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ON q-HAHN'S THEOREM: A q-EXTENSION

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Abstract. We give a q-analogue of Hahn's theorem and a q-extension for it; we show that an orthogonal polynomial sequence, whose $m-th$ associated sequence of $k-th$ q-derivative sequence is orthogonal, is necessarily a H_q-classical one, where H_q is the q-difference operator.

1. Introduction and preliminary results

In [8] the authors give a new proof of Hahn's theorem [1, 2, 5, 10], which states that an orthogonal sequence, whose derivative sequence of any finite order is orthogonal, is necessarily a D-classical sequence where D is the derivative operator. They also give an extension to Hahn's theorem making orthogonality assumption on $m-th$ associated sequence of $k-th$ derivative sequence and this still leads to D-classical forms.

In this paper, first we give a q-analog theorem to Hahn's one with replacing the derivative operator D by the q-difference operator H_q (see [3]). This theorem (see Theorem 1 below) characterizes the H_q-classical orthogonal polynomials: Al Salam-Carlitz, q-Laguerre, q-Meixner, ... [3]. Second, we establish an extension to that theorem making orthogonality assumption on m-th associated sequence of k-th q-derivative sequence and this still leads to H_q-classical forms (see Theorem 2 below).

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. Let us introduce some useful operations in $\mathcal{P}'$. For any form u, any polynomial g and any $(a, b, c) \in \mathbb{C} - \{0\} \times \mathbb{C}^2$, we let gu, $h_a u$,$\tau_b u$ and δ_c, be the forms defined by duality

$$\langle gu, f\rangle := \langle u, gf\rangle, \langle h_a u, f\rangle := \langle u, h_a f\rangle, f \in \mathcal{P},$$

$$\langle \tau_b u, f\rangle := \langle u, \tau_{-b} f\rangle, \langle \delta_c, f\rangle := f(c), f \in \mathcal{P},$$

where $(h_a f)(x) = f(ax)$ and $(\tau_{-b} f)(x) = f(x + b)$ [7].

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n$, $n \geq 0$ (polynomial sequence : PS) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by $\langle u_n, P_m\rangle := \delta_{n,m}$, $n, m \geq 0$.

We call associated sequence of $\{P_n\}_{n\geq 0}$ (with respect to u_0), the sequence $\{P_n^{(1)}\}_{n\geq 0}$ defined by

$$P_n^{(1)}(x) := \left\langle u_0, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x - \xi} \right\rangle, n \geq 0.$$

Any polynomial $P_n^{(1)}$ is monic and $\deg P_n^{(1)} = n$.

The Hahn's operator H_q is defined in the linear space $\mathcal{P}$ in the following way [3]

$$(H_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, f \in \mathcal{P}, q \in \widetilde{\mathbb{C}}, \tag{1}$$

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where $\widetilde{\mathbb{C}} := \mathbb{C} - \left(\{0\} \bigcup \left(\bigcup_{n\geq 0} \{z \in \mathbb{C}, z^n = 1\} \right) \right)$. By duality, the image of a form using this operator H_q is a form such that [3]

$$\langle H_q u, f \rangle = - \langle u, H_q f \rangle , \forall f \in \mathcal{P}. \tag{2}$$

In particular, this yields

$$(H_q u)_n = - [n]_q \, (u)_{n-1} \, , n \geq 0, \tag{3}$$

where $(u)_{-1} = 0$ and $[n]_q := \dfrac{q^n - 1}{q - 1}, n \geq 0$. The dual sequence $\left\{ u_n^{[1]} (q) \right\}_{n\geq 0}$ of $\left\{ P_n^{[1]} (.; q) \right\}_{n\geq 0}$ where $P_n^{[1]} (.; q) := \dfrac{H_q P_{n+1}}{[n+1]_q}$, $n \geq 0$, is given by [3]

$$H_q \left(u_n^{[1]} (q) \right) = - [n+1]_q \, u_{n+1}, n \geq 0. \tag{4}$$

More generally, we can define $\left\{ P_n^{[k]} \right\}_{n\geq 0}$ where $P_n^{[k]} (x) := \dfrac{\left(H_q P_{n+1}^{[k-1]} \right)(x)}{[n+1]_q}$, $n \geq 0$ for $k \geq 1$ and we have

$$H_q \left(u_n^{[k]} \right) = - [n+1]_q \, u_{n+1}^{[k-1]}, n \geq 0, k \geq 1.$$

Likewise, the dual sequence $\{\widetilde{u}_n\}_{n\geq 0}$ of $\left\{ \widetilde{P}_n \right\}_{n\geq 0}$ with $\widetilde{P}_n (x) := a^{-n} P_n (ax)$, $n \geq 0$, $a \neq 0$ is given by

$$\widetilde{u}_n = a^n h_{a^{-1}} u_n, n \geq 0. \tag{5}$$

In the sequel, we shall need the following formulas [3]:

Lemma 1. *We have*

$$H_q (gu) = \left(h_{q^{-1}} g \right) H_q u + q^{-1} \left(H_{q^{-1}} g \right) u, u \in \mathcal{P}', g \in \mathcal{P}, \tag{6}$$

$$H_q \circ h_{q^{-1}} = q^{-1} H_{q^{-1}}, h_{q^{-1}} \circ H_q = H_{q^{-1}} in \mathcal{P}. \tag{7}$$

$$H_q \circ h_a = a h_a \circ H_q \ in \ \mathcal{P}, \in \mathbb{C} - \{0\} \, . \tag{8}$$

The form u is called *regular* if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ such that

$$\langle u, P_m P_n \rangle = r_n \delta_{n,m}, n, m \geq 0; r_n \neq 0, n \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. Necessarily, $u = \lambda u_0$, $\lambda \neq 0$ and $\{P_n\}_{n\geq 0}$ is an (OPS) whose any polynomial can be supposed monic (MOPS). In this case, we have $u_n = r_n^{-1} P_n u_0, n \geq 0$ and conversely. Also, the (MOPS) $\{P_n\}_{n\geq 0}$ fulfils a second-order recurrence relation (see (15) below) [6].

Let Φ monic and Ψ be two polynomials , $\deg \Phi = t, \deg \Psi = p \geq 1$ and let $s = \max (p-1, t-2)$. We suppose that the pair (Φ, Ψ) is *admissible*, i.e. when $p = t - 1$, writing $\Psi (x) = a_p x^p + \ldots$, then $a_p \neq [n+1]_q$, $n \in \mathbb{N}$.

A form u is called H_q-semiclassical when it is regular and satisfies the equation

$$H_q (\Phi u) + \Psi u = 0. \tag{9}$$

Also, the corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called H_q-semiclassical. When $s = 0$, we have H_q-classical orthogonal polynomials (see [3, 4]).

2. $q-$Hahn's theorem

First two lemmas.

Lemma 2. *For any $u \in \mathcal{P}'$, $g \in \mathcal{P}$, we have*

$$H_q^n\left(\left(h_{q^n} g\right) u\right)=\sum_{\nu=0}^n \begin{bmatrix} n \\ \nu \end{bmatrix}_q q^{\frac{1}{2}(\nu-1) \nu}\left(H_q^\nu g\right)\left(H_q^{n-\nu} u\right), n \geq 0, \tag{10}$$

where

$$\begin{bmatrix} n \\ \nu \end{bmatrix}_q:=\frac{[n]_q!}{[\nu]_q![n-\nu]_q!}, 0 \leq \nu \leq n. \tag{11}$$

It is easy to prove this lemma by induction on account of lemma 1.

Lemma 3. [7, 9] *Let $\{P_n\}_{n\geq 0}$ be a H_q-semiclassical sequence, orthogonal with respect to u_0. Suppose that u_0 fulfils the two equations*

$$\left\{\begin{array}{l} H_q\left(\Phi_1 u_0\right)+\Psi_1 u_0=0 \\ H_q\left(\Phi_2 u_0\right)+\Psi_2 u_0=0 \end{array}\right. \tag{12}$$

and there exist an integer $m \geq 0$ and four polynomials E, F, G, H such that

$$\left\{\begin{array}{l} \Phi_1(x)=E(x) P_{m+1}(x)+F(x) P_m(x) \\ \Phi_2(x)=G(x) P_{m+1}(x)+H(x) P_m(x) \end{array}\right. . \tag{13}$$

Let Δ be the determinant of the system (13)

$$\Delta(x)=\left|\begin{array}{cc} E(x) & F(x) \\ G(x) & H(x) \end{array}\right| . \tag{14}$$

Then if one of the following conditions is fulfilled, the form u_0 is H_q-classical:
a) $\exists i=1,2$ *such that* $\deg \Psi_i \leq \deg \Phi_i-1$ *and* $\deg \Delta=2$.
b) $\exists i=1,2$ *such that* $\deg \Psi_i=\deg \Phi_i$ *and* $\deg \Delta=1$.
c) $\exists i=1,2$ *such that* $\deg \Psi_i=\deg \Phi_i+1$ *and* $\deg \Delta=0$.

Proof. From (13), we have

$$\Delta(x) P_{m+1}(x)=\left|\begin{array}{cc} \Phi_1(x) & F(x) \\ \Psi_1(x) & H(x) \end{array}\right|, \ \Delta(x) P_m(x)=\left|\begin{array}{cc} E(x) & \Phi_2(x) \\ G(x) & \Psi_2(x) \end{array}\right| .$$

This implies that any common factor of Φ_1 and Φ_2 is a factor of Δ; in particular, the highest common factor of Φ_1, Φ_2, say Φ, is a factor of Δ. But, we know [4] that there exists a polynomial Ψ such that $H_q(\Phi u_0)+\Psi u_0=0$, where Ψ is given by $\Psi_i+q^{-1}\Phi\left(H_{q^{-1}}\overset{\vee}{\Phi}_i\right)=\Psi\left(h_{q^{-1}}\overset{\vee}{\Phi}_i\right)$, $\Phi_i=\Phi\overset{\vee}{\Phi}_i$.

We have $\max\left(\deg\Phi+\deg\overset{\vee}{\Phi}_i-1, \deg\Psi_i\right)=\deg\Psi+\deg\overset{\vee}{\Phi}_i$, with $\deg\Phi_i=\deg\Phi+\deg\overset{\vee}{\Phi}_i$, $\max\left(\deg\Phi_i-1,\deg\Psi_i\right)=\deg\Psi+\deg\Phi_i-\deg\Phi$.
In the case *a)*, we have $\deg\Phi_i-1=\deg\Psi+\deg\Phi_i-\deg\Phi$. It follows $\deg\Psi=\deg\Phi-1\geq 1$, since u_0 is regular, therefore $\deg\Phi\geq 2$. But $\deg\Phi\leq 2$ by virtue of the assumption. Consequently $\deg\Phi=2$ and $\deg\Psi=1$. The form u_0 is either *Alternative q-Charlier, Stieltjes-Wigert, Little q-Jacobi, q-Charlier-II-, Generalized Stieltjes-Wigert or Big q-Jacobi, Bi-generalized Stieltjes-Wigert* [3].
In the case *b)*, we have $\deg\Phi_i=\deg\Psi+\deg\Phi_i-\deg\Phi$, hence $\deg\Psi=\deg\Phi\geq 1$. But $\deg\Phi\leq 1$, therefore $\deg\Phi=1$ and $\deg\Psi=1$. The form u_0 is either *Big q-Laguerre, q-Meixner or Wall, q-Laguerre, Little q-Laguerre* [3].

Finally, in the third case, we get $\deg\Phi_i + 1 = \deg\Psi + \deg\Phi_i - \deg\Phi$, this implies $\deg\Psi = \deg\Phi + 1$ with $\deg\Phi = 0$. It is the Al-Salam Carlitz case [3]. □

Then the claim is the following [2, 8]

Theorem 1. *Let* $\{P_n\}_{n\geq 0}$ *be an orthogonal sequence; there exists an integer* $k \geq 1$ *such that* $\left\{P_n^{[k]}\right\}_{n\geq 0}$ *is also orthogonal. Then* $\{P_n\}_{n\geq 0}$ *is a* H_q*-classical sequence.*

Proof. For the sake of simplicity, let us denote $Q_n(x) := P_n^{[k]}(x)$ and $\{v_n\}_{n\geq 0}$ the dual sequence of $\{Q_n\}_{n\geq 0}$ ($v_n = u_n^{[k]}$).
On account of assumptions, we can write the following recurrence relations

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x),\ n \geq 0. \end{cases} \tag{15}$$

$$\begin{cases} Q_0(x) = 1,\ Q_1(x) = x - \zeta_0, \\ Q_{n+2}(x) = (x - \zeta_{n+1}) Q_{n+1}(x) - \rho_{n+1} Q_n(x),\ n \geq 0. \end{cases} \tag{16}$$

Equivalently, we also have [7]

$$u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n u_0, n \geq 0, \tag{17}$$

$$(x - \zeta_n) v_n = v_{n-1} + \rho_{n+1} v_{n+1},\ n \geq 0,\ v_{-1} = 0. \tag{18}$$

By applying H_q k times the two sides of (18) and with (10), we get

$$q^{-k}[k]_q H_q^{k-1} v_n = H_q^k v_{n-1} + \rho_{n+1} H_q^k v_{n+1} - \left(q^{-k}x - \zeta_n\right) H_q^k v_n,\ n \geq 0.$$

But, since $v_n = u_n^{[k]}$, we easily see that

$$H_q^k v_n = (-1)^k \prod_{\mu=1}^{k} [n+\mu]_q\, u_{n+k},\ n \geq 0. \tag{19}$$

Therefore

$$(-1)^k q^{-k}[k]_q H_q^{k-1} v_n = \prod_{\mu=1}^{k} [n+\mu]_q \left\{ \frac{[n]_q}{[n+k]_q} u_{n-1+k} + \right.$$
$$\left. + \frac{[n+k+1]_q}{[n+1]_q} \rho_{n+1} u_{n+1+k} - \left(q^{-k}x - \zeta_n\right) u_{n+k} \right\}.$$

Taking account of (17) and (15), we obtain

$$H_q^{k-1} v_n = N_n^k \Phi_{n+k+1} u_0,\ n \geq 0, \tag{20}$$

where Φ_{n+k+1} is monic and

$$N_n^k \Phi_{n+k+1}(x) = L_n^k \left\{ \left(\frac{[n+k+1]_q}{[n+1]_q} \rho_{n+1} \gamma_{n+1+k}^{-1} - \frac{[n]_q}{[n+k]_q} \right) P_{n+k+1}(x) - \right.$$
$$\left. - \left(q^{-k} \frac{[k]_q}{[n+k]_q} x + \frac{[n]_q}{[n+k]_q} \beta_{n+k} - \zeta_n \right) P_{n+k}(x) \right\},$$

$$L_n^k = (-1)^k q^k \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\langle u_0, P_{n+k}^2\rangle\right)^{-1}.$$

From (20) and (19), we get

$$N_n^k H_q\left(\Phi_{n+k+1} u_0\right) = (-1)^k \prod_{\mu=1}^{k} [n+\mu]_q \left(\langle u_0, P_{n+k}^2\rangle\right)^{-1} P_{n+k} u_0.$$

Hence

$$H_q\left(\Phi_{n+k+1}u_0\right)+\lambda_n^k P_{n+k}\,u_0=0\ ,\ n\geq 0, \tag{21}$$

with

$$\lambda_n^k=(-1)^{k-1}\prod_{\mu=1}^{k}[n+\mu]_q\left(\left\langle u_0,P_{n+k}^2\right\rangle\right)^{-1}\left(N_n^k\right)^{-1}.$$

Without going into details, we can read

$$\Phi_{n+k+1}(x)=A_n^k P_{n+k+1}(x)-\left(B_n^k x+C_n^k\right)P_{n+k}(x)\,,\ n\geq 0. \tag{22}$$

In particular, for $n=0$ and $n=1$

$$\Phi_{k+1}(x)=A_0^k P_{k+1}(x)-\left(B_0^k x+C_0^k\right)P_k(x)\,, \tag{23}$$

$$\Phi_{k+2}(x)=A_1^k P_{k+2}(x)-\left(B_1^k x+C_1^k\right)P_{k+1}(x)\,. \tag{24}$$

Taking into account (15), (24) becomes

$$\Phi_{k+2}(x)=\left\{\left(A_1^k-B_1^k\right)x-\left(A_1^k\beta_{k+1}+C_1^k\right)\right\}P_{k+1}(x)-A_1^k\gamma_{k+1}P_k(x)\,. \tag{25}$$

Let us introduce the determinant Δ of (23), (25) (see (14)). Since $\deg\Delta\leq 2$, the form u_0 is H_q-classical by virtue of lemma 3. □

3. An extension of $q-$ Hahn's theorem

First a lemma.

Lemma 4. [8] *Let $\{Q_n\}_{n\geq 0}$ be any sequence with its dual sequence $\{\,v_n\}_{n\geq 0}$. Then, for any integer $m\geq 1$, the dual sequence $\left\{\,v_n^{(m)}\right\}_{n\geq 0}$ of the associated sequence $\left\{Q_n^{(m)}\right\}_{n\geq 0}$ fulfils*

$$v_n^{(m)}v_{m-1}=xv_{n+m},\ n\geq 0. \tag{26}$$

When $\left\{Q_n^{(m)}\right\}_{n\geq 0}$ is orthogonal, the sequence $\{\,v_n\}_{n\geq 0}$ fulfils

$$s_n^{(m)}v_{n+m}=Q_n^{(m)}v_m-Q_{n-1}^{(m+1)}v_{m-1},\ n\geq 0, \tag{27}$$

where

$$s_n^{(m)}=\left\langle v_0^{(m)},\left(Q_n^{(m)}\right)^2\right\rangle,\ n\geq 0,\ m\geq 1. \tag{28}$$

Now, we are going to characterize all orthogonal sequences $\{P_n\}_{n\geq 0}$ for which there exist two integer k, $m\geq 1$ such that, putting $P_n^{[k]}=Q_n$, $n\geq 0$, the associated sequence $\left\{Q_n^{(m)}\right\}_{n\geq 0}$ is also orthogonal. When $m=0$, it is $q-$ Hahn's problem. When $m\geq 1$, the answer is giving by the following theorem.

Theorem 2. [8] *Let $\{P_n\}_{n\geq 0}$ be an orthogonal sequence; for any integer $k\geq 1$ fixed, let us put $P_n^{[k]}:=Q_n$. Suppose that there exists an integer $m\geq 1$ such that the associated sequence $\left\{Q_n^{(m)}\right\}_{n\geq 0}$ is orthogonal. Then $\{P_n\}_{n\geq 0}$ is a H_q-classical sequence.*

Proof. For simplifying, we put $Q_n^{(m)} = R_n$ and $Q_n^{(m+1)} = S_n$. By applying H_q k times both sides of (27) where $n \to n+1$, and taking into account (10) we have

$$\sum_{\nu=1}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_q q^{\frac{1}{2}(\nu-1)\nu} \left(H_q^{\nu} \circ h_{q^{-k}} R_{n+1}\right) \left(H_q^{k-\nu} v_m\right) - \sum_{\nu=1}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_q q^{\frac{1}{2}(\nu-1)\nu} \left(H_q^{\nu} \circ h_{q^{-k}} S_n\right) \left(H_q^{k-\nu} v_{m-1}\right)$$
$$= s_{n+1}^{(m)} H_q^k v_{n+1+m} - \left(h_{q^{-k}} R_{n+1}\right)\left(H_q^k v_m\right) + \left(h_{q^{-k}} S_n\right)\left(H_q^k v_{m-1}\right).$$

With (19), we obtain

$$\sum_{\nu=1}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_q q^{\frac{1}{2}(\nu-1)\nu} \left(H_q^{\nu} \circ h_{q^{-k}} R_{n+1}\right) \left(H_q^{k-\nu} v_m\right) - \sum_{\nu=1}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_q q^{\frac{1}{2}(\nu-1)\nu} \left(H_q^{\nu} \circ h_{q^{-k}} S_n\right) \left(H_q^{k-\nu} v_{m-1}\right) = A_{n+1+m+k} u_0,\ n \geq 0, \tag{29}$$

where

$$A_{n+1+m+k} = (-1)^k \left(\left\langle u_0, P_{m-1+k}^2 \right\rangle\right)^{-1} \Big\{ L_n^{(m)}(k)\, P_{n+1+m+k} - \frac{[m+k]_q!}{[m]_q!} \gamma_{m+k}^{-1} \left(h_{q^{-k}} R_{n+1}\right) P_{m+k} + \frac{[m+k-1]_q!}{[m-1]_q!} \left(h_{q^{-k}} S_n\right) P_{m-1+k} \Big\},\ n \geq 0, \tag{30}$$

$$L_n^{(m)}(k) = \prod_{\mu=1}^{k} [n+1+m+\mu]_q \frac{\left\langle u_0, P_{m-1+k}^2 \right\rangle}{\left\langle u_0, P_{n+1+m+k}^2 \right\rangle} \left\langle v_0^{(m)}, R_{n+1}^2 \right\rangle,\ n \geq 0. \tag{31}$$

Taking $n = 0$ in (29) we get

$$q^{-k} [k]_q H_q^{k-1} v_m = A_{m+1+k} u_0. \tag{32}$$

By virtue of (32) the equality in (29) becomes

$$\sum_{\nu=2}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_q q^{\frac{1}{2}(\nu-1)\nu} \left(H_q^{\nu} \circ h_{q^{-k}} R_{n+1}\right) \left(H_q^{k-\nu} v_m\right) - \sum_{\nu=2}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_q q^{\frac{1}{2}(\nu-1)\nu} \left(H_q^{\nu} \circ h_{q^{-k}} S_n\right) \left(H_q^{k-\nu} v_{m-1}\right) = \left\{ A_{n+1+m+k} - q^k \left(H_q \circ h_{q^{-k}} R_{n+1}\right) A_{m+1+k} \right\} u_0. \tag{33}$$

The choice $n = 1$ in (33) leads to

$$[k]_q [k-1]_q q^{1-2k} H_q^{k-2} v_m - [k]_q q^{-k} H_q^{k-1} v_{m-1} = \left\{ A_{m+2+k} - q^k \left(H_q \circ h_{q^{-k}} R_2\right) A_{m+1+k} \right\} u_0. \tag{34}$$

Applying the operator H_q to (32) and taking into account (17) and (19) we get

$$H_q\left(\Phi_1 u_0\right) + \lambda_1 P_{m+k} u_0 = 0, \tag{35}$$

where

$$N_1 \Phi_1 = A_{m+1+k},$$

$$\lambda_1 = (-1)^{k+1} q^{-k} [k]_q \frac{[m+k]_q!}{[m]_q!} \left(\langle u_0, P_{m+k}^2 \rangle\right)^{-1} N_1^{-1}.$$

Now, after applying H_q both sides of (34), we have

$$[k]_q [k-1]_q q^{1-2k} H_q^{k-1} v_m - [k]_q q^{-k} H_q^k v_{m-1} = H_q \left(\left\{A_{m+2+k} - q^k \left(H_q \circ h_{q^{-k}} R_2\right) A_{m+1+k}\right\} u_0\right).$$

Putting $N_2 \Phi_2 = A_{m+2+k} - q^k \left(H_q \circ h_{q^{-k}} R_2\right) A_{m+1+k}$ and on account of (32) and (19), we get

$$H_q (\Phi_2 u_0) + \left\{\lambda_2 P_{m-1+k} - [k-1]_q q^{1-k} N_2^{-1} A_{m+1+k}\right\} u_0 = 0 \tag{36}$$

where

$$\lambda_2 = (-1)^k [k]_q q^{-k} \frac{[m-1+k]_q!}{[m-1]_q!} \left(\langle u_0, P_{m-1+k}^2 \rangle\right)^{-1} N_2^{-1}.$$

Finally, with (15) we can express Φ_1, Φ_2 as

$$\Phi_1(x) = E(x) P_{m+k}(x) + F(x) P_{m-1+k}(x) \ ; \ \Phi_2(x) = G(x) P_{m+k}(x) + H(x) P_{m-1+k}(x)$$

where

$$E(x) = (-1)^k \left(\langle u_0, P_{m-1+k}^2 \rangle\right)^{-1} N_1^{-1} \times \left\{(x - \beta_{m+k}) L_0^{(m)}(k) - \frac{[m+k]_q!}{[m]_q!} \gamma_{m+k}^{-1} \left(h_{q^{-k}} R_1\right)(x)\right\},$$

$$F(x) = (-1)^k \left(\langle u_0, P_{m-1+k}^2 \rangle\right)^{-1} N_1^{-1} \left\{\frac{[m-1+k]_q!}{[m-1]_q!} - \gamma_{m+k} L_0^{(m)}(k)\right\},$$

$$G(x) = (-1)^k \left(\langle u_0, P_{m-1+k}^2 \rangle\right)^{-1} N_2^{-1} \times$$

$$\left\{(x - \beta_{m+k}) \left((x - \beta_{m+k+1}) L_1^{(m)}(k) - q^k \left(H_q \circ h_{q^{-k}} R_2\right)(x) L_0^{(m)}(k)\right) - \gamma_{m+k+1} L_1^{(m)}(k)\right.$$
$$\left. + \frac{[m+k]_q!}{[m]_q!} \gamma_{m+k}^{-1} \left(q^k \left(H_q \circ h_{q^{-k}} R_2\right)(x) \left(h_{q^{-k}} R_1\right)(x) - \left(h_{q^{-k}} R_2\right)(x)\right)\right\},$$

$$H(x) = (-1)^k \left(\langle u_0, P_{m-1+k}^2 \rangle\right)^{-1} N_2^{-1} \times$$
$$\left\{\frac{[m-1+k]_q!}{[m-1]_q!} \left(\left(h_{q^{-k}} S_1\right)(x) - q^k \left(H_q \circ h_{q^{-k}} R_2\right)(x)\right)\right.$$
$$\left. - \gamma_{m+k} \left((x - \beta_{m+k+1}) L_1^{(m)}(k) - q^k \left(H_q \circ h_{q^{-k}} R_2\right)(x) L_0^{(m)}(k)\right)\right\}.$$

Since $\deg \Delta \leq 2$ with Δ given by (14), the linear form u is H_q-classical. □

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