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POLYNOMIAL COEFFICIENTS AND APPROXIMATION ERRORS OF ENTIRE SERIES

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Abstract. In this paper we consider the maximum of entire function $f(z)$ over a certain lemniscate instead of considering the maximum of $f(z)$ on $|f(z)| = r$ and obtained analogous results for entire functions of the form $f(z) = \sum_{k=1}^{\infty} q_k(z)[\gamma(z)]^{k-1}$, where $\gamma(z)$ is a polynomial of degree m and $q_k(z)$ is of degree $(m-1)$. The (p, q)-order and generalized (p, q)-type have been characterized in terms of Polynomial Coefficients and L^s-approximation errors, $1 \leq s \leq \infty$. Finally, a saturation theorem for $f(z)$ which can be extended to a entire function of (p, q)-order 0 or 1 and for entire functions of minimal generalized (p, q)-type have been obtained.

1. Introduction

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a nonconstant entire function and assume that $a_k \neq 0$ for $k = 1, 2, ...$ We set $M(r, f) = \max_{|z|=r} |f(z)|$ and $\mu(r, f) = \max_{n\geq 0}\{|a_k|r^k\}$, $M(r, f)$ and $\mu(r, f)$ are called the maximum modulus and maximum term respectively.

For classifying entire functions their growth, the concept of order was introduced. If the order is a (finite) positive number, then the concept of type permits a subclassification. For the class of order $\rho = 0$ and $\rho = \infty$ no subclassification is possible. For example all entire functions that grow at least as fast as $\exp(\exp(z))$ have to be kept in one class. For this reason, numerous attempts have been made to refine the concept of order and type. It is known that the order ρ $(0 \leq \rho \leq \infty)$ and the type T $(0 \leq T \leq \infty)$ are give by [1, p. 9-11]

$$\limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r} = \rho = \limsup_{k\to\infty} \frac{k \log k}{\log |a_k|^{-1}} \tag{1}$$

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{r^\rho} = T = \limsup_{k\to\infty} \Big(\frac{1}{e\rho}\ k|a_k|^{\rho/k}\Big). \tag{2}$$

In this paper we have picked up a concept of (p, q)-order and(p, q)-type which was introduced by Juneja et al ([4, 5]) for further classification of entire function of order $\rho = 0$ and $\rho = \infty$ and show that instead of considering the maximum of $|f(z)|$, for $|z| = r$, we can consider the maximum of $f(z)$ over a certain lemniscate and obtain the analogous results for the entire function $f(z)$ when it is expanded in the form [12, pp. 56], $f(z) = \sum_{k=1}^{\infty} q_k(z)[\gamma(z)]^{k-1}$ where $\gamma(z)$ is a polynomial of degree m and $q_k(z)$ is of degree $m - 1$ and the equipotential curve $|\gamma(z)| = R$ defines the lemniscate mentioned above.

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2. Definitions And Auxiliary Results

First we introduce the concept of (p,q)-scale, $p \geq q \geq 1$, and certain notations which will be frequently used in the text

$$\exp^{[m]}x = \log^{[-m]}x = \exp(\exp^{[m-1]}x) = \log(\log^{[-m-1]}x)\ , m = \pm 1, \pm 2, \ldots,$$

$$\Lambda_{[r]}(x) = \Pi_{i=0}^{r}\log^{[i]}x \text{ for } r = 0, 1, \ldots,$$

$$P(L(p,q)) = \begin{cases} L(p,q) & if\ q < p < \infty, \\ 1 + L(p,q) & if\ p = q = 2, \\ \max(1, L(p,q)) & if\ 3 \leqslant p \leqslant q, \\ \infty & if\ p = q = \infty, \end{cases}$$

Definition 1. *An entire function $f(z)$ is said to be of (p,q)-order $\rho(p,q)$ if it is of index-pair (p,q) such that*

$$\limsup_{r\to\infty} \frac{\log^{[p]}M(r,f)}{\log^{[q]}r} = \rho(p,q),$$

and the function $f(z)$ having (p,q)-order $\rho(p,q)$ $(b < \rho(p,q) < \infty)$ is said to be of (p,q)-type $T(p,q)$ if

$$\limsup_{r\to\infty} \frac{\log^{[p-1]}M(r,f)}{\left(\log^{[q-1]}r\right)^{\rho(p,q)}} = T(p,q),\ \ 0 \leq T(p,q) \leq \infty,$$

where $b = 1$ if $p = q$, $b = 0$ if $p > q$.

These concepts are inadequate to compare the growth of those functions which are of the same order and of infinite type. Hence, for a refinement of the above growth scale, one may utilize proximate order the concept of which is extended by Nandan et al [9] to entire functions with (p,q)-growth as

Definition 2. *A positive function $\rho_{p,q}(r)$ defined on $[r_0, \infty)$, $r_0 > \exp^{[q-1]} 1$, is said to be the proximate order of an entire function with index- pair (p,q) if*
(i) $\rho_{p,q}(r) \longrightarrow \rho(p,q)$ as $r \longrightarrow \infty$, $(b < \rho(p,q) < \infty)$,
(ii) $\Lambda_{[q]}(r)\ \rho'_{p,q}(r) \longrightarrow 0$ as $r \longrightarrow \infty$; $\rho'_{p,q}(r)$ are denotes the derivative of $\rho_{p,q}(r)$.

It is known that [9, Thm. 4] that $\left(\log^{[q-1]} r\right)^{\rho_{p,q}(r)-A}$ is a monotonically increasing function of r for $r > r_0$, where $A = 1$ of $(p,q) = (2,2)$ and $A = 0$ otherwise. Hence we can define the function $\phi(x)$ for $x > x_0$ to be the unique solution of equation

$$x = \left(\log^{[q-1]} r\right)^{\rho_{p,q}(r)-A} \Leftrightarrow \phi(x) = \log^{[q-1]}r.$$

Definition 3. *Let $f(z)$ be an entire function of (p,q)-order $\rho(p,q$ $(b < \rho(p,q < \infty)$ such that*

$$\limsup_{r\to\infty} \frac{\log^{[p-1]}M(r,f)}{\left(\log^{[q-1]}r\right)^{\rho_{p,q}(r)}} = T^*(p,q),\ \ 0 \leq T^*(p,q) \leq \infty.$$

If the quantity $T^*(p,q)$ is different from zero and infinite then $\rho_{p,q}(r)$ said to be the proximate order of a given entire function $f(z)$ and $T^*(p,q)$ as its generalized (p,q)-type.

Definition 4. *An entire function with index-pair (p,q) is said to be of minimal, normal and maximal (p,q)-type with respect to a proximate order according as $T^*(p,q)$ as zero, positive finite and infinite respectively.*

Rice [10] has extended the results (1) and (2) by considering the polynomial expansion of $f(z)$ of the form

$$f(z) = \sum_{k=1}^{\infty} q_k(z)[\gamma(z)]^{k-1}. \tag{3}$$

If Γ_R be the lemniscate $\Gamma_R = \{z : |\gamma(z)| = R\}$, $\|\Gamma_R\|$ be the length of Γ_R and $M(\Gamma_R, f) = \|f\|_{\Gamma_R} = \max_{z\in\Gamma_R} |f(z)|$, then using the estimate

$$\|\Gamma_R\| = 2\pi R^{\frac{1}{m}}(1 + o(1)) \; as \; R \longrightarrow \infty,$$

be showed that $f(z)$ given by (3) is an entire function of order ρ, if and only if,

$$\limsup_{R\to\infty} \frac{\log\log M(\Gamma_R, f)}{\log R} = \rho/m,$$

and that $f(z)$ is of order $\rho > 0$ and type $T(0 < T < \infty)$, if and only if,

$$\limsup_{R\to\infty} \frac{\log M(\Gamma_R, f)}{R^{\rho/m}} = T.$$

The generalization of (1) and (2) read as follows:

Let α be fixed. Then $f(z)$, given by (3), is an entire function of order $\rho > 0$, if and only if,

$$\rho = m \limsup_{R\to\infty} \frac{k \log k}{\log\left(\|q_k(z)\|_{\Gamma_\alpha}\right)^{-1}}, \tag{4}$$

and that it is of order $\rho > 0$ and type $T(0 < T < \infty)$, if and only if

$$e\rho T = m \limsup_{R\to\infty} \log\left(\|q_k(z)\|_{\Gamma_\alpha}\right)^{\rho/mn} \tag{5}$$

Results (4) and (5) depict the influence of the rate of decreasebof $\|q_k(z)\|_{\Gamma_\alpha}$ on the growth of $f(z)$. But, as in the case of power series, these results also do not give any precise information about the fast and slow growth and if $f(z)$ has same order and infinite type. For this purpose, in the present paper, we obtain the formulae for (p, q)-order and generalized (p, q)-type of $f(z)$ given by (5) in terms of the polynomial coefficients $\|q_k(z)\|_{\Gamma_\alpha}$ and polynomial approximation error in L^s-norm, $1 \le s \le \infty$. Finally we have obtained a saturation theorem for $f(z)$ which can be extended to be an entire function of (p, q)-order 0 or 1 and for entire function of minimal generalized (p, q)-type order. Our results include some results of Sato [11], Rice [10], Juneja [2], and Juneja and Kapoor [3]. Recently Kumar [7] studied the results for (α, α) order and Kumar and Kaur [8] obtained the results for (α, β)- orders for analytic functions. However, they have to study separately the entire functions of slow and fast growth. That is why in our studies the (p, q)-growth has been preferred to the (α, α) and (α, β)-growths.

Consider the function

$$H_\alpha(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k \;, \; \alpha < R,$$

where $\|q_k(z)\|_{\Gamma_\alpha} = \max_{z\in\Gamma_R}\{|q_k(z)|\}$ as $k \to \infty$. It is known [10, Lemma 2] that if $f(z)$ is analytic in Γ_R, then there exists a polynomial $Q(z)$ of degree $m-1$ independent of k and R such that for $\alpha < R$ and $k = 1, 2, \ldots$

$$\|q_k(z)\|_{\Gamma_\alpha} \le \frac{\|\Gamma_R\| M(\Gamma_R, f)}{2\pi R^k} \|Q(z)\|_{\Gamma_R}. \tag{6}$$

Using (6) we can easily seen that $H_\alpha(w)$ is entire if and only if

$$\left[\|q_k(z)\|_{\Gamma_\alpha}\right]^{1/k} = 0. \tag{7}$$

Moreover, $H_\alpha(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k$, holds in the whole complex plane.

Lemma 1. *$f(z) = \sum_{k=1}^{\infty} q_k(z)[\gamma(z)]^{k-1}$ is an entire function of (p,q)-order $\rho(p,q,f)$ if and only if*

$$\limsup_{R\to\infty} \frac{\log^{[p]} M(\Gamma_R, f)}{\log^{[q]} R} = \rho(p,q,f)/m.$$

Further, if the (p,q)-order of $f(z)$ is $(b < \rho(p,q,f) < \infty)$, then it is of (p,q)-type $T^*(p,q,f)$, if and only if,

$$\limsup_{R\to\infty} \frac{\log^{[p-1]} M(\Gamma_R, f)}{\left(\log^{[q-1]} R\right)^{\rho_{p,q(R)/m}}} = T^*(p,q,f)/m.$$

Proof. The lemma follows on the lines similar to those of Rice [10, Lemma 3].

Let B^* be a component of the complement of the closure of the Caratheodory domain B that contains the point ∞. Set $B_R = \{z : |\bar{\phi}(z)| = R\}$, $R > 1$ where that function $w^* = \bar{\phi}(z)$ Maps B^* conformally onto $|w^*| > 1$ such that $\bar{\phi}(\infty) = \infty$ and $\bar{\phi}'(\infty) > 0$. Here B_R is the largest equipotential curve of the modulus of the mapping function associated with the domain B, B_1, correspond to the boundary of B.

Given $\varepsilon > 0$ there is a lemniscate $\Gamma_\alpha = \{z : |\gamma(z)| = \alpha\}$ so that Γ_α is interior to $B_{1+\varepsilon}$ and exterior to B_1. □

Lemma 2. *$f(z) = \sum_{k=0}^{\infty} q_k(z)[\gamma(z)]^{k-1}$ be an entire function of (p,q)-order $\rho(p,q,f)$ then*

$$\limsup_{R\to\infty} \frac{\log^{[p]} \bar{M}(\Gamma_R, f)}{\log^{[q]} R} = \rho(p,q,f)/m.$$

Further, if the (p,q)-order of $f(z)$ is $(b < \rho(p,q,f) < \infty)$, then it is of (p,q)-type $T^*(p,q,f)$, such that,

$$\limsup_{R\to\infty} \frac{\log^{[p-1]} \bar{M}(\Gamma_R, f)}{\left(\log^{[q-1]} R\right)^{\rho_{p,q}(R)/m}} = T^*(p,q,f)/m,$$

where $\bar{M}(\Gamma_R, f) = \max_{z\in B_R} |f(z)|$.

Proof. Let z_0 be a fixed point of the set B and $R > 1$. Then using [14], we get

$$R - 2|B| - |z_0| \le |z| \le R + |B| + |z_0|,\ z \in B_R.$$

For $p \ge q \ge 1$, $\xi^* < 1$ and $\eta > 1$, using $\log^{[q]} Kx \simeq \log^{[q]} x$ as $x \to \infty$, $0 < K < \infty$,

$$\frac{\log^{[p]} M(\Gamma_{\xi^* R}, f)}{\log^{[q]} R} \le \frac{\log^{[p]} \bar{M}(\Gamma_R, f)}{\log^{[q]} R} \le \frac{\log^{[p]} M(\Gamma_{\eta R}, f)}{\log^{[q]} R}.$$

Also, for $\rho(p,q,f)$ $(b < \rho(p,q,f) < \infty)$, using $\log^{[q]}(K + x) \simeq \log^{[q]} x$ as $x \to \infty$,

$$\frac{\log^{[p-1]} M(\Gamma_{R-a}, f)}{\left(\log^{[q-1]} R\right)^{\rho_{p,q}(R)/m}} \le \frac{\log^{[p-1]} \bar{M}(\Gamma_R, f)}{\left(\log^{[q-1]} R\right)^{\rho_{p,q}(R)/m}} \le \frac{\log^{[p-1]} M(\Gamma_{R+c}, f)}{\left(\log^{[q-1]} R\right)^{\rho_{p,q}(R)/m}},$$

where $a = 2|B| + z_0$, $c = |B| + |z_0|$.

After passing to the limits and taking the Lemma 1 into account this lemma is immediate. □

Let $L^s(B)$, $1 \le s \le \infty$, be the class of all functions f holomorphic on B and satisfying

$$\|f\|_{B,s} = \left[\frac{1}{A^*}\int\int_B |f(z)|^s dx\ dy\right]^{1/s} < \infty.$$

where the last inequality is understood to be $\sup_{z\in B}|f(z)| < \infty$ for $s = \infty$. Then $\|,\|_{B,s}$ is called the L^s-norm on $L^s(B)$. For $f \in L^s(B)$, we define $E^s_\xi(f)$, the error in approximating the function f by polynomial of degree at most $\xi = mn$ in L^s-norm as

$$E^s_\xi(f) = E^s_\xi(f, B) = \inf_{t\in\pi_\xi} \|f - t\|_{B,s},\ n = 0, 1, 2, \dots$$

where π_ξ consists of all polynomials of degree at most $\xi = mn$.

3. Main Results

In this section we shall prove our main results.

Theorem 1. *If $f \in L^s(B)$, $1 \le s \le \infty$, can be extended to an entire function with index-pair (p, q),(p, q)-order $\rho(p, q, f)$ $(b < \rho(p, q, f) < \infty)$ and generalized (p, q)-order $T^*(p, q, f)$, then for every $\|q_k(z)\|_{\Gamma_\alpha}$, there exists an entire function $H_\alpha(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k$, such that*

$$\rho(p, q, f)/m = \rho(p, q, H_\alpha) \text{ and } T^*(p, q, f) = T^*(p, q, H_\alpha) \tag{8}$$

Proof. By virtue of (7) H_α is an entire function. From [13, p.77] for $R > \alpha$, we have

$$\|q_k(z)\|_{\Gamma_R} \le \|q_k(z)\|_{\Gamma_\alpha} R^{m-1},$$

for $z \in \Gamma_R$

$$|f(z)| \le \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_R}\ \|\gamma(z)\|^{k-1}_{\Gamma_R}$$

or

$$\begin{aligned}\bar{M}(\Gamma_R, f) &\le \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_R} R^{k+m-2},\ z \in B_k\\ &= R^{m-2}\sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_R} R^k\\ &= R^{m-2}\ H_\alpha(R),\ R > 1.\end{aligned}$$

Using Lemma 1, we observe that for all index-pair (p, q).

$$\rho(p, q, f)/m \le \rho(p, q, H_\alpha) \text{ and } T^*(p, q, f) \le T^*(p, q, H_\alpha). \tag{9}$$

Consider the power series expansion of $H_\alpha(w)$, we have from Lemma 1, Lemma 2 and (6) for every $\varepsilon > 0$,

$$\begin{aligned}H_\alpha(R/e^\varepsilon) &= \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} (R/e^\varepsilon)^k\\ &\le \sum_{k=1}^{\infty} \frac{\bar{M}(\Gamma_R, f)\ \|\Gamma_R\|\ \|Q\|_{\Gamma_\alpha} (R/e^\varepsilon)^k}{2\pi R^k}\end{aligned}$$

$$= \bar{M}(\Gamma_R, f) R^{1/m}(1 + O(1)) \ \|Q\|_{\Gamma_\alpha} \sum_{k=1}^{\infty} \frac{1}{e^{k\varepsilon}}$$

$$= \bar{M}(\Gamma_R, f) R^{1/m}(1 + O(1)) \ \|Q\|_{\Gamma_\alpha} \frac{1}{e^{\varepsilon - 1}}.$$

Thus in view of above inequality and Lemma 1, lemma 2 for $p \geq 2$ and $q \geq 1$,

$$\rho(p, q, H_\alpha) \leq \rho(p, q, f)/m \text{ and } T^*(p, q, H_\alpha) \leq T^*(p, q, f). \tag{10}$$

Combining (9) and (10), the proof is completed for $s = \infty$. The result can be proved for $0 \leq s < \infty$ in a similar manner. □

Theorem 2. *If $f \in L^s(B)$, $1 \leq s \leq \infty$, can be extended to be an entire function with index-pair (p, q), (p, q)-order $\rho(p, q, f)$ $(b < \rho(p, q, f) < \infty)$ and generalized (p, q)-type $T^*(p, q, f)$, then for every $E^s_\xi(f)$, there exists an entire function $\tilde{H}(t) = \sum_{k=1}^{\infty} E^s_\xi(f) t^k$, such that*

$$\rho(p, q, f)/m = \rho(p, q, \tilde{H}) \text{ and } T^*(p, q, f) = T^*(p, q, \tilde{H}). \tag{11}$$

Proof. Define the function

$$\tilde{f}(z) = \sum_{k=0}^{\infty} \Big(P_{k+1}(z) - P_k(z) \Big), \tag{12}$$

since

$$|P_{k+1}(z) - P_k(z)| \leq \|P_{k+1}(z) - P_k(z)\| \leq 2 \ \|f - P_k(z)\|, \ z \in B.$$

Using Walsh inequality, [13, p.77], we get

$$|P_{k+1}(z) - P_k(z)| \leq 2 \ \|f - P_k(z)\|^2_{B,1} {R'}^k, \ z \in B_{k'} \ , \ R' > 1.$$

Applying Holder inequality, we get

$$\|P_{k+1}(z) - P_k(z)\| {R'}^k \leq 2 \ {A^*}^{q'} \|f - P_k(z)\|_{B_{R', q'}},$$

where A^* is defined as earlier and $q' = 1 - 1/s$, $1 \leq s \leq \infty$. Since above inequality holds for any polynomial $P_k(z)$, we have

$$\|P_{k+1}(z) - P_k(z)\| {R'}^k \leq 2 \ {A^*}^{q'} E^s_{k-1}(f) \ , \ 1 \leq s < \infty \ . \tag{13}$$

From [12], we get

$$E^s_\xi(f) \leq \sum_{k=n}^{\infty} \|q_k(z)\|_{\Gamma_R} \ \alpha^{k-1}.$$

Using (6) it given

$$\leq \sum_{k=n}^{\infty} \frac{\|\Gamma_R\| \bar{M}(\Gamma_R, f)}{2\pi R^k} \|Q(z)\|_{\Gamma_R} \ \alpha^{k-1}.$$

For $\alpha > 1$ be fixed constant and $R > \alpha$, we obtain

$$E^s_\xi(f) \leq \gamma^* \bar{M}(\Gamma_R, f) \Big(\frac{\alpha}{R}\Big)^n \Big(\frac{1}{1 - \alpha/R}\Big) \ R^{1/m}(1 + O(1)) \ , \tag{14}$$

for sufficiently large R.

Using (12) and (14), we obtain from (11) that

$$|\tilde{f}(z)| \leq \sum_{k=0}^{\infty} \|P_{k+1}(z) - P_k(z)\|$$

or

$$\bar{M}(\Gamma_R, f) \le |a_0| + 2A^{*^{q'}} \sum_{k=0}^{\infty} E_{k-1}^{s}(f)(RR')^{R} \ , \ z \in B_R.$$

$$\le |a_0| + 2A^{*^{q'}}(RR')^{1/R} \ \mu(RR^*, \tilde{H}) \tag{15}$$

The right hand side of series (12) converges for every R and therefore, the series on the right of (15) converges uniformly on every compact subset of complex plane and so $\tilde{f}(z)$ is entire and $\tilde{f}(z) = f(z)$. Since $\lim_{\xi\to\infty} \left[E_{\xi}^{s}(f)\right]^{1/\xi} = 0$ by (14) it gives that $\tilde{H}(t)$ is entire. In view of Lemma 1, Lemma 2 and (15) for all index pair (p, q), we have

$$\rho(p, q, f)/m \le \rho(p, q, \tilde{H}) \text{ and } T^*(p, q, f) \le T^*(p, q, \tilde{H}). \tag{16}$$

From (13), we have

$$\mu(R, \alpha\tilde{H}) \le \gamma^* \bar{M}(\Gamma_R, f)\Big(\frac{R^{1+1/m}}{R-\alpha}\Big)(1 + O(1)). \tag{17}$$

Again in view of Lemma 1, Lemma 2 and (16) we obtain for $p \ge 2$ and $q \ge 1$,

$$\rho(p, q, f)/m \ge \rho(p, q, \tilde{H}) \textit{ and } T^*(p, q, f) \ge T^*(p, q, \tilde{H}). \tag{18}$$

Combining (16) and (18) the proof is complete. □

Theorem 3. *Let $f(z) \in L^s(B)$, $1 \le s \le \infty$. Then $f(z)$ can be extended to be an entire function of (p, q)-order $\rho(p, q, f)$ $(b < \rho(p, q, f) < \infty)$ if and only if*

$$\rho(p, q, f)/m = P(L(p, q, \tilde{H})) = P(L(p, q, \tilde{H}),$$

where

$$L(p, q, \tilde{H}) = \limsup_{k\to\infty} \frac{\log^{[p-1]} k}{\log^{[q-1]} \Big(\|q_k(z)\|_{\Gamma_\alpha}\Big)^{-1/k}}$$

and

$$L(p, q, \tilde{H}) = \limsup_{k\to\infty} \frac{\log^{[p-1]} k}{\log^{[q-1]} \Big(E_{\xi}^{s}(f)\Big)^{-1/k}} \ , \ \xi = mk.$$

Proof. In Theorem 1 and Theorem 2 we have concluded that $f \in L^s(B)$ can be extended to an entire function if and only if $H_\alpha(w)$ and $\tilde{H}(t)$ are entire functions. Applying corollarly 1 by Juneja at al [4, p. 62] to the functions $H_\alpha(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k$ and $\tilde{H}(t) = \sum_{k=1}^{\infty} E_{\xi}^{s}(f)t^k$, $\xi = mk$, with (8) and (11) Theorem 3 follows. □

Theorem 4. *If $f \in L^s(B)$, $1 \le s \le \infty$. Then $f(z)$ can be extended to an entire function of (p, q)-order $\rho(p, q, f)$ $(b < \rho(p, q, f) < \infty)$ and generalized (p, q)-type $T^*(p, q, f)$ $(0 < T^*(p, q) < \infty)$ if and only if*

$$\frac{T^*(p, q, f)}{M(p, q)} = \limsup_{k\to\infty} \left[\frac{\phi\Big(\log^{[p-2]} k\Big)}{\log^{[q-1]} \Big(\|q_k(z)\|_{\Gamma_\alpha}\Big)^{-1/k}}\right]^{\frac{\rho(p,q,f)}{m} - A}$$

$$= \limsup_{k\to\infty} \left[\frac{\phi\Big(\log^{[p-2]} k\Big)}{\log^{[q-1]} \Big(E_{\xi}^{s}(f)\Big)^{-1/k}}\right]^{\frac{\rho(p,q,f)}{m} - A} \ , \ \xi = mk \ ,$$

where A is defined as earliar and

$$M(p,q)=\begin{cases}\dfrac{(\zeta-1)^{(\zeta-1)}}{\zeta\zeta} & \text{if } (p,q)=(2,2),\ \zeta=\dfrac{\rho(2,2)}{m}\\ \dfrac{m}{e\,\rho(2,1)} & \text{if } (p,q)=(2,1)\\ 1 & \text{otherwise.}\end{cases}$$

Proof. To prove this theorem apply Theorem 1 by Kasana [6] to the functions $H_\alpha(w)$ and $\tilde{H}(t)$ defined in Theorem 1 and Theorem 2 and the resulting characterization of $T^*(p,q,H_\alpha)$ and $T^*(p,q,\tilde{H})$ in terms of $\|q_k(z)\|_{\Gamma_\alpha}$ and $E^s_\xi(f)$ respectively with relations $T^*(p,q,f)=T^*(p,q,H_\alpha)=T^*(p,q,\tilde{H})$ prove the theorem.

Taking $\rho_{p,q}(R)=\rho(p,q)\ \forall\ R>R_0$ and $\phi(x)=\frac{1}{x\left(\rho(p,q)/m\right)-A}$, we get the following corollary which gives a formula for (p,q)-type $T(p,q,f)$ in terms of the polynomial coefficients and approximation errors of an entire function $f(z)=\sum_{k=1}^{\infty} q_k(z)[\gamma(z)]^{k-1}$ corollary. Let $f(z)\in L^s(B)$. Then $f(z)$ is the restriction to B of an entire function having (p,q)-order $\rho(p,q,f)$ $(b<\rho(p,q,f)<\infty)$ and (p,q)-type $T(p,q,f)$ $(0<T(p,q,f)<\infty)$ if and only if

$$\frac{T(p,q,f)}{M(p,q)}=\limsup_{k\to\infty}\frac{\log^{[p-2]}k}{\left[\log^{[q-1]}\left(\|q_k(z)\|_{\Gamma_\alpha}\right)^{-1/k}\right]^{\frac{\rho(p,q,f)}{m}-A}}$$
$$=\limsup_{k\to\infty}\frac{\log^{[p-2]}k}{\left[\log^{[q-1]}\left(E^s_\xi(f)\right)^{-1/k}\right]^{\frac{\rho(p,q,f)}{m}-A}}.$$

□

Theorem 5. *If $f(z)\in L^s(B)$ can be extended to be an entire function of (p,q)-order $\rho(p,q,f)$ such that $\rho(p,q,f)=b$, then for every $\delta>0$,*

$$\limsup_{k\to\infty}\frac{\left(\log^{[p-2]}k\right)^{\delta}}{\log^{[q-1]}\left(\|q_k(z)\|_{\Gamma_\alpha}\right)^{-1/k}}=0.$$

Further, if $\rho(p,q)>b$ and $f(z)$ is of minimal generalized (p,q)-type, then

$$\limsup_{k\to\infty}\frac{\phi\left(\log^{[p-2]}k\right)}{\log^{[q-1]}\left(\|q_k(z)\|_{\Gamma_\alpha}\right)^{-1/k}}=0.$$

Proof. Since $\rho(p,q,f)=b$, using Lemma 2 for given $\varepsilon>0$ and $r>r_0$

$$\log\bar{M}(\Gamma_R,f)<\exp^{[p-2]}\left(\log^{[q-1]}\right)^{(b/m)+\varepsilon}. \tag{19}$$

we have

$$H_\alpha(R/e^\varepsilon)\le\bar{M}(\Gamma_R,f)R^{1/m}(1+O(1))\ \|Q\|_{\Gamma_\alpha}\frac{1}{(e^\varepsilon-1)}.$$

or

$$\|q_k(z)\|_{\Gamma_\alpha}(R/e^\varepsilon)^k\le\bar{M}(\Gamma_R,f)R^{1/m}(1+O(1))\ \|Q\|_{\Gamma_\alpha}\frac{1}{(e^\varepsilon-1)}$$

$$\|q_k(z)\|_{\Gamma_\alpha} \leq \bar{M}(\Gamma_R, f) R^{1/m}(1+O(1))\ \|Q\|_{\Gamma_\alpha} \frac{e^{k\varepsilon}}{(e^\varepsilon - 1)} R^{-k}.$$

Using (19), we get

$$\log \|q_k(z)\|_{\Gamma_\alpha} < \exp^{[p-2]} \left(\log^{[q-1]} R \right)^{(b/m)+\varepsilon} + \frac{1}{m} \log R - R \log R + \log \left(\|Q\|_{\Gamma_\alpha} \frac{e^{k\varepsilon}}{(e^\varepsilon - 1)} \right)$$

or

$$\log \left(\|q_k(z)\|_{\Gamma_\alpha} \right)^{1/k} < \frac{\exp^{[p-2]} (\log^{[q-1]} R)^{(b/m)+\varepsilon}}{k} + \frac{1}{km} \log R - \log R + \frac{1}{k} \log \left(\|Q\|_{\Gamma_\alpha} \frac{e^{k\varepsilon}}{(e^\varepsilon - 1)} \right). \tag{20}$$

Choose the value of R satisfying

$$R = \exp^{[q-1]} \left(\log^{[p-2]} \frac{k}{(b/m+\varepsilon)} \right)^{1/(b/m+\varepsilon)}. \tag{21}$$

For $(p,q) = (2,1)$, (21) gives $R = (\frac{k}{\varepsilon})^{1/\varepsilon}$ and using the value in (20) we obtain

$$\left\| q_k(z) \right\|_{\Gamma_\alpha}^{1/k} < \frac{e^{k/\varepsilon} \left(\frac{k}{\varepsilon} \right)^{1/km\varepsilon} \|Q\|_{\Gamma_\alpha} \left(\frac{e^{k\varepsilon}}{e^\varepsilon - 1} \right)}{\left(k/\varepsilon \right)^{1/\varepsilon}}$$

or

$$k^{1/\varepsilon} \left(\|q_k(z)\|_{\Gamma_\alpha} \right)^{1/k} < \infty. \tag{22}$$

For $(p,q) = (2,2)$ we see that $\log R = (\frac{k}{1/m+\varepsilon})^{1/(1/m+\varepsilon)}$ satisfies (20) and (21) is reduced to

$$\log \left(\|q_k(z)\|_{\Gamma_\alpha} \right)^{-1/k} > (1+O(1)) \Big(\frac{k}{1/m+\varepsilon} \Big)^{1/(1/m+\varepsilon)},$$

which gives

$$\limsup_{k\to\infty} \frac{k^{1/(1/m+\varepsilon)}}{\log \left(\|q_k(z)\|_{\Gamma_\alpha} \right)^{-1/k}} \leq 1. \tag{23}$$

Finally, for $p,q \neq (2,1)$ and $p,q \neq (2,2)$, (20) and (21) yield

$$\log^{[q-1]} \left(\|q_k(z)\|_{\Gamma_\alpha} \right)^{-1/k} > (1+O(1)) \Big(\log^{[p-2]} \frac{k}{\varepsilon} \Big)^{1/\varepsilon},\ p > q\ ,$$

or

$$\log^{[q-1]} \left(\|q_k(z)\|_{\Gamma_\alpha} \right)^{-1/k} > (1+O(1)) \Big(\log^{[p-2]} \frac{k}{(1/m+\varepsilon)} \Big)^{\frac{1}{1/m+\varepsilon}},\ p = q\ .$$

Thus for all $p \geq q \geq 3$,

$$\limsup_{k\to\infty} \frac{\left(\log^{[p-2]} k \right)^{1/(1/m+\varepsilon)}}{\log^{[q-1]} \left(\|q_k(z)\|_{\Gamma_\alpha} \right)^{-1/k}} \leq 1. \tag{24}$$

Combining (22), (23) and (24) we get

$$\limsup_{k\to\infty} \frac{\left(\log^{[p-2]} k \right)^{\delta}}{\log^{[q-1]} \left(\|q_k(z)\|_{\Gamma_\alpha} \right)^{-1/k}} \leq \infty, \tag{25}$$

for every $\delta > 0$.

If limit superior in (25) is finite and positive for some $\delta > 0$ then for every $\delta^* > 0$, we have

$$\limsup_{k\to\infty} \frac{\left(\log^{[p-2]} k\right)^{\delta+\delta^*}}{\log^{[q-1]}\left(\|q_k(z)\|_{\Gamma_\alpha}\right)^{-1/k}} = \infty.$$

This is a contradiction to what we obtain in (25) and hence the first part is proved.

For the second part putting $T^*(p,q,f) = 0$ in Theorem 4 and we get the required result. Hence the proof is complete. □

Similarly we can prove the following theorem.

Theorem 6. *If $f(z) \in L^s(B)$ can be extended to be an entire function of (p,q)-order $\rho(p,q,f)$ such that $\rho(p,q,f) = b$, then for ever $\delta > 0$,*

$$\limsup_{k\to\infty} \frac{\left(\log^{[p-2]} k\right)^{\delta}}{\log^{[q-1]}\left(E^s_\xi(f)\right)^{-1/k}} = 0.$$

Further, if $\rho(p,q,f) > b$ and $f(z)$ is of minimal generalized (p,q)-type, then

$$\limsup_{k\to\infty} \frac{\phi\left(\log^{[p-2]} k\right)}{\log^{[q-1]}\left(E^s_\xi(f)\right)^{-1/k}} = 0.$$

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