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CONVERGENCE OF A GENERALIZED ITERATIONS FOR A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

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Abstract. In this paper we propose a new modified Mann iterations with certain control conditions for a countable family of nonexpansive mappings $\{T_n\}$ in Banach spaces. The sequence $\{x_n\}$ generated by the iteration converges strongly to a common fixed point in $\bigcap_{n=1}^{\infty} F(T_n)$ which is a solution of certain variational inequality. Moreover, we get a specific conclusion from main results.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E. Recall that a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{x \in C \mid x = Tx\}$ and we use Π_C to denote the collection of all contractions on C, that is $\Pi_C = \{f : C \to C \mid f \text{ is a contraction with a constant } \alpha\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C, and for any fixed element $x_0 \in C$, Picard's iteration $x_{n+1} = f^n(x_0)$ converges strongly to a unique fixed point of f. However, a simple example shows that Picard's iteration cannot be used in the case of nonexpansive mappings. One method in [2] used for nonexpansive mappings is to employ a Halpern-type iterative scheme which produces a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \beta_n u + (1 - \beta_n)Tx_n, \ n \geq 1, \end{cases} \tag{1}$$

where $u \in C$ is arbitrary and $\{\beta_n\} \subset [0, 1]$. In 2000, Moudafi [1] introduced a viscosity approximation method for a nonexpansive mapping as follows:

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)Tx_n, \ n \geq 1, \end{cases} \tag{2}$$

where $f \in \Pi_C$ and $\{\beta_n\} \subset [0, 1]$. In a real Hilbert space and under certain control conditions, he proved the sequence $\{x_n\}$ defined by (2) converges strongly to a fixed point of T which is the unique solution to the variational inequality $\langle (I - f)x^*, x - x^*\rangle \geq 0$ for all $x \in F(T)$. Kim and Xu [3] investigated the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \ n \geq 1, \\ x_{n+1} = \beta_n u + (1 - \beta_n)y_n, \ n \geq 1, \end{cases} \tag{3}$$

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where $u \in C$ is arbitrary and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. In 2008 Yao et al [4] introduced the iterative scheme defined by

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \ n \geq 1, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)y_n, \ n \geq 1, \end{cases} \tag{4}$$

where $f \in \Pi_C$ and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. They proved that the iterative scheme (4) strongly convergence to a fixed point of T. Recently, Plubtieng and Wangkeeree [5] studied the iterative scheme (5) for a countable family of nonexpansive mappings $\{T_n\}$ as follows:

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \ n \geq 1, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)y_n, \ n \geq 1, \end{cases} \tag{5}$$

where $f \in \Pi_C$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings. They proved that the iterative scheme (5) strongly convergence to a common fixed point of $\{T_n\}$.

On the other hand, Mainge and Moudafi [6] introduced an iterative scheme for approximating a specific solution of a fixed point problem as follows:

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n Sx_n + (1-\alpha_n)Tx_n, \ n \geq 1, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)y_n, \ n \geq 1, \end{cases} \tag{6}$$

where $f \in \Pi_C$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and S and T are nonexpansive mappings. They proved that if the sequence $\{x_n\}$ given by scheme (6) is bounded, then $\{x_n\}$ strongly convergence to the fixed point of a nonexpansive mapping T with respect to a nonexpansive mapping S under some control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$.

In this paper, inspired and motivated by the above iterative schemes, we introduced and studied a new composite iterative scheme for countable family of nonexpansive mappings as follows:

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n Sx_n + (1-\alpha_n)T_n x_n, \ n \geq 1, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)y_n, \ n \geq 1, \end{cases} \tag{7}$$

where $f \in \Pi_C$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and S and T_n $(n \in \mathbb{N})$ are nonexpansive mappings. The main results improve and complement the corresponding results of [2, 3, 7, 6, 1, 5, 4]. In particular, It should be noticed that the iterative scheme (6) has been investigated by kimura et.al. in [7] where $Fix(S) \cap Fix(T) \neq \emptyset$ with several control conditions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$. In this case the iteration converges strongly to an element in $Fix(S) \cap Fix(T)$. In Hilbert Space, Mainge and Moudafi in [6] proved that if the sequence $\{x_n\}$ given by scheme (6) is bounded, then $\{x_n\}$ strongly converges to the fixed point of a nonexpansive mapping T with respect to a nonexpansive mapping S under some control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$. Now, we get the similar result as a conclusion in uniformly smooth Banach space with different control conditions.

2. Preliminaries

Let E be a real Banach space, with norm $\| \ \|$ and E^* be its dual. The value of $x^* \in E^*$ will be denoted by $\langle x^*, x \rangle$. The duality mapping J from E into the family of nonempty w^*-compact subsets of its dual E^* is defined by

$$J(x) = \{x^* \in E^* \ : \ \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\} \tag{8}$$

for each $x \in E$ [8].

The norm of E is said to be Gateaux differentiable (and E is said to be smooth) if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{9}$$

exists for each $x,\ y\in U:=\{z\in E\ :\ \|z\|=1\}$. The norm is said to be uniformly Gateaux differentiable if for $y\in U$, the limit is attained uniformly for $x\in U$. The space E is said to have a uniformly Frechet differentiable norm (and E is said to be uniformly smooth) if the limit in (9) is attained uniformly for $(x,y)\in U\times U$. It is known that E is smooth if and only if each duality mapping J is single-valued. It is also well known that if E has a uniformly Gateaux differentiable norm, J is uniformly norm to weak continuous on each bounded subset of E.

Lemma 1. [9] *Let E be a real Banach space and J be the duality mapping. Then, for each $x,y\in E$, one has*

$$\|x+y\|^2\le\|x\|^2+2\langle y,J(x+y)\rangle. \tag{10}$$

Lemma 2. [10] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1}\le(1-\gamma_n)a_n+\delta_n,\ n\ge 1$, where $\{\gamma_n\}\subseteq(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(i) $\sum\limits_{n=1}^{\infty}\gamma_n=\infty$,

(ii) $\limsup\limits_{n\to\infty}\frac{\delta_n}{\gamma_n}\le 0$ *or* $\sum\limits_{n=1}^{\infty}|\delta_n|<\infty$.

Then $\lim\limits_{n\to\infty}a_n=0$.

Lemma 3. [11] *Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E, $T:C\to C$ be a nonexpansive mapping with $F(T)\neq\emptyset$, and $f\in\Pi_C$. Then the unique fixed point $x_t\in C$ of the contraction $t\to tf(x)+(1-t)Tx$ converges strongly to a fixed point of T as $t\to 0^+$. If we define $Q:\Pi_C\to F(T)$ by $Q(f)=\lim\limits_{t\to 0^+}x_t$, then $Q(f)$ solves the variational inequality*

$$\langle(I-f)Q(f),J(Q(f)-q)\rangle\le 0\ \forall q\in F(T).$$

Lemma 4. [12] *Let C be a nonempty closed convex subset of E. Suppose that*

$$\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z-T_nz\|\ ;\ z\in C\}<\infty$$

Then for each $y\in C$, $\{T_ny\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty=\lim\limits_{n\to\infty}T_ny$ for all $y\in C$. Then

$$\limsup_{n\to\infty}\{\|Tz-T_nz\|\ ;\ z\in C\}=0$$

3. Main results

In this section, we prove several strong convergence theorems of the iterative scheme (7). Throughout this section, C is a nonempty closed convex subset of E, T_n for each $n\in\mathbb{N}$ and S are nonexpansive mappings of C into itself such that $\bigcap\limits_{n=1}^{\infty}F(T_n)$ is nonempty and $\sum\limits_{n=1}^{\infty}\sup\{\|T_{n+1}z-T_nz\|\ ;\ z\in B\}<\infty$ for any bounded subset B of C.

Theorem 1. *Let Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm such that every weakly compact convex subset of E has the fixed point property for non expansive mapping. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ which satisfy in conditions*

$$(\mathrm{C1})\ \lim_{\mathrm{n}\to\infty}\frac{\alpha_{\mathrm{n}}}{\beta_{\mathrm{n}}}=0,$$

$$(\mathrm{C2})\ \lim_{\mathrm{n}\to\infty}\beta_{\mathrm{n}}=0,\ \sum_{\mathrm{n}=1}^{\infty}\beta_{\mathrm{n}}=\infty.$$

Let $f\in\Pi_C$, $x_1\in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (7). Consider T be a mapping of C into itself defined by $Tz=\lim_{n\to\infty}T_nz$ for all $z\in C$ and suppose that $F(T)=\bigcap_{n=1}^{\infty}F(T_n)$. If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to $p\in F(T)=\bigcap_{n=1}^{\infty}F(T_n)$, where p is the unique solution of the variational inequality

$$\langle (I-f)(p),J(p-q)\rangle\leq 0,\ \forall q\in F(T). \tag{11}$$

Proof. First, we claim that $\{x_n\}$ is bounded. Indeed, take an arbitrary fixed $p\in\bigcap_{n=1}^{\infty}F(T_n)$ so using the definition of $\{y_n\}$, we have

$$\begin{aligned}
\|y_n-p\| &= \|\alpha_nSx_n+(1-\alpha_n)T_nx_n-p\|\\
&\leq \alpha_n\|Sx_n-p\|+(1-\alpha_n)\|T_nx_n-p\|\\
&\leq \alpha_n\|Sx_n-Sp\|+\alpha_n\|Sp-p\|+(1-\alpha_n)\|T_nx_n-T_np\|\\
&\leq \alpha_n\|x_n-p\|+\alpha_n\|Sp-p\|+(1-\alpha_n)\|x_n-p\|\\
&= \|x_n-p\|+\alpha_n\|Sp-p\|,
\end{aligned}\tag{12}$$

and hence by the definition of $\{x_n\}$, we obtain

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\beta_nf(x_n)+(1-\beta_n)y_n-p\|\\
&\leq \beta_n\|f(x_n)-p\|+(1-\beta_n)\|y_n-p\|\\
&\leq \beta_n\|f(x_n)-f(p)\|+\beta_n\|f(p)-p\|+(1-\beta_n)\|y_n-p\|\\
&\leq \alpha\beta_n\|x_n-p\|+\beta_n\|f(p)-p\|+(1-\beta_n)\|y_n-p\|.
\end{aligned}\tag{13}$$

From (C1), we can assume, without loss of generality, that $\alpha_n\leq\beta_n$ for each $n\geq 1$. Substituting (12) into (13) to obtain

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \alpha\beta_n\|x_n-p\|+\beta_n\|f(p)-p\|+(1-\beta_n)\|x_n-p\|+(1-\beta_n)\alpha_n\|Sp-p\|\\
&\leq [1-(1-\alpha)\beta_n]\|x_n-p\|+\beta_n\|f(p)-p\|+\beta_n\|Sp-p\|\\
&= [1-(1-\alpha)\beta_n]\|x_n-p\|+(1-\alpha)\beta_n[\frac{\|f(p)-p\|+\|Sp-p\|}{1-\alpha}]\\
&\leq \max\{\|x_n-p\|,\frac{\|f(p)-p\|+\|Sp-p\|}{1-\alpha}\}.
\end{aligned}$$

By induction on n, we obtain that $\|x_n-p\|\leq\max\{\|x_1-p\|,\frac{\|f(p)-p\|+\|Sp-p\|}{1-\alpha}\}$ for all $n\in\mathbb{N}$ and all $p\in\bigcap_{n=1}^{\infty}F(T_n)$. Hence, the sequence $\{x_n\}$ is bounded and so $\{y_n\}$, $\{T_nx_n\}$,$\{Sx_n\}$

and $\{f(x_n)\}$ are bounded sequences. Moreover, it follows from conditions (C1) and (C2) that

$$\|y_n - T_n x_n\| = \alpha_n \|Sx_n - T_n x_n\| \to 0 \ as \ n \to \infty$$

and

$$\|x_{n+1} - y_n\| = \beta_n \|f(x_n) - y_n\| \to 0 \ as \ n \to \infty$$

and hence by asymptotic regularity of $\{x_n\}$ implies that

$$\|T_n x_n - x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - T_n x_n\| \to 0 \ as \ n \to \infty.$$

Therefore, by lemma 4 we obtain

$$\begin{aligned} \|Tx_n - x_n\| &\le \|Tx_n - T_n x_n\| + \|T_n x_n - x_n\| \\ &\le \sup\{\|Tz - T_n z\| \ : \ z \in \{x_n\}\} + \|T_n x_n - x_n\| \to 0 \ as \ n \to \infty. \end{aligned}$$

Let x_t be the unique fixed point of the contraction mapping $tf(x)+(1-t)Tx$. By Lemma 3 $x_t \to p = Q(f) \in Fix(T)$ as $t \to 0^+$ and p is the unique solution of the variational inequality (11). By the same argument as in the proof of Theorem 1 in [4], we obtain

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_n - p)\rangle \le 0.$$

Finally, we claim that $\{x_n\}$ strongly convergence to p. Indeed, using Lemma 1 we obtain that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-\beta_n)(y_n - p) + \beta_n(f(x_n) - p)\|^2 \\ &\le (1-\beta_n)^2\|(y_n - p)\|^2 + 2\beta_n\langle f(x_n) - p, J(x_{n+1} - p)\rangle \\ &\le (1-\beta_n)^2(\|x_n - p\| + \alpha_n\|Sp - p\|)^2 + 2\beta_n\langle f(x_n) - f(p), J(x_{n+1} - p)\rangle \\ &+ 2\beta_n\langle f(p) - p, J(x_{n+1} - p)\rangle \\ &\le (1-\beta_n)^2(\|x_n - p\|^2 + \alpha_n^2\|Sp - p\|^2 + 2\alpha_n\|x_n - p\|\|Sp - p\|) \\ &+ 2\alpha\beta_n\|x_n - p\|\|x_{n+1} - p\| + 2\beta_n\langle f(p) - p, J(x_{n+1} - p)\rangle \\ &\le (1-\beta_n)^2[\|x_n - p\|^2 + \alpha_n^2\|Sp - p\|^2 + \alpha_n(\|x_n - p\|^2 + \|Sp - p\|^2)] \\ &+ \alpha\beta_n\|x_n - p\|^2 + \alpha\beta_n\|x_{n+1} - p\|^2 + 2\beta_n\langle f(p) - p, J(x_{n+1} - p)\rangle \\ &\le [(1-\beta_n)^2(1+\alpha_n) + \alpha\beta_n]\|x_n - p\|^2 + (1-\beta_n)^2(\alpha_n + \alpha_n^2)\|Sp - p\|^2 \\ &+ \alpha\beta_n\|x_{n+1} - p\|^2 + 2\beta_n\langle f(p) - p, J(x_{n+1} - p)\rangle. \end{aligned}$$

Then we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \frac{1}{1-\alpha\beta_n}[(1-\beta_n)^2(1+\alpha_n) + \alpha\beta_n]\|x_n - p\|^2 + (1-\beta_n)^2(\alpha_n + \alpha_n^2)\|Sp - p\|^2 \\ &+ 2\beta_n\langle f(p) - p, J(x_{n+1} - p)\rangle \\ &= (1 - \frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n})\|x_n - p\|^2 + [\frac{\alpha_n(1-2\beta_n) + \beta_n^2(1+\alpha_n)}{1-\alpha\beta_n}\|x_n - p\|^2 \\ &+ \frac{2\beta_n}{1-\alpha\beta_n}\langle f(p) - p, J(x_{n+1} - p)\rangle + \frac{(1-\beta_n)^2(\alpha_n + \alpha_n^2)}{1-\alpha\beta_n}\|Sp - p\|^2] \\ &= (1-\gamma_n)\|x_n - p\|^2 + \delta_n, \end{aligned}$$

where $\gamma_n = \frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n}$ and $\delta_n = \frac{\alpha_n(1-2\beta_n)+\beta_n^2(1+\alpha_n)}{1-\alpha\beta_n}\|x_n - p\|^2 + \frac{2\beta_n}{1-\alpha\beta_n}\langle f(p) - p, J(x_{n+1} - p)\rangle + \frac{(1-\beta_n)^2(\alpha_n+\alpha_n^2)}{1-\alpha\beta_n}\|Sp - p\|^2$. It is easily seen that $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \le 0$. Then Lemma 2 implies that $\{x_n\}$ convergence strongly to $p \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. □

Theorem 2. *Let E be a uniformly smooth Banach space. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ which satisfy in conditions* (C1) and (C2). Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (7). Consider T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to $p \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, where p is the unique solution of the variational inequality (11).

Proof. Since E is a uniformly smooth Banach space, E is reflexive and the norm is uniformly Gateaux differentiable norm and every nonempty weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Thus, the conclusion of the theorem follows from Theorem 1 immediately. □

Theorem 3. *Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm Such that every weakly compact convex subset of E has the fixed point property for nonexpansive mapping. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ which satisfy in conditions* (C1), (C2) and

$$\text{(C3)} \sum_{\mathrm{n}=1}^{\infty} |\alpha_{\mathrm{n}+1} - \alpha_{\mathrm{n}}| < \infty, \sum_{\mathrm{n}=1}^{\infty} |\beta_{\mathrm{n}+1} - \beta_{\mathrm{n}}| < \infty.$$

Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (7). Consider T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $p \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, where p is the unique solution of the variational inequality (11).

Proof. From the definition of $\{y_n\}$ for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\alpha_{n+1} S x_{n+1} + (1-\alpha_{n+1}) T_{n+1} x_{n+1} - \alpha_n S x_n - (1-\alpha_n) T_n x_n\| \\
&\le \alpha_{n+1} \|S x_{n+1} - S x_n\| + (1-\alpha_{n+1}) \|T_{n+1} x_{n+1} - T_n x_n\| \\
&+ |\alpha_{n+1} - \alpha_n| \|S x_n - T_n x_n\| \\
&\le \alpha_{n+1} \|x_{n+1} - x_n\| + (1-\alpha_{n+1}) \|T_{n+1} x_{n+1} - T_{n+1} x_n\| \\
&+ (1-\alpha_{n+1}) \|T_{n+1} x_n - T_n x_n\| + |\alpha_{n+1} - \alpha_n| \|S x_n - T_n x_n\| \\
&\le \|x_{n+1} - x_n\| + (1-\alpha_{n+1}) \|T_{n+1} x_n - T_n x_n\| + |\alpha_n - \alpha_{n+1}| \|S x_n - T_n x_n\|
\end{aligned}$$

Since C is bounded then $\{S x_n - T_n x_n\}$ and $\{f(x_n) - y_n\}$ are bounded. Let $M = \sup\{\|S x_n - T_n x_n\|, \ \|f(x_n) - y_n\| \ : \ n \in \mathbb{N}\}$. By the same argument as in the proof of Lemma 3.2 in [5], we obtain $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Hence $\{x_n\}$ is asymptotic regular, then by Theorem 1 the proof is complete. □

Theorem 4. *Let E be a uniformly smooth Banach space. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ which satisfy in* (C1), (C2) and (C3). Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (7). Consider T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $p \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, where p is the unique solution of the variational inequality (11).

Proof. It follows from Theorem 2 and Theorem 3.

□

Example 1. [5] *Let E be a uniformly smooth Banach space. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $[0,1]$ *which satisfy in conditions* (C1), (C2) and (C3). Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (5). Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to a common fixed point in $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. It is sufficient that assume $Sx = x$ in the Theorems 4. □

Theorem 5. *Let E be a uniformly smooth Banach space. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $[0,1]$ *which satisfy in condition* (C2), (C3) and

$$\text{(C4)} \sum_{\mathrm{n=1}}^{\infty} \alpha_{\mathrm{n}} < \infty.$$

Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (7). Cosider T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $p \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, where p is the unique solution of the variational inequality 11

Proof. From conditions (C2) and (C4) one can see that (C1) is satisfied. Then the conclusion implies that from Theorem 4. □

The following corollary is the similar result of [7] and [6] as a conclusion from Theorem 5 in uniformly smooth Banach space with different control conditions.

Example 2. *Let E be a uniformly smooth Banach space. Consider T and S are nonexpansive mappings of C into itself such that $F(T)$ is nonempty. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $[0,1]$ *which satisfy in* (C2), (C3) and (C4). Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (6). Then $\{x_n\}$ converges strongly to a common fixed point in $F(T)$.

Proof. It is sufficient that assume $T_n = T$ for all $n \in \mathbb{N}$ and using Theorem 5. □

Kim and Xu [3] in uniformly smooth Banach space proved that the sequence $\{x_n\}$ generated by (3) converges to a fixed point of T under control conditions (C2), (C3), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. The following corollary show that the sequence $\{x_n\}$ generated by (3) converges to a fixed point of T while $\{\alpha_n\}, \{\beta_n\}$ satisfy in conditions (C2), (C3) and $\sum_{n=1}^{\infty} \alpha_n < \infty$.

Example 3. *Let E be a uniformly smooth Banach space. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $[0,1]$ *which satisfy in conditions* (C2), (C3) and (C4). Let $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (3). Then $\{x_n\}$ converges strongly to a fixed point in T.

Proof. Let $T_n = T$ for all $n \in \mathbb{N}$, $S = I$ and $f(x) = u$ for all $x \in C$. Then all conditions of Theorem 5 hold and hence we have the required result. □

4. Applications to zero of m-accretive operator

Let E be a real Banach space. An operator $A : E \to E$ is said to be *accretive* if for each $(x_1, y_1), (x_2, y_2) \in Gph(A)$ there exists a $j \in J(x_2 - x_1)$ such that $\langle y_2 - y_1, j\rangle \geq 0$. An accretive operator A is m-accretive if $R(I + rA) = E$ for each $r \geq 0$. Throughout this section we always assume that A is m-accretive and has a zero. The set of zeros of A is denoted by $A^{-1}(0)$, that is $A^{-1}(0) = \{z \in D(A) \ : \ 0 \in Az\}$. For each $r \geq 0$, we denote by J_r the resolvent of A, that is $J_r = (I + rA)^{-1}$. Note that, if A is m-accretive, then $J_r : E \to E$ is a nonexpansive mapping and $F(J_r) = A^{-1}(0)$ for all $r \geq 0$. We also know the following [13]: For each $r, s \geq 0$ and $x \in R(I + rA) \cap R(I + sA)$, it holds that

$$\|J_r x - J_s x\| \leq \frac{|r - s|}{r}\|x - J_r x\|.$$

We apply Theorem 4 and 1 for finding a zero of an m-accretive operator. By the proof of Theorem 4.3 in [12] we have the following lemma.

Lemma 5. *Let E be a Banach space and C be a nonempty closed convex subset of E. Let $A : E \to E$ be an accretive operator such that $A^{-1}(0) \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Suppose that $\{r_n\}$ is a sequence of $(0, +\infty)$ such that $\inf\{r_n \ : \ n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then*

(i) $\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}} z - J_{r_n} z\| \ ; \ z \in B\} < \infty$ *for any bounded subset B of C.*

(ii) $\lim J_{r_n} z = J_r z$ *for all $z \in C$ and $F(J_r) = \bigcap_{n=1}^{\infty} F(J_{r_n})$, where $r_n \to r$ as $n \to \infty$.*

By Lemma 5, Theorem 4 and 1, we obtain the following results.

Theorem 6. *Let E be a uniformly smooth Banach space and C be a nonempty closed convex subset of E. Given sequences $\{\alpha_n\}$ in $(0, 1)$, $\{\beta_n\}$ in $[0, 1]$ and $\{r_n\}$ in $(0, \infty)$ which satisfy conditions* (C1)-(C3) or (C2)-(C4), $r_n \geq \varepsilon$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n J_r x_n + (1 - \alpha_n) J_{r_n} x_n, \ n \geq 1, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, \ n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to $p \in A^{-1}(0)$, where p is the unique solution of the variational inequality

$$\langle (I - f)(p), J(p - q)\rangle \leq 0, \ \forall q \in A^{-1}(0). \tag{14}$$

Theorem 7. *Let E be a uniformly smooth Banach space and C be a nonempty closed convex subset of E. Given a point $u \in C$ and given sequences $\{\alpha_n\}$ in $(0, 1)$, $\{\beta_n\}$ in $[0, 1]$ and $\{r_n\}$ in $(0, \infty)$ which satisfy conditions* (C1)-(C3) or (C2)-(C4), $r_n \geq \varepsilon$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n J_r x_n + (1 - \alpha_n) J_{r_n} x_n, \ n \geq 1, \\ x_{n+1} = \beta_n u + (1 - \beta_n) y_n, \ n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to $p \in A^{-1}(0)$, where p is the unique solution of the variational inequality

$$\langle p-u, J(p-q)\rangle \leq 0, \ \forall q \in A^{-1}(0). \tag{15}$$

Proof. It is sufficient that assume $f(x) = u$ for all $x \in C$ in Theorem 6. □

Theorem 8. *Let E be a uniformly smooth Banach space and C be a nonempty closed convex subset of E. Given sequences $\{\beta_n\}$ in $[0,1]$ and $\{r_n\}$ in $(0,\infty)$ which satisfy conditions* (C2), (C3), $r_n \geq \varepsilon$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) J_{r_n} x_n, \ n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to $p \in A^{-1}(0)$, where p is the unique solution of the variational inequality (14)

Proof. It is sufficient that assume $\alpha_n = 0$ in Theorem 6. □

Theorem 9. *Let E be a uniformly smooth Banach space and C be a nonempty closed convex subset of E. Given a point $u \in C$ and given sequences $\{\beta_n\}$ in $[0,1]$ and $\{r_n\}$ in $(0,\infty)$ which satisfy conditions* (C2), (C3), $r_n \geq \varepsilon$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $f \in \Pi_C$, $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \beta_n u + (1-\beta_n) J_{r_n} x_n, \ n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to $p \in A^{-1}(0)$, where p is the unique solution of the variational inequality (15).

Proof. It is sufficient that assume $f(x) = u$ for all $x \in C$ and $\alpha_n = 0$ in Theorem 6. □

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