^{n}(x)v\right\|$$

from where

$$\|v\| \leq MD2^{d+\omega}\frac{2n+1}{n+1}\left(\frac{n+1}{m+1}\right)^{d}\left\|\Phi_{m}^{n}(x)v\right\| \leq$$

$$\leq MD2^{d+\omega+1}\left(\frac{n+1}{m+1}\right)^{d}\left\|\Phi_{m}^{n}(x)v\right\|$$

for all $(x,v) \in X \times V$.
If $n \leq m \leq 2n$, then from (ii) obtain immediately that

$$\left(\frac{m+1}{n+1}\right)^{d}\|v\| \leq M\left(\frac{m+1}{n+1}\right)^{d+\omega}\left\|\Phi_{m}^{n}(x)v\right\| \leq M2^{d+\omega}\left\|\Phi_{m}^{n}(x)v\right\|$$

for all $(x,v) \in X \times V$. Therefore, the system $(A,\varphi)$ is u.p.is. □

Now we present a Lyapunov-type theorem for uniformly polynomially instable property, which is required to be preceded by the definition of Lyapunov function for polynomial instability associated of discrete variational system $(A,\varphi)$ given in

**Definition 2.** *An application $L : \Delta \times X \times V \to \mathbb{R}$ is called to be a* ***Lyapunov function for polynomial instability*** *associated system $(A,\varphi)$ if there are $l > 1$ such that:*

$$L(n,p,x,v) + \sum_{k=n}^{m}\frac{(m+1)^{l}}{(k+1)^{l+1}}\left\|\Phi_{k}^{p}(x)v\right\| \leq L(m,p,x,v) \tag{1}$$

*for all $(m,n,p,x,v) \in T \times X \times V$, with $m > n$.*

**Theorem 2.** *The system $(A,\varphi)$ is uniformly polynomially instable if and only if there exists a Lyapunov function for polynomial instability associated system $(A,\varphi)$ and a constant $K \geq 1$ such that:*

$$L(m,n,x,v) \leq K\|v\| \tag{2}$$

*for all $(m,n,x,v) \in \Delta \times X \times V$.*

*Proof. Necessity.* We define the function $L : \Delta \times X \times V \to \mathbb{R}$ by

$$L(m,n,x,v) = \sum_{k=n}^{m}\frac{(m+1)^{d}}{(k+1)^{d+1}}\left\|\Phi_{k}^{n}(x)v\right\|$$

where $d > 0$ is the constant given in Theorem 1. We show that this is a Lyapunov function for polynomial instability associated system $(A,\varphi)$:

$$L(n,p,x,v)+\sum_{k=n}^{m}\frac{(m+1)^{d}}{(k+1)^{d+1}}\left\|\Phi_{k}^{p}(x)v\right\| = \sum_{k=p}^{n}\frac{(n+1)^{d}}{(k+1)^{d+1}}\left\|\Phi_{k}^{p}(x)v\right\|+\sum_{k=n}^{m}\frac{(m+1)^{d}}{(k+1)^{d+1}}\left\|\Phi_{k}^{p}(x)v\right\| \leq$$

$$\leq \sum_{k=p}^{n}\frac{(m+1)^{d}}{(k+1)^{d+1}}\left\|\Phi_{k}^{p}(x)v\right\| + \sum_{k=n}^{m}\frac{(m+1)^{d}}{(k+1)^{d+1}}\left\|\Phi_{k}^{p}(x)v\right\| \leq$$

$$\leq \sum_{k=p}^{m}\frac{(m+1)^{d}}{(k+1)^{d+1}}\left\|\Phi_{k}^{p}(x)v\right\| = L(m,p,x,v)$$

for all $(m,n,p,x,v) \in T \times X \times V$, with $m > n$. From Theorem 1 we have that there exists $D \geq 1$ such that

$$L(m,n,x,v) = \sum_{k=n}^{m} \frac{(m+1)^d}{(k+1)^{d+1}} \left\| \Phi_k^n(x)v \right\| \leq D \left\| \Phi_m^n(x)v \right\|$$

for all $(m,n,x,v) \in \Delta \times X \times V$ and inequality (2) is satisfied.
*Sufficiency.* For $p = n$, from relations (1) and (2) it result that there exist $l > 1$ and $K \geq 1$ such that

$$\sum_{k=n}^{m} \frac{(m+1)^l}{(k+1)^{l+1}} \left\| \Phi_k^n(x)v \right\| \leq L(m,n,x,v) \leq K \left\| \Phi_m^n(x)v \right\|$$

for all $(m,n,x,v) \in \Delta \times X \times V$, which implies, according to Theorem 1 that the system $(A,\varphi)$ is u.p.is. □

A sufficient condition for uniform polynomial instability of discrete variational systems is

**Theorem 3.** *If there are $b > 0$ and $B \geq 1$ such that*

$$\sum_{k=n}^{m} \left(\frac{k+1}{n+1}\right)^b \left\| \Phi_m^k(x)v \right\| \leq B \left\| \Phi_m^n(x)v \right\|$$

*for all $(m,n,x,v) \in \Delta \times X \times V$, then the system $(A,\varphi)$ is uniformly polynomially instable.*

*Proof.* For all $(m,n,x,v) \in \Delta \times X \times V$ we have that

$$\left(\frac{m+1}{n+1}\right)^b \|v\| \leq \sum_{k=n}^{m} \left(\frac{k+1}{n+1}\right)^b \left\| \Phi_m^k(x)v \right\| \leq B \left\| \Phi_m^n(x)v \right\|.$$

Hence $(A,\varphi)$ is u.p.is. □

Next we present the connection between the continuous and discrete case for uniform polynomial instability property of discrete variational sistems studied.

**Proposition 1.** *A continuous variational system $(A_c,\varphi_c)$ with the property that there are $M \geq 1$and $\omega > 0$ such that*

$$\|v\| \leq M \left(\frac{t+1}{s+1}\right)^\omega \left\| \Phi_t^s(x)v \right\| \tag{3}$$

*for all $(t,s,x,v) \in \Delta_c \times X \times V$ is uniformly polynomially instable if and only if the associated discrete variational system $(A,\varphi)$ is uniformly polynomially instable.*

*Proof. Necessity* is immediate because $\Delta \subset \Delta_c$.
*Sufficiency.* If $s \leq t < s+1$, then under the assumption there are $M \geq 1$ and $\omega > 0$ such that

$$\left(\frac{t+1}{s+1}\right)^\alpha \|v\| \leq M \left(\frac{t+1}{s+1}\right)^{\alpha+\omega} \left\| \Phi_t^s(x)v \right\| \leq 2^{\alpha+\omega} \left\| \Phi_t^s(x)v \right\|$$

for all $(x,v) \in X \times V$.
Otherwise, $t \geq s+1$ and will be denote $m = [t]$ and $n = [s]+1$. From relation (3) and Definition 1 it result that

$$\left\| \Phi_t^s(x)v \right\| = \left\| \Phi_t^m(\varphi(m,s,x)) \Phi_m^n(\varphi(n,s,x)) \Phi_n^s(x)v \right\| \geq$$
$$\geq \frac{1}{M} \left(\frac{m+1}{s+1}\right)^\omega \frac{1}{N} \left(\frac{m+1}{n+1}\right)^\alpha \frac{1}{M} \left(\frac{s+1}{n+1}\right)^\omega \|v\| =$$

$$= \frac{1}{M^2N}\left(\frac{t+1}{s+1}\right)^{\alpha}\left(\frac{s+1}{t+1}\right)^{\alpha+\omega}\left(\frac{m+1}{n+1}\right)^{\alpha+\omega}\|v\| \geq$$

$$\geq \frac{1}{M^2N}\left(\frac{t+1}{s+1}\right)^{\alpha}\left(\frac{n}{t+1}\right)^{\alpha+\omega}\left(\frac{t}{n+1}\right)^{\alpha+\omega}\|v\| \geq$$

$$\geq \frac{1}{M^2N}\left(\frac{t+1}{s+1}\right)^{\alpha}\frac{1}{2^{2(\alpha+\omega)}}\|v\|$$

therefore

$$\left(\frac{t+1}{s+1}\right)^{\alpha}\|v\| \leq N'\left\|\Phi_t^s(x)v\right\|$$

for all $(x, v) \in X \times V$, where $N' = M^2N2^{2(\alpha+\omega)}$. □

## 3. Nonuniform polynomial instability

Let $(A, \varphi)$ be a discrete variational system associated to the discrete evolution semiflow $\varphi : \Delta \times X \to X$ and to the sequence of mappings $A = (A_m)$, where $A_m : X \to \mathcal{B}(V)$, for all $m \in \mathbb{N}$.

**Definition 3.** *The system $(A, \varphi)$ is said to be **polynomially instable** (and denote p.is.) if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that:*

$$\left(\frac{m+1}{n+1}\right)^{\alpha}\|v\| \leq N(n+1)^{\beta}\left\|\Phi_m^n(x)v\right\|$$

*for all $(m, n, x, v) \in \Delta \times X \times V$.*

**Remark 3.** *The system $(A, \varphi)$ is polynomially instable if and only if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that:*

$$\left(\frac{m+1}{n+1}\right)^{\alpha}\left\|\Phi_n^p(x)v\right\| \leq N(n+1)^{\beta}\left\|\Phi_m^p(x)v\right\|$$

*for all $(m, n, p, x, v) \in T \times X \times V$.*

**Remark 4.** *It is obvious that*

$$u.p.is. \Rightarrow p.is.$$

The following example shows that the converse implication is not valid.

**Example 3.** *Let $(X, d)$ be a metric space, $V$ the Banach space and $\varphi$ the discrete evolution semiflow given as in Example 2.*
*We define the sequence of mappings $A_m : X \to \mathcal{B}(\mathbb{R})$ as*

$$A_m(x)v = \frac{u(m+1)x(\tau)}{u(m)x(\tau+1)}$$

*for all $(m, x, v) \in \mathbb{N} \times X \times \mathbb{R}$, where the sequence $u : \mathbb{N} \to \mathbb{R}$ is given by*

$$u(m) = m^2\left(m + 1 + m\cos\frac{m\pi}{2}\right).$$

*So,*

$$\Phi_m^n(x)v = \frac{m^2(m+1+m\cos\frac{m\pi}{2})x(\tau)}{n^2(n+1+n\cos\frac{n\pi}{2})x(m-n+\tau)}v$$

*hence*

$$\left|\Phi_m^n(x)v\right| \geq \frac{m^2 l}{n^2(2n+1)x(0)}|v| \geq \frac{1}{N}\left(\frac{m+1}{n+1}\right)^2\frac{1}{n+1}|v|$$

*and further*

$$\left(\frac{m+1}{n+1}\right)^2 |v| \leq N(n+1)\left|\Phi_m^n(x)v\right|$$

*for all* $(m,n,x,v) \in \Delta \times X \times \mathbb{R}$*, where* $N = \frac{2x(0)}{l}$*. We show that* $(A,\varphi)$ *is p.is.*
*Let us suppose that the system* $(A,\varphi)$ *is u.p.is. According to Definition 1, there are* $N \geq 1$ *and* $\alpha > 1$ *such that*

$$\left(\frac{m+1}{n+1}\right)^\alpha \leq N \frac{m^2(m+1+m\cos\frac{m\pi}{2})x(\tau)}{n^2(n+1+n\cos\frac{n\pi}{2})x(m-n+\tau)}$$

*for all* $(m,n,x) \in \Delta \times X$*. If we consider* $m = 4k+2$ *şi* $n = 4k$*,* $k \in \mathbb{N}$*, we have that*

$$\left(\frac{4k+3}{4k+1}\right)^\alpha \leq N \frac{(4k+2)^2}{16k^2(8k+1)} \frac{x(\tau)}{x(\tau+2)}$$

*which, for* $k \to \infty$*, leads to a contradiction. This proove that* $(A,\varphi)$ *is not u.p.is.*

An important result for polynomial instability of discrete variational systems is given by

**Theorem 4.** *The system* $(A,\varphi)$ *is polynomially instable if and only if there are* $d > 2\gamma \geq 0$ *and* $D \geq 1$ *such that:*

$$\sum_{k=n}^{m} \left(\frac{m+1}{k+1}\right)^d \left\|\Phi_k^n(x)v\right\| \leq D(m+1)^\gamma \left\|\Phi_m^n(x)v\right\| \tag{4}$$

*for all* $(m,n,x,v) \in \Delta \times X \times V$*.*

*Proof. Necessity.* From Remark 3 it results that there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that, for all $d \in (0,\alpha]$ we have

$$\sum_{k=n}^{m} \left(\frac{m+1}{k+1}\right)^d \left\|\Phi_k^n(x)v\right\| \leq$$

$$\leq N \left\|\Phi_m^n(x)v\right\| \sum_{k=n}^{m} \left(\frac{k+1}{m+1}\right)^{\alpha-d} (k+1)^\beta \leq$$

$$\leq N(m+1)^\beta \left\|\Phi_m^n(x)v\right\| \sum_{k=n}^{m} \left(\frac{k+1}{m+1}\right)^{\alpha-d} \leq$$

$$\leq N(m+1)^\beta \frac{m-n+1}{m+1} \left\|\Phi_m^n(x)v\right\| \leq$$

$$\leq N(m+1)^\beta \left\|\Phi_m^n(x)v\right\|$$

for all $(m,n,x,v) \in \Delta \times X \times V$.
*Sufficiency.* The inequality (4) implies that there are $d > 2\gamma \geq 0$ and $D \geq 1$ such that:

$$\|v\| \leq D\left(\frac{n+1}{k+1}\right)^d (k+1)^\gamma \left\|\Phi_k^n(x)v\right\|$$

for all $(k,n,x,v) \in \Delta \times X \times V$. From this and using again the relation (4) it follows that

$$(m-n+1)\|v\| = \sum_{k=n}^{m} \|v\| \leq$$

$$\leq D \sum_{k=n}^{m} \left(\frac{n+1}{k+1}\right)^d (k+1)^\gamma \left\|\Phi_k^n(x)v\right\| =$$

$$= D\sum_{k=n}^{m}\left(\frac{m+1}{k+1}\right)^{d}\left(\frac{n+1}{m+1}\right)^{d}(k+1)^{\gamma}\left\|\Phi_k^n(x)v\right\| \leq$$

$$\leq D\left(\frac{n+1}{m+1}\right)^{d}(m+1)^{\gamma}\sum_{k=n}^{m}\left(\frac{m+1}{k+1}\right)^{d}\left\|\Phi_k^n(x)v\right\| \leq$$

$$\leq D^2\left(\frac{n+1}{m+1}\right)^{d}(m+1)^{2\gamma}\left\|\Phi_m^n(x)v\right\|$$

hence

$$\|v\| \leq D^2\left(\frac{n+1}{m+1}\right)^{d}\frac{(m+1)^{2\gamma}}{m-n+1}\left\|\Phi_m^n(x)v\right\| \leq$$

$$\leq D^2\left(\frac{n+1}{m+1}\right)^{d}(m+1)^{2\gamma}\left\|\Phi_m^n(x)v\right\| =$$

$$= D^2\left(\frac{n+1}{m+1}\right)^{d-2\gamma}(n+1)^{2\gamma}\left\|\Phi_m^n(x)v\right\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$. □

The previous Datko-type theorem gives us the possibility to prove the following theorem of Lyapunov-type:

**Theorem 5.** *The system $(A, \varphi)$ is polynomially instable if and only if there are a Lyapunov function for polynomial instability associated system $(A, \varphi)$, the constants $K \geq 1$ and $\delta > 0$ such that:*

$$L(m, n, x, v) \leq K(m+1)^{\delta}\left\|\Phi_m^n(x)v\right\| \tag{5}$$

*for all $(m, n, x, v) \in \Delta \times X \times V$.*

*Proof. Necessity.* From Theorem 2 we have that the application $L : \Delta \times X \times V \to \mathbf{R}_+$ defined by

$$L(m, n, x, v) = \sum_{k=n}^{m}\frac{(m+1)^{d}}{(k+1)^{d+1}}\left\|\Phi_k^n(x)v\right\|$$

is a Lyapunov function for polynomial instability associated system $(A, \varphi)$. The relation (5) it result from following sequence of inequalities:

$$L(m, n, x, v) = \sum_{k=n}^{m}\frac{(m+1)^{d}}{(k+1)^{d+1}}\left\|\Phi_k^n(x)v\right\| \leq$$

$$\leq \frac{1}{n+1}\sum_{k=n}^{m}\left(\frac{m+1}{k+1}\right)^{d}\left\|\Phi_k^n(x)v\right\| \leq$$

$$\leq \frac{D}{n+1}(m+1)^{\gamma}\left\|\Phi_m^n(x)v\right\| \leq D(m+1)^{\gamma}\left\|\Phi_m^n(x)v\right\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$, where D and $\gamma$ are constants from Theorem 4.
*Sufficiency* it result from Theorem 4 thus:

$$\sum_{k=n}^{m}\left(\frac{m+1}{k+1}\right)^{d}\left\|\Phi_k^n(x)v\right\| = (m+1)\sum_{k=n}^{m}\frac{(m+1)^{d}}{(k+1)^{d}(m+1)}\left\|\Phi_k^n(x)v\right\| \leq$$

$$\leq (m+1)L(m, n, x, v)\left\|\Phi_m^n(x)v\right\| \leq K(m+1)^{\delta+1}\left\|\Phi_m^n(x)v\right\|$$

which implies that the system $(A, \varphi)$ is p.is. □

Further we give a sufficient condition for polynomial instability of discrete variational systems.

**Theorem 6.** *If there are $b > 0$, $\delta \geq 0$ and $B \geq 1$ such that*

$$\sum_{k=n}^{m} \left(\frac{k+1}{n+1}\right)^{b} \left\|\Phi_m^k(x)v\right\| \leq B(n+1)^{\delta} \left\|\Phi_m^n(x)v\right\|$$

*for all $(m, n, x, v) \in \Delta \times X \times V$, then the system $(A, \varphi)$ is polynomially instable.*

*Proof.* For all $(m, n, x, v) \in \Delta \times X \times V$ we have that there exist $b > 0$, $\delta \geq 0$ and $B \geq 1$ such that

$$\left(\frac{m+1}{n+1}\right)^{b} \|v\| \leq \sum_{k=n}^{m} \left(\frac{k+1}{n+1}\right)^{b} \left\|\Phi_m^k(x)v\right\| \leq B(n+1)^{\delta} \left\|\Phi_m^n(x)v\right\|$$

hence the system $(A, \varphi)$ is p.s. □

Discrete-continuous connection for the polynomial instability property of discrete variational systems is given in

**Proposition 2.** *A continuous variational system $(A_c, \varphi_c)$ with the property that there are $M \geq 1$ and $\delta > \gamma > 0$ such that*

$$\|v\| \leq M(t+1)^{\gamma}(s+1)^{-\delta} \left\|\Phi_t^s(x)v\right\| \tag{6}$$

*for all $(t, s, x, v) \in \Delta_c \times X \times V$ is polynomially instable if and only if the associated discrete variational sistem $(A, \varphi)$ is polynomially instable.*

*Proof. Necessity* is immediate as in the uniform case.
*Sufficiency.* If $s \leq t < s + 1$, then the hypothesis implies that

$$\begin{aligned}\left(\frac{t+1}{s+1}\right)^{\alpha} \|v\| &\leq M \left(\frac{t+1}{s+1}\right)^{\alpha} (t+1)^{\gamma}(s+1)^{-\delta} \left\|\Phi_t^s(x)v\right\| = \\ &= M \left(\frac{t+1}{s+1}\right)^{\alpha+\gamma} (s+1)^{\gamma-\delta} \left\|\Phi_t^s(x)v\right\| \leq \\ &\leq M 2^{\alpha+\gamma}(s+1)^{\beta} \left\|\Phi_t^s(x)v\right\|\end{aligned}$$

for all $(x, v) \in X \times V$ and all $\beta > 0$.
If $t \geq s + 1$, then we denote $m = [t]$ şi $n = [s] + 1$ and from hypothesis we have that

$$\begin{aligned}&\left(\frac{t+1}{s+1}\right)^{-\alpha} (s+1)^{\beta} \left\|\Phi_t^s(x)v\right\| = \\ &= \left(\frac{s+1}{t+1}\right)^{\alpha} \left\|\Phi_t^m(\varphi(m, s, x))\Phi_m^n(\varphi(n, s, x))\Phi_n^s(x)v\right\| \geq \\ &\geq \left(\frac{s+1}{t+1}\right)^{\alpha} \frac{1}{M} \frac{(m+1)^{\delta}}{(t+1)^{\gamma}} \frac{1}{N} \left(\frac{m+1}{n+1}\right)^{\alpha} \frac{1}{(n+1)^{\beta}} \frac{1}{M} \frac{(s+1)^{\delta}}{(n+1)^{\gamma}} (s+1)^{\beta} \|v\| = \\ &= \frac{1}{M^2 N} \left(\frac{s+1}{n+1}\right)^{\alpha+\beta} \left(\frac{m+1}{t+1}\right)^{\alpha} \frac{(s+1)^{\delta}}{(n+1)^{\gamma}} \frac{(m+1)^{\delta}}{(t+1)^{\gamma}} \|v\| \geq \\ &\geq \frac{1}{M^2 N} \left(\frac{n}{n+1}\right)^{\alpha+\beta+\gamma} \left(\frac{t}{t+1}\right)^{\alpha+\gamma} \|v\| \geq \frac{1}{M^2 N} \frac{1}{2^{2\alpha+\beta+2\gamma}} \|v\|\end{aligned}$$

for all $(x, v) \in X \times V$, which implies that

$$\left(\frac{t+1}{s+1}\right)^{\alpha} \|v\| \leq N'(s+1)^{\beta} \left\|\Phi_t^s(x)v\right\|$$

where $N' = M^2 N 2^{2\alpha+\beta+2\gamma}$, consequently, $(A_c, \varphi_c)$ is p.is. □

**Remark 5.** *Theorem 3 and Theorem 6 are generalizations for the case of uniform (respectively nonuniform) polynomial instability of discrete variational systems of the classic result proved by E. A. Barbashin in* [1] *for uniform exponential stability.*

**Remark 6.** *Propositions 1 and Propositions 2 emphasizes that the results of discrete case for variational systems leads to some conclusions for continuous case of these systems.*

## References

[1] E. A. Barbashin, *Introduction in Stability Theory*, Izd. Nauka, Moskva, 1967.
[2] L. Barreira, C. Valls, *Stability of Nonautonomous Differential Equations*, Lecture Notes in Math., 1926, Springer, (2008).
[3] L. Barreira, C. Valls, *Polynomial growth rates*, Nonlinear Analysis, 7(2009), 5208-5219.
[4] R. Datko, *Uniform asymptotic stability of evolutionary processes in a Banach spaces*, SIAM J.Math. Anal., 3(1973), 428-455.
[5] S. Elaydi, *Asymptotics for linear difference equations*, J.Diff. Equ. Appl. 5 (1999), 563-589.
[6] A. Lyapunov, *The General Problem of the Stability of Motion*, Taylor & Francis, 1992.
[7] J. L. Massera, J. J. Shaffer, *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.
[8] Mihail Megan, Traian Ceaușu, L.M. Rămneanțu, *Polynomial instability of evolution operators in Banach spaces*, Editura Carpahatian J. Math. 26 Vol. 2, 2010.
[9] M. Megan, C. Stoica, *Discrete asymptotic behaviors for skew-evolution semiflows on Banach spaces*, Carpathian J. Math. 24 (2008) 348-355.
[10] M. Megan, C. Stoica, L. Buliga, *On asymptotic behaviours for skew-evolution semiflows in Banach spaces*, Carpathian J. Math. 23 (2007) pp. 117-125.
[11] M. L. Rămneanțu, *Uniform polynomial dichotomy of evolution operators in Banach spaces*, Analele Universității de Vest Timișoara, seria Matematică-Informatică XLIX, 1(2011), 107-116.
[12] M. A. Tomescu, *On exponential instability of variational nonautonomous difference equations in Banach spaces*, Int. J. of Pure and Appl. Math., Vol. 76 4(2012), 615-624.

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