TJMM
3 (2011), No. 2, 103-110

ON THE BEST POLYNOMIAL APPROXIMATION OF GENERALIZED BIAXISYMMETRIC POTENTIALS IN L^p-NORM, $p \geq 1$

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Abstract. The real valued regular solution of generalized biaxially symmetric potential equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{(2\alpha+1)}{x}\frac{\partial F}{\partial x} + \frac{(2\beta+1)}{y}\frac{\partial F}{\partial y} = 0, \alpha > \beta > -\frac{1}{2}$$

are called generalized biaxisymmetric potentials. In this paper, the characterization of lower order and lower type of entire GBASP F in term of their approximation error $E_n^p(F_\sigma)$ in L^p-norm, $p \geq 1$ have been obtained. The analysis utilizes the Bergman and Gilbert Integral Operator Method to extend results from classical function theory on the best polynomial approximation of analytic functions of one complex variable.

1. Introduction

Let $F = F(x, y)$ be a real valued regular solution to the generalized biaxially symmetric potential equation

$$\Big[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{(2\alpha+1)}{x}\frac{\partial}{\partial x} + \frac{(2\beta+1)}{y}\frac{\partial}{\partial y}\Big]F = 0, \alpha > \beta > -\frac{1}{2}, \tag{1}$$

(α, β) fixed in a neighbourhood of the origin where the analytic Cauchy data

$$F_x(0, y) = F_y(x, 0) = 0$$

is satisfied along the singular lines in $\Sigma^{(\alpha,\beta)}$, the open unit hypersphere. These solutions called the generalized biaxially symmetric potentials (GBASP) can be expanded in $\Sigma^{(\alpha,\beta)}$ uniquely as

$$F(x, y) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x, y) \tag{2}$$

in terms of the complete set

$$R_n^{(\alpha,\beta)}(x, y) = (x^2 + y^2)^n \; P_n^{(\alpha,\beta)}\Big[(x^2 - y^2)r^{-2}\Big]/P_n^{(\alpha,\beta)}(1), n = 0, 1, 2, \ldots \tag{3}$$

of biaxisymmetric harmonic potentials, where $P_n^{(\alpha,\beta)}$ are Jacobi polynomials ([1],[14]), $r^2 = x^2 + y^2$.

After quadratic transformation [1] $R_n^{(\alpha,\beta)}$ gives various special functions for suitable limits of α and β. For example $\alpha = \beta = 0$ gives the zonal harmonics, so that F interprets as an axisymmetric potential on E^3 and $\alpha = \beta = -\frac{1}{2}$ gives the even circular harmonics on E^2 where the interpretation in $F = R_e f$, f is real analytic.

The Euler-Poisson-Darboux equation, arising in gas dynamics, is viewed in terms of equation (1) after a transformation (see Gilbert and Roger [5] [pp.223]). The GBASP

2010 *Mathematics Subject Classification.* 30B10, 41A10.
Key words and phrases. L^p-Norm, lower order, lower type, Best polynomial approximation error, entire function and generalized biaxisymmetric potentials.

$F(x, y)$ then suggest generalizations of analytic functions and have a variety of Physical interpolations ([2, 3, 5, 7]). Some examples also can be found in Askey [1].

Mccoy [10] developed an operator and its inverse, which associates with each GBASP, a unique even analytic function. Thus, let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{2n}, z = x + iy \ \in \ \mathbf{C} \tag{4}$$

be the unique associated even analytic function. The operator mapping f onto GBASP

$$F(x, y) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x, y), \alpha > \beta > -\frac{1}{2},$$

uniquely is given by

$$F(x, y) = H_{(\alpha,\beta)}(f) = \int_0^1 \int_0^{\pi} f(\xi) \ \mu_{\alpha,\beta}(t, s) ds dt \tag{5}$$

$$\xi^2 = x^2 - y^2 t^2 + 2i(xyt \ \cos s)$$

$$\mu_{(\alpha,\beta)}(t, s) = \gamma_{(\alpha,\beta)}(1 - t^2)^{\alpha-\beta-1} t^{2\beta+1}(sin \ s)^{2\alpha}$$

$$\gamma_{(\alpha,\beta)} = 2\Gamma(\alpha + 1)/\Gamma(\frac{1}{2})\Gamma(\alpha - \beta)\Gamma(\beta + 1/2).$$

The inverse operator $H^{-1}_{(\alpha,\beta)}$ is given by

$$f(z) = H^{-1}_{(\alpha,\beta)}(F) = \int_{-1}^{1} F[r\xi, r(1 - \xi)^{1/2}] S^*_{(\alpha,\beta)}(z/r, \xi) d\xi \tag{6}$$

$$S^*_{(\alpha,,\beta,)}(\tau, \xi) = S_{(\alpha,\beta)}(\tau, \xi)(1 - \xi)^{\alpha}(1 + \xi)^{\beta},$$

the kernal $S_{(\alpha,\beta)}$ is given by

$$S_{(\alpha,\beta)}(\tau, \xi) = \eta_{\alpha,\beta} \frac{(1 - r)}{(1 + \tau)\alpha + \beta + 2} \ {}_2F_1\Big(\frac{\alpha + \beta + +2}{2}; \frac{\alpha + \beta + 3}{2}; \beta + 1; \frac{2\tau(1 + \xi)}{(1 + \tau)^2}\Big)$$

$$\eta_{(\alpha,\beta)} = \Gamma(\alpha + \beta + 2)/2^{\alpha+\beta,+1}\Gamma(\alpha + 1)\Gamma(\beta + 1).$$

The normalizations $H_{(\alpha,\beta)}(1) = H^{-1}_{(\alpha,\beta)}(1) = 1$ are taken. The kernal $S_{(\alpha,\beta)}(\tau, \xi)$ ia analytic on $|\tau| < 1$ for $-1 \leq \xi \leq 1$. The local function elements F and f are harmonically/analytically, continued by contour deformation by the Envelope Method [3]. It was proved [3, 4], that GBASP F is regular in the hypersphere $\sum_{\sigma}^{(\alpha,\beta)} : x^2 + y^2 \leq \sigma^2$ if and only if its associate f is analytic in the disk $D_{\sigma} : x^2 + y^2 \leq \sigma^2$. Further we have

$$f(x + i0) = F(x, 0), |x| < \sigma$$

which can be analytically continued as

$$f(z) = F(z, 0), |z| < \sigma.$$

The order, type, lower order and lower type of an entire function GBASP are defined from the maximum modulus $M_r(F) = \sup\{|F(x, y)| : x^2 + y^2 \leq r^2\}$ as in function theory [8], respectively.

$$\begin{matrix} \rho(F) \\ \lambda(F) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M_r(F)}{\log r}$$

$$\begin{matrix} T(F) \\ t(F) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M_r(F)}{r^{\rho(F)}}.$$

We now have

Remark 1. *The orders, types,lower orders and lower types of the GBASP and the associate are respectively equal* [10]. *Hence, the notations*

(i) $\rho = \rho(F) = \rho(f)$
(ii) $\pi = T(F) = T(f)$
(iii) $\lambda = \lambda(F) = \lambda(f)$
(iv) $t = t(F) = t(f)$.

Further we define the approximation errors in L^p-norms.

Let $A_p(\sum_\sigma^{(\alpha,\beta)})$ and $a_p(D_\sigma)$ denote the spaces of GBASP and its associate that remain regular and respectively analytic on $\sum_\sigma^{(\alpha,\beta)} : x^2 + y^2 \leq \sigma^2$ and $D_\sigma : x^2 + y^2 \leq \sigma^2$ with finite norms

$$\|F_\sigma\|_\sigma = \Big(\int\int_{\bar{\sum}_\sigma^{(\alpha,\beta)}} |F_\sigma|^p dxdy\Big)^{1/q} \tag{7}$$

$$\|f_\sigma\|_\sigma = \Big(\int\int_{\bar{D}_\sigma} |F_\sigma|^p dx\ dy\Big)^{1/q}, \tag{8}$$

for fixed $p \geq 1$. For each integer n, define the sets of polynomials

$$\Phi_n = \Big\{\sum_{k=0}^{n} a_k\ z^{2k} : a_k - real\Big\}, G_n = \Big\{H_{(\alpha,\beta)}(P_n) : P_n \in \Phi_n\ \Big\}$$

with

$$\Phi_{\sigma,n} = \Big\{p_{\sigma,n}(z) = p_n(z/\sigma) : p_n \in \Phi_n\Big\} \text{ and } G_{\sigma,n} = \Big\{H_{(\alpha,\beta)}(p_{\sigma,n}) : p_{\sigma,n}\ \in \Phi_{\sigma,n}\ \Big\}.$$

Now the best polynomial approximates to the GBASP and associate are defined as

$$e_n^{(p)}(f\sigma) = \inf\Big\{\|f_\sigma - p_{\sigma,n}\|_\sigma : p_{\sigma,n} \in \Phi_{\sigma,n}\Big\}$$

$$E_n^{(p)}(F\sigma) = \inf\Big\{\|F_\sigma - P_{\sigma,n}\|_\sigma : P_{\sigma,n} \in G_{\sigma,n}\Big\}.$$

It is known [15] that for each n there is an extremal polynomial $p^*_{\sigma,n} \in \Phi_{\sigma,n}$ for which $e_n^{(p)}(f\sigma) = \{\|f_\sigma - p^*_{\sigma,n}\|_\sigma$. In the following it will be evident that for each n, $E_n^{(p)}(F\sigma) = \|F_\sigma - P^*_{\sigma,n}\|_\sigma, P^*_{\sigma,n}\ \in G_{\sigma,n}$. It is obvious that when $\sigma = 1$ in the above notations, the subscripts σ are dropped.

Mccoy [10] studied the growth parameters of F in terms of the errors $E_n(F)$ obtaining global existence criterion for GBASP of Sato index k using the results obtained by Reddy [12]. Also in a subsequent paper [11] he studied order and type of F in terms of $E_n^{(p)}(F_\sigma)$. Srivastava [13], studied the growth and polynomial approximation of GBASP in sub norm. However they did not consider other important growth parameters such as lower order, lower type in terms of $E_n^{(p)}(F_\sigma)$, which play very significant role in the study of growth of an entire GBASP. In this paper we have tried to fill this gap.

2. Preliminaries

Let $w = \phi(z)$ be the univalent function mapping the complement of $\bar{D}$ onto $|w| > 1$ such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. Set $D_r = \{z : |\phi(z)| = r\}, r > 1$, then we have

Lemma 1. *Let the $F \in A_p(\sum), p \geq 1$, be an entire function GBASP of order ρ, type T, lower order λ, lower type t, then*

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \begin{matrix}\rho(F) \\ \lambda(F)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log \bar{M}_r(F)}{\log r} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log \bar{M}_r(f)}{\log r},$$

$$\begin{matrix}T\sigma^{\rho} \\ t\sigma^{\rho}\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \bar{M}_r(F)}{r^{\rho(F)}} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log \bar{M}_r(f)}{r^{\rho(f)}},$$

where $\bar{M}_r(F) = \sup_{z\in D_r}\{|F(z),0|\}, r > 1$ and $\bar{M}_r(f) = \sup_{|z|=r} |f(z)|$

Proof. Taking the definition of order type, lower order and lower type of entire GBASP into account with Remark 1 the proof proceeds on the lines of Lemma 3.1 [16]. □

Lemma 2. *Let the $f \in a_p(D), p \geq 1$, be an entire function and $r^{'} (\geq 1)$ be a fixed number, then*

$$e_n^{(p)}(f_\sigma) \leq K\bar{M}_r(f)(r'/r)^n, r > 2r', \tag{9}$$

For all sufficiently large values of n. Here K is a constant depending on D, and p but independent of n and r.

Proof. Since f is entire [11], it follows that [9] [p.114], there exists a sequence of polynomials $\{p_n\}$, p_n being of degree almost n, such that

$$\Big|f(z) - p_n(z)\Big| \leq \frac{3}{2}\bar{M}_r(f)\frac{(r'/r)^{n+1}}{1-(r'/r)}, z \in \bar{D}, \tag{10}$$

for all sufficiently large values of n and all $r > r'$.

It is well known [9] [Chapter XII] that there exist polynomials $p_{\sigma,n}^{(p)} \in \Phi_{\sigma,n}$ such that

$$e_n^{(p)}(f_\sigma) = \Big\|f_\sigma - p_{\sigma,n}^{(\sigma)}\Big\|_\sigma, n = 0, 1, 2, \ldots \tag{11}$$

Lemma now follows from (8), (11) and (10). □

3. Main results

In this section we shall prove our main results.

Theorem 1. *The entire function GBASP $F \in A_p(\sum^{\alpha,\beta}), p \geq 1$, is of lower order λ if and only if*

$$\lambda = \max_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log E_{n_k}^{(p)}(F_\sigma)}. \tag{12}$$

Here maximum is taken over all increasing sequences $\{n_k\}$ of positive integers.

Proof. Let $F \in A_p(\sum^{\alpha,\beta})$ be an entire function, then for $0 < \sigma < 1$, $F_\sigma \in A_p(\sum^{\alpha,\beta})$. Hence, the f_σ is entire as is

$$F_\sigma = H_{(\alpha,\beta)}(f\sigma)$$

or

$$|F_\sigma| = \int_0^1 \int_0^\pi |H_{(\alpha,\beta)}|\ |f\sigma| dsdt,$$

using Holder's inequality to get

$$|F_\sigma|^p \leq \gamma_{(\alpha,\beta)}^p \int_0^1 \int_0^\pi |f\sigma|^p dsdt.$$

Now Applying Fubini's theorem, we get

$$\|F_\sigma\|_\sigma^p \leq \gamma_{(\alpha,\beta)}^p \int_0^1 \int_0^\pi \Big[\int\int_{x^2+y^2\leq\sigma^2} |f\sigma|^p dxdy\Big] dsdt,$$

$$\leq \gamma^p_{(\alpha,\beta)} \int_0^1 \int_0^\pi \|f\sigma\|^p \; ds \; dt$$

$$= \pi\gamma^p_{(\alpha,\beta)}\|f\sigma\|^p$$

or

$$\|F_\sigma\|_\sigma \leq \pi^{1/p}\gamma_{(\alpha,\beta)}\|f\sigma\|^p$$

it gives

$$E^p_n(F_\sigma) \leq \pi^{1/p}\gamma_{(\alpha,\beta)}e^p_n(f_\sigma). \tag{13}$$

Let for any increasing sequence $\{n_k\}$,

$$\liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log E^p_{n_k}(F\sigma)} = \theta(n_k) = \theta.$$

First, let $\theta > 0$. Then for arbitrary ε, $\theta > \varepsilon > 0$, we have

$$E^p_{n_k}(F\sigma) > \Big[n_{k-1}\Big]^{-n_k/(\theta-\varepsilon)}, k > k_0(\varepsilon).$$

Define the sequence

$$r_k = e\Big[n_{k-1}\Big]^{1/(\theta-\varepsilon)}, \; k = 1, 2, \ldots$$

for $r_k \leq r \leq r_{k+1}$, $k > k_0$, we have from (9), (13) and above inequality.

$$\log \bar{M}_r(f) \geq \log E^p_{n_k}(F\sigma) - \log(\pi^{1/p}\gamma(\alpha,\beta) - \log K + n_k \log(r/r')$$

$$> \frac{-n_k \log n_{k+1}}{\theta - \varepsilon} + n_k \log r_k - n_k \log r'_k - O(1)$$

$$= -\Big(\frac{r_{k+1}}{e}\Big)^{\theta-\varepsilon} \log(r_k/e) + \Big(\frac{r_{k+1}}{e}\Big)^{\theta-\varepsilon} \log r_k - \Big(\frac{r_{k+1}}{e}\Big)^{\theta-\varepsilon} \log r' - O(1)$$

$$> \Big(\frac{r_{k+1}}{e}\Big)^{\theta-\varepsilon}\Big(1 - \log r'\Big) - O(1).$$

Hence for $k > k_0$

$$\log\log \bar{M}_r(f) > (\theta - \varepsilon)\log(r_{k+1}) + O(1)$$

$$> (\theta - \varepsilon)\log r + O(1)$$

or

$$\lambda = \liminf_{k\to\infty} \frac{\log\log \bar{M}_r(f)}{\log r} \geq \theta.$$

The inequality obviously holds if $\theta = 0$. Since $\{n_k\}$ was any increasing sequence, we obtain

$$\lambda = \lambda(F\sigma) \geq \max \theta(n_k) = \max_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log E^p_{n_k}(F\sigma)}. \tag{14}$$

To obtain the reverse inequality, we have

$$f_\sigma = H^{-1}_{(\alpha,\beta)}(F\sigma)$$

$$|f_\sigma| \leq \int_{-1}^1 |S(\alpha,\beta)| \; |(F\sigma)| \; d\xi.$$

Applying Holder's inequality in the above estimate, we get

$$|f_\sigma|^p \leq \omega^p_{\alpha,\beta} \int_{-1}^1 |(F\sigma)|^p \; d\xi$$

where

$$\omega_{\alpha,\beta} = \sup\Big\{\|S_{\alpha,\beta}(\tau,\xi)\| : |\tau| \le \sigma, |\xi| \le 1\Big\}.$$

Inview of Fubinis theorem,

$$\|f_\sigma\|_\sigma^p \le \omega_{(\alpha,\beta)}^p \int_0^1 \int_0^\pi \Big[\int\int_{x^2+y^2\le\sigma^2} |F\sigma|^p \, dx \, dy\Big] d\xi,$$

$$\le 2\omega_{(\alpha,\beta)}^p \|F\sigma\|_\sigma^p$$

or

$$\|f_\sigma\|_\sigma^p \le 2^{1/p}\omega_{(\alpha,\beta)}\|F\sigma\|_\sigma.$$

So that $f_\sigma \in a_p(D_\sigma)$. Thus we get

$$e_n^p(f_\sigma) \le 2^{1/p}\ \omega_{(\alpha,\beta)} E_n^p(F_\sigma), \text{ for any } n.$$

Thus for any sequence$\{n_k\}$,

$$\liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log e_{n_k}^p(f\sigma)} \le \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log E_n^p(F\sigma)}.$$

By using theorem 1 of Juneja [6] for $q = 2$ in view of (11), we have

$$\lambda = \max_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log e_{n_k}^p(f\sigma)} \le \max_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{-\log E_{n_k}^p(F\sigma)}. \tag{15}$$

Combining (14) and (15) we get (12). Hence the proof is complete. □

We can prove the above theorem by imposing, a weaker condition on the sequence $\{e_n^p(f\sigma)\}$. Hence we have

Theorem 2. *Let $F \in A_p(\sum^{\alpha,\beta})$, $p \ge 1$ be an entire function GBASP of lower order λ and suppose that $\{e_n^p(f\sigma)/e_{n+1}^p(f\sigma)\}$ form a non-decreasing function of n for $n > n_0$. Then*

$$\lambda = \liminf_{k\to\infty} \frac{n \log n}{-\log E_n^p(F\sigma)}. \tag{16}$$

Proof. Proceeding as the proof of first part of Theorem 1, we can easily show that

$$\lambda \ge \liminf_{k\to\infty} \frac{n \log n}{-\log E_n^p(F\sigma)}\ .$$

We now apply theorem 2A and 2B of Reddy [12] [p.133-134] with (11) for entire function f_σ. Then

$$\lambda = \liminf_{k\to\infty} \frac{n \log n}{-\log e_n^p(f\sigma)}.$$

Since $e_n^p(f_\sigma) \le 2^{1/p} w_{(\alpha,\beta)} E_n^p(F_\sigma)$, we get

$$\lambda \ge \liminf_{k\to\infty} \frac{n \log n}{-\log E_n^p(F\sigma)} \ge \liminf_{k\to\infty} \frac{n \log n}{-\log e_n^p(f\sigma)} = \lambda.$$

Hence the proof is complete. □

Finally we obtain characterization for the lower type of F

Theorem 3. *Let $F \in A_p(\sum^{\alpha,\beta})$, $p \ge 1$ be an entire function GBASP of lower order ρ and lower type t. If $\{e_n^p(f\sigma)/e_{n+1}^p(f\sigma)\}$ forms a non-decreasing function of n Then*

$$t\sigma^{-\rho} = \liminf_{k\to\infty} \frac{n}{e\rho}\Big[E_n^p(F_\sigma)\Big]^{\rho/n}. \tag{17}$$

Proof. Let $\lim_{n\to\infty}\inf \frac{n}{e\rho}\Big[E_n^p(F_\sigma)\Big]^{\rho/n} = \eta$, $0<\eta<\infty$. For arbitrary ε, $0<\varepsilon<\eta$ and all sufficiently large $n > n_0 = n_0(\varepsilon)$, we have

$$E_n^p(F_\sigma) > \Big(\frac{e\rho(\eta-\varepsilon)}{n}\Big)^{n/\rho},$$

from (9), (13) and above inequality, we get

$$\log \bar{M}_r(f) \geq \frac{n}{\rho}\log\Big(\frac{e\rho(\eta-\varepsilon)}{n}\Big) + n\log(r/r') - \log(\pi^{1/p}\gamma_{\alpha,\beta}) - \log K.$$

Choose $n = \rho(\eta-\varepsilon)(r/r')^\rho$. Then for large values of r, $n > n_0$, in above, we get

$$\log \bar{M}_r(f) \geq (\eta-\varepsilon)(r/r')^\rho - O(1).$$

Dividing by r^ρ and proceeding to limits as $r\to\infty$, we get

$$t\sigma^{-\rho} \geq \liminf_{k\to\infty} \frac{n}{e\rho}\Big[E_n^p(F_\sigma)\Big]^{\rho/n}. \tag{18}$$

The above inequality holds if $\eta = 0$. To prove reverse inequality in (17), under the given condition on $e_n^p(f_\sigma)$, we have by a result of Reddy [12][Theorem 4A] in of (11).

$$\liminf_{k\to\infty} \frac{n}{e\rho}\Big[e_n^p(f_\sigma)\Big]^{\rho/n} \geq \sigma^{-\rho}\ t$$

or

$$t\sigma^{-\rho} = \liminf_{k\to\infty} \frac{n}{e\rho}\Big[e_n^p(f_\sigma)\Big]^{\rho/n} \leq \liminf_{k\to\infty} \frac{n}{e\rho}\Big[E_n^p(F_\sigma)\Big]^{\rho/n}.$$

Hence the proof is complete. □

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