TJMM
4 (2012), No. 1, 09-14

AN APPLICATION OF UNIVALENT SOLUTIONS TO FRACTIONAL VOLTERRA EQUATION IN COMPLEX PLANE

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Abstract. In this article, we discuss the existence and uniqueness of solution to fractional Volterra equation in complex plane. We apply our results on the single species model of Volterra type. Fixed point theorems are the main tool used here to establish the existence and uniqueness results. First we use Banach contraction principle and then Krasnoselskii's fixed point theorem under certain conditions. Moreover, we prove that the solution can be extended to maximal interval of existence.

1. Introduction

In these coming years, problems concerning the physical process of fractional order becoming popular among the researchers. The fractional calculus has allowed the operations of integration and differentiation to be applied upon any fractional order. The order may take on any real or imaginary value. Recently theory of fractional differential equations attracted many scientists and mathematicians to work on [4, 16, 17, 18, 19]. For the existence of solutions for fractional differential equations, one can see [6, 5, 7, 8, 9, 10, 11, 12] and references therein. The results have been obtained by using fixed point theorems like Picard's, Schauder fixed-point theorem and Banach contraction mapping principle. About the development of existence theorems for fractional functional differential equations, many contributions exist and can be referred to [1, 7, 2, 4]. Many applications of fractional calculus amounting to replace the time derivative in a given evolution equation by a derivative of fractional order. Recently, interesting attempts have been made to give the physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives which were proposed in [17, 18].

Many equivalent definitions of fractional derivative and fractional integral are introduced and presented by many authors see for example [13] and [20]. However, we state the ones given by [14].

Definition 1. [14]. *The fractional integral of order α is defined* [14] *for a function f by*

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{1-\alpha}} dt, \qquad \alpha > 0, \tag{1}$$

where f is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

Definition 2. [14]. *The fractional derivative of order α is defined* [14] *for a function f by*

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt, \qquad 0 \le \alpha < 1, \tag{2}$$

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Fractional differential equation; univalent function; Srivastava-Owa operators; fixed point theorems; maximum interval of existence.

where f is an analytic function in a simply connected domain of the z-plane containing the origin and the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

In this article, our aim is to investigate the existence and uniqueness of univalent solution for fractional Volterra equation

$$u(z) = g(z) + \frac{\lambda}{\Gamma(\alpha)} \int_0^z f(t,u(t))(z-t)^{\alpha-1}dt, \qquad \alpha > 0, \tag{3}$$

where Γ is the gamma function and λ is an arbitrary parameter, $U := \{z : |z| < 1\}$, $f : U \times C \longrightarrow C$ is a continuous function and $g : U \longrightarrow C$ is a given increasing continuous function on U, by using fixed point theorem for generalized contractions due to Pathak and Shahzad [15].

Let $B = C[U, C]$ be the Banach space of continuous functions from $U \longrightarrow C$, endowed with max norm.

The study for fractional Volterra equation has been investigated by some other authors include [21] and [3].

Define the operator $T : B \longrightarrow B$ by

$$Tu(z) = g(z) + \frac{\lambda}{\Gamma(\alpha)} \int_0^z f(t,u(t))(z-t)^{\alpha-1}dt, \qquad \alpha > 0, z \in U. \tag{4}$$

Let the function f be bounded by M, we assume that our function f is Lipschitz continuous with respect to u with Lipschitz constant L_f.

Consider

$$\begin{aligned}
|Tu(t) - g(z)| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^z |f(t,u(t))|(z-t)^{\alpha-1}dt \\
&\leq \frac{M\lambda}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1}dt \\
&\leq \frac{M\lambda}{\Gamma(\alpha)} \int_0^z t^{\alpha-1}dt \\
&\leq \frac{M\lambda}{\Gamma(\alpha+1)} z^{\alpha} \\
&\leq \frac{M\lambda}{\Gamma(\alpha+1)} T^{\alpha}.
\end{aligned}$$

Thus for u bounded, continuous, Tu is also bounded, continuous. For $u, v \in B$, we have

$$\begin{aligned}
|Tu(t) - Tv(t)| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^z |f(t,u(t)) - f(t,v(t))|(z-t)^{\alpha-1}dt \\
&\leq \frac{\lambda L_f}{\Gamma(\alpha)} \int_0^z |u(t) - v(t)|(z-t)^{\alpha-1}dt \\
&\leq \frac{\lambda L_f}{\Gamma(\alpha)} \sup_{u \in [0,T]} |u(t) - v(t)| \left(\int_0^z (z-t)^{\alpha-1}dt \right) \\
&\leq \frac{\lambda L_f}{\Gamma(\alpha)} \|u - v\| \left(\int_0^z t^{\alpha-1}dt \right) \\
&\leq \frac{\lambda L_f}{\Gamma(\alpha+1)} \|u - v\| T^{\alpha}.
\end{aligned}$$

Thus for

$$\frac{\lambda L_f T^\alpha}{\Gamma(\alpha+1)} < 1,$$

we have $\|Tu - Tv\| < \|u - v\|$.

By the contraction mapping principle, we therefore know that T has a unique fixed point in B. This implies that our problem has a unique solution in B. Hence we summarize our result in the following theorem.

Theorem 1. *Problem* (3) *has a unique solution in B provided that*

$$\frac{\lambda L_f T^\alpha}{\Gamma(\alpha+1)} < 1.$$

Now our next target is to use Krasnoselskii's fixed point theorem to prove the existence of solution of the problem (3).

First, we mention statement of Krasnoselskii's fixed point theorem.

Theorem 2. *Let B be a nonempty closed convex subset of a Banach space. Suppose that Λ_1 and Λ_2 map B into X such that*

(i) *for any $u, v \in B$, $\Lambda_1 u + \Lambda_2 v \in B$,*
(ii) *Λ_1 is a contraction,*
(iii) *Λ_2 is continuous and $\Lambda_2(B)$ is contained in a compact set.*

Then there exists $z \in B$ such that $z = \Lambda_1 z + \Lambda_2 z$.

Now we prove existence of the solutions for the fractional Volterra equation (3) using Krasnoselskii's fixed point theorem. We begin with the assumption that our function f is Lipschitz continuous function with Lipschitz constant L_f.

We denote by U_r the disk $\{z : |z| < r\}$ where $0 < r \leq 1$, by $U = U_1$ the open unit disk of the complex plane where r satisfies

$$\frac{\lambda M r}{\Gamma(\alpha+1)} T^\alpha \leq r.$$

Moreover for $u \in U_r$, we obtain

$$\begin{aligned} |Tu(t)| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} |f(t, u(t))| dt \\ &\leq \frac{\lambda M \|u\|}{\Gamma \alpha} \int_0^z (z-t)^{\alpha-1} dt \\ &\leq \frac{\lambda M r}{\Gamma(\alpha+1)} T^\alpha \leq r. \end{aligned}$$

By using (4), it is easy to prove the continuity of Tu.

Let us consider a sequence u_n converging to u. Taking the norm of

$$Tu_n(t) - Tu(t),$$

we have

$$\begin{aligned}
&|Tu_n(t) - Tu(t)| \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^z |f(t, u_n(t)) - f(t, u(t))|(z-t)^{\alpha-1} dt \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^z |u_n(t) - u(t)|(z-t)^{\alpha-1} dt \\
&\leq \frac{\lambda L_f}{\Gamma(\alpha)} \Big(\int_0^z t^{\alpha-1} dt \Big) \|u_n - u\| \\
&\leq \frac{\lambda L_f}{\Gamma(\alpha+1)} T^\alpha \|u_n - u\|.
\end{aligned}$$

From the above analysis we obtain

$$\|Tu_n(t) - Tu(t)\| \leq \frac{\lambda L_f}{\Gamma(\alpha+1)} T^\alpha \|u_n - u\|,$$

and hence whenever $u_n \to u$, $Tu_n \to Tu$. This proves the continuity of Tu.

Now for $z_1 \leq z_2 \leq T$, (remember z is real here). we have

$$\begin{aligned}
&|Tu(z_2) - Tu(z_1)| \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \Big| \int_0^{z_2} (z_2 - t)^{\alpha-1} f(t, u(t) dt \\
&\quad - \int_0^{z_1} (z_1 - t)^{\alpha-1} f(t, u(t) dt \Big| \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \Big| \int_0^{z_1} (z_2 - t)^{\alpha-1} f(t, u(t) dt \\
&\quad + \int_{z_1}^{z_2} (z_2 - t)^{\alpha-1} f(t, u(t) dt \\
&\quad - \int_0^{z_1} (z_1 - t)^{\alpha-1} f(t, u(t) dt \Big| \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^{z_1} |((z_2 - t)^{\alpha-1} - (z_1 - t)^{\alpha-1})| \times |f(t, u(t)| dt \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_{z_1}^{z_2} |(z_2 - t)^{\alpha-1}| \times |f(t, u(t)| dt \\
&\leq \frac{\lambda M_1 r}{\Gamma(\alpha)} \int_0^{z_1} |(z_2 - t)^{\alpha-1} - (z_1 - t)^{\alpha-1}| dt + \frac{\lambda M_2 r}{\Gamma(\alpha)} \int_{z_1}^{z_2} |(z_2 - t)^{\alpha-1}| dt \\
&\leq \frac{r\lambda}{\Gamma(\alpha+1)} \max\{M_1, M_2\} \Big| -2(z_2 - z_1)^\alpha + z_2^\alpha - z_1^\alpha \Big| \\
&\leq \frac{r\lambda}{\Gamma(\alpha+1)} \max\{M_1, M_2\} (z_2 - z_1)^\alpha.
\end{aligned}$$

The right-hand side of above expression does not depend on u. Thus we conclude that $Tu(U_r)$ is relatively compact and hence Tu is compact by Arzela-Ascoli theorem. Using Krasnoselskiis fixed point theorem, we obtain that there exists $z \in U_r$ such that $Tu = Tu_1(z) + Tu_2(z) = z$, which is a fixed point of Tu. Hence the problem (3) has at least one solution in U_r. We summarize the above results in the form of the following theorem.

Theorem 3. *Model* (3) *has a solution in the set* U_r *provided* $\frac{\lambda L_f T^\alpha}{\Gamma(\alpha+1)} < 1$, *and*

$$\frac{Mr\lambda}{\Gamma(\alpha+1)}T^\alpha \leq r\,.$$

Consider the function $f(t,u(t)) = u(t)(r(t) - a(t))$ and let us denote

$$f_1(t,x(t)) = x(t)r(t), \quad f_2(t,x(t)) = -a(t)u(t).$$

It is easy to see that

$$|f_1(t,x(t))| \leq r^*|x(t)|,$$
$$|f_2(t,x(t))| \leq a^*|x(t)|.$$

Using fractional calculus, (3) can be representable as an integral form of the type

$$x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}u(t)\Big(r(s) - a(s)\Big)dt$$
$$g(z) = \phi(z).$$

Define a mapping Λ by

$$\Lambda u(t) = \Lambda_1 u(t) + \Lambda_2 u(t),$$

where

$$\Lambda_1 u(t) = \frac{\lambda}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}u(t)r(t)dt,$$
$$\Lambda_2 u(t) = -\frac{\lambda}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}u(t)a(t)dt.$$

One can easily see that in this case our operator Tu_1 coincide with Λ_1 and Tu_2 coincides with Λ_2. Thus our model systems (3) have at least one solution.

Acknowledgement. The work presented here was partially supported by UKM-ST-06-FRGS0244-2010.

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