GENERALIZED TYPE OF ENTIRE MONOGENIC FUNCTIONS OF SLOW GROWTH

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Abstract. In the present paper we study the generalized growth of entire monogenic functions having slow growth. The characterizations of generalized type of entire monogenic functions have been obtained in terms of their Taylor’s series coefficients.

1. Introduction

Clifford analysis offers possibility of generalizing complex function theory to higher dimensions. It considers Clifford algebra valued functions that are defined in open subsets of \( \mathbb{R}^n \) for arbitrary finite \( n \in \mathbb{N} \) and that are solutions of higher dimensional Cauchy-Riemann systems. These are often called Clifford holomorphic or monogenic functions. In order to make calculations more concise we use following notations, where \( \delta_{ij} \) where \( \delta_{ij} = 1 \) if \( i = j \), and \( \delta_{ij} = 0 \) otherwise.

\[
x^m = x_1^{m_1} \cdots x_n^{m_n}, \quad m! = m_1! \cdots m_n!, \quad |m| = m_1 + \cdots + m_n.
\]

Following Constales, Almeida and Krausshar (see [1] and [2]), we give some definitions and associated properties.

By \( \{e_1, e_2, \ldots, e_n\} \) we denote the canonical basis of the Euclidean vector space \( \mathbb{R}^n \). The associated real Clifford algebra \( Cl_{0n} \) is the free algebra generated by \( \mathbb{R}^n \) modulo \( x^2 = -||x||^2 e_0 \), where \( e_0 \) is the neutral element with respect to multiplication of the Clifford algebra \( Cl_{0n} \). In the Clifford algebra \( Cl_{0n} \) following multiplication rule holds:

\[
e_{i}e_{j} + e_{j}e_{i} = -2\delta_{ij}e_0, \quad i, j = 1, 2, \ldots, n,
\]

where \( \delta_{ij} \) is Kronecker symbol. A basis for Clifford algebra \( Cl_{0n} \) is given by the set \( \{e_A : A \subseteq \{1, 2, \ldots, n\} \} \) with \( e_A = e_{l_1}e_{l_2} \cdots e_{l_r} \), where \( 1 \leq l_1 < l_2 < \cdots < l_r \leq n \) and \( e_0 = e_0 = 1 \). Each \( a \in Cl_{0n} \) can be written in the form \( a = \sum_{A} a_{A}e_{A} \) with \( a_{A} \in \mathbb{R} \). The conjugation in Clifford algebra \( Cl_{0n} \) is defined by \( \bar{a} = \sum_{A} a_{A}\bar{e}_{A} \), where \( e_{A} = \bar{e}_{l_1}e_{l_2} \cdots e_{l_r} \) and \( \bar{e}_{j} = -e_{j} \) for \( j = 1, 2, \ldots, n \). Here \( x_0 = \text{Sc}(z) \) is scalar part and \( x = x_1e_{1} + x_2e_{2} + \cdots + x_ne_{n} = \text{Vec}(z) \) is vector part of para vector \( z \). The Clifford norm of an arbitrary \( a = \sum_{A} a_{A}e_{A} \) is given by

\[
||a|| = \left( \sum_{A} |a_{A}|^2 \right)^{1/2}.
\]

Each para vector \( z \in \mathbb{R}^{n+1}\setminus\{0\} \) has an inverse element in \( \mathbb{R}^{n+1} \) which can be represented in the form \( z^{-1} = \frac{1}{z} ||z||^2 \).
The generalized Cauchy-Riemann operator in \( \mathbb{R}^{n+1} \) is given by
\[
D \equiv \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}.
\]

If \( U \subseteq \mathbb{R}^{n+1} \) is an open set, then a function \( g : U \to Cl_{0n} \) is called left (right) monogenic at a point \( z \in U \) if \( Dg(z) = 0 \) \((gD(z) = 0)\). The functions which are left (right) monogenic in the whole space are called left (right) entire monogenic functions.

Let \( A_{n+1} \) be \( n \)-dimensional surface area of \( n+1 \)-dimensional unit ball and \( q_0(z) = \frac{z}{||z||} \) be Cauchy kernel function. Then every function \( g \) which is monogenic in a neighborhood of closure \( \overline{G} \) of domain \( G \) satisfies the following equation (see [2], p. 766)
\[
g(z) = \frac{1}{A_{n+1}} \int_{\partial G} q_0(z - \zeta) d\tau(\zeta) g(\zeta), \text{ for all } z \in G,
\]
where
\[
d\tau(\zeta) = \sum_{j=0}^{n} (-1)^j e_j d\zeta_j
\]
with
\[
d\zeta_j = d\zeta_0 \wedge \cdots \wedge d\zeta_{j-1} \wedge d\zeta_{j+1} \wedge \cdots \wedge d\zeta_n
\]
is the oriented outer normal surface measure. If \( g \) is a left monogenic function in a ball \( ||z|| < R \), then for all \( ||z|| < r \) with \( 0 < r < R \)
\[
g(z) = \sum_{|m|=0}^{\infty} V_m(z) a_m. \tag{1}
\]
In (1) \( V_m(z) \) are called Fueter polynomials and are given as
\[
V_m(z) = \frac{m!}{|m|!} \sum_{\pi \in \text{perm}(m)} z_{\pi(m_1)} \cdots z_{\pi(m_n)},
\]
where \( \text{perm}(m) \) is the set of all permutations of the sequence \( (m_1, m_2, \ldots, m_n) \) and \( z_i = x_i - x_0 e_i \) for \( i = 1, \ldots, n \) and \( V_0(z) = 1 \). Also in [1], \( \{a_m\} \) are Clifford numbers which are defined by
\[
a_m = \frac{1}{m! A_{n+1}} \int_{||\zeta|| < r} q_m(\zeta) d\tau(\zeta) g(\zeta)
\]
and satisfy the inequality
\[
||a_m|| \leq c(n, m) M(r, g) \frac{r^{|m|}}{|m|!}.
\]
Here \( M(r, g) = \max \{||g(z)||\} \) denotes the maximum modulus of the function \( g \) in the closed ball of radius \( r \) and
\[
q_m(z) = \frac{\partial^{m_0+m_1+\cdots+m_n}}{\partial x_0^{m_0} \partial x_1^{m_1} \cdots \partial x_n^{m_n}} q_0(z), \quad c(n, m) = \frac{n(n+1) \cdots (n+|m|-1)}{m!}.
\]

The concept of generalized order and generalized type for entire transcendental functions was given by Seremeta [3], Kapoor and Nautiyal [3]. Hence, let \( L^0 \) denote the class of functions \( h(x) \) satisfying the following conditions:
(i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to $\infty$ as $x \to \infty$,

(ii) $\lim_{x \to \infty} \frac{h((1+1/\psi(x))x)}{h(x)} = 1$, for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let $\Lambda$ denote the class of functions $h(x)$ satisfying conditions (i) and

(iii) $\lim_{x \to \infty} \frac{h(cx)}{h(x)} = 1$, for every $c > 0$, that is $h(x)$ is slowly increasing.

Let $\Omega$ be the class of functions $h(x)$ satisfying conditions (i) and

(iv) there exist a function $\delta(x) \in \Lambda$ and constants $x_0$, $K_1$ and $K_2$ such that

$$0 < K_1 \leq \frac{d(h(x))}{d(\delta(\log x))} \leq K_2 < \infty,$$

for all $x > x_0$.

Let $\Omega$ be the class of functions $h(x)$ satisfying (i) and

(v) $\lim_{x \to \infty} \frac{d(h(x))}{d(\log x)} = K$, $0 < K < \infty$.

Kapoor and Nautiyal [3] showed that classes $\Omega$ and $\Omega$ are contained in $\Lambda$ and $\Omega \cap \Omega = \phi$.

For an entire monogenic function $g(z)$ and functions $\alpha(x)$ either belongs to $\Omega$ or to $\Omega$, we define the generalized order $\rho(\alpha, g)$ of $g(z)$ as

$$\rho(\alpha, g) = \lim_{r \to \infty} \sup_{|m| \leq r} \frac{\alpha(|m|)}{\alpha(r)} \frac{\log M(r, g)}{\log r}.$$  \hspace{1cm} (2)

Further, for $\alpha(x)$ either belongs to $\Omega$ or to $\Omega$, we define the generalized type $\sigma(\alpha, \rho, g)$ of $g(z)$ as

$$\sigma(\alpha, \rho, g) = \lim_{r \to \infty} \sup_{|m| \leq r} \frac{\alpha(|m|)}{\alpha(r)^{\rho}} \frac{\log M(r, g)}{\log r}^{\rho},$$  \hspace{1cm} (3)

where $\rho = \rho(\alpha, g)$ and $1 < \rho < \infty$.

2. Main results

Now we prove

**Theorem 1.** Let $g : \mathbb{R}^{n+1} \to Cl_{0, n}$ be an entire monogenic function whose Taylor’s series representation is given by $g(z) = \sum_{|m|=0}^{\infty} V_m(z) a_m$. Also if $\alpha(x)$ either belongs to $\Omega$ or to $\Omega$ and $1 < \rho < \infty$, then the generalized type $\sigma(\alpha, \rho, g)$ of $g(z)$ is given as

$$\sigma(\alpha, \rho, g) - 1 = \lim_{|m| \to \infty} \sup_{|m| \leq r} \frac{\alpha(|m|/\rho)}{\alpha\left(\frac{\rho}{\rho-1} \log \left(||a_m/c(n, m)||\right)^{-1/|m|}\right)^{\rho-1}}.$$ \hspace{1cm} (4)

**Proof.** Write $\sigma = \sigma(\alpha, \rho, g)$ and

$$\eta = \lim_{|m| \to \infty} \sup_{|m| \leq r} \frac{\alpha(|m|/\rho)}{\alpha\left(\frac{\rho}{\rho-1} \log \left(||a_m/c(n, m)||\right)^{-1/|m|}\right)^{\rho-1}}.$$

First we prove that $\eta \leq \sigma - 1$. The coefficients of an entire monogenic Taylor’s series satisfy Cauchy’s inequality, that is
\[ \|a_m/c(n,m)\| \leq r^{-|m|} M(r,g). \] (5)

Also from (3), for \( \varepsilon > 0 \) and \( r > r_0(\varepsilon) \), we have

\[ M(r,g) \leq \exp(\alpha^{-1}[\sigma(\alpha \log r)]^\rho), \]

where \( \sigma = \sigma + \varepsilon \) provided \( r \) is sufficiently large. So from (5), we have

\[ \|a_m/c(n,m)\| \leq r^{-|m|} \exp(\alpha^{-1}[\sigma(\alpha \log r)]^\rho) \]

or

\[ \|a_m/c(n,m)\| \leq \exp(-|m| \log r + \alpha^{-1}[\sigma(\alpha \log r)]^\rho). \] (6)

Let \( r = r(|m|) \) be unique root of the equation

\[ \alpha \left( \frac{|m| \log r}{\rho} \right) = (\sigma - 1) \{\alpha(\log r)\}^\rho. \] (7)

Then for all large values of \( |m| \), we have

\[ \log r \simeq \alpha^{-1} \left\{ \frac{1}{(\sigma - 1)} \alpha(|m|/\rho) \right\}^{1/\rho - 1} = G(|m|/\rho, 1/((\sigma - 1), \rho - 1)). \] (8)

Using (7) and (8) in (6), we get

\[ \|a_m/c(n,m)\| \leq \exp[-|m|G + (|m|/\rho)G] \]

or

\[ \frac{\rho}{\rho - 1} \log \|a_m/c(n,m)\|^{-1/|m|} \geq \alpha^{-1} \left\{ \frac{1}{(\sigma - 1)} \alpha(|m|/\rho) \right\}^{1/\rho - 1} \]

or

\[ \alpha(|m|/\rho) \left[ \alpha \left\{ \frac{\rho}{\rho - 1} \log \|a_m/c(n,m)\|^{-1/|m|} \right\} \right]^{\rho - 1} \leq (\sigma - 1). \]

Proceeding to limits as \( |m| \to \infty \), we get

\[ \eta = \lim_{|m| \to \infty} \sup_{\alpha(|m|/\rho)} \alpha \left\{ \frac{\rho}{\rho - 1} \log \|a_m/c(n,m)\|^{-1/|m|} \right\}^{\rho - 1} \leq (\sigma - 1). \]

Since \( \varepsilon > 0 \) is arbitrarily small, we finally get

\[ \eta \leq (\sigma - 1). \] (9)

Now we will prove that \( \sigma - 1 \leq \eta \). If \( \eta = \infty \), then there is nothing to prove. So let us assume that \( 0 \leq \eta < \infty \). Therefore for all \( \varepsilon > 0 \) there exist \( n_0 \in N \) such that for all multi-indices \( m \) with \( |m| > n_0 \), we have

\[ 0 \leq \frac{\alpha(|m|/\rho)}{\left[ \alpha \left\{ \frac{\rho}{\rho - 1} \log \|a_m/c(n,m)\|^{-1/|m|} \right\} \right]^{\rho - 1} < \eta + \varepsilon = \eta. \]

or
\[ ||a_m|| \leq c(n, m) \exp \left( -\frac{\rho - 1}{\rho} |m|^{\alpha - 1} \left[ \left\{ \frac{1}{\eta} \alpha \left( |m|/\rho \right) \right\}^{1/\rho - 1} \right] \right). \]

Now from the property of maximum modulus, we have
\[ M(r, g) \leq \sum_{|m|=0}^{\infty} ||a_m|| r^{|m|} \]
or
\[ M(r, g) \leq \sum_{|m|=0}^{n_0} ||a_m|| r^{|m|} + \sum_{|m|=n_0+1}^{\infty} c(n, m) r^{|m|} \exp \left( -\frac{\rho - 1}{\rho} |m|^{\alpha - 1} \left[ \left\{ \frac{1}{\eta} \alpha \left( |m|/\rho \right) \right\}^{1/\rho - 1} \right] \right). \]

Now for \( r > 1 \), we have
\[ M(r, g) \leq B_1 r^{n_0} + \sum_{|m|=n_0+1}^{\infty} c(n, m) r^{|m|} \exp \left( -\frac{\rho - 1}{\rho} |m|^{\alpha - 1} \left[ \left\{ \frac{1}{\eta} \alpha \left( |m|/\rho \right) \right\}^{1/\rho - 1} \right] \right), \]
where \( B_1 \) is a positive real constant. We take
\[ N(r) = \left\lfloor \rho^{-1} \left( \alpha \left[ \frac{\rho}{\rho - 1} \log \{(n+1)r\} \right] \right)^{\rho - 1} \right\rfloor, \]
where \( \lfloor x \rfloor \) denotes the integer part of \( x \geq 0 \). Since \( \alpha(x) \) either belongs to \( \Omega \) or to \( \Omega^* \), the integer \( N(r) \) is well defined. Now if \( r \) is sufficiently large, then from (10) we have
\[ M(r, g) \leq B_1 r^{n_0} + r^{N(r)\times} \sum_{n_0+1 \leq |m| \leq N(r)} c(n, m) \exp \left( -\frac{\rho - 1}{\rho} |m|^{\alpha - 1} \left[ \left\{ \frac{1}{\eta} \alpha \left( |m|/\rho \right) \right\}^{1/\rho - 1} \right] \right) \]
or
\[ M(r, g) \leq B_1 r^{n_0} + r^{N(r)\times} \sum_{|m|>N(r)} c(n, m) r^{|m|} \exp \left( -\frac{\rho - 1}{\rho} |m|^{\alpha - 1} \left[ \left\{ \frac{1}{\eta} \alpha \left( |m|/\rho \right) \right\}^{1/\rho - 1} \right] \right). \]

Now the first series in (11) can be rewritten as
\[ \sum_{p=1}^{\infty} \left( \sum_{|m|=p} c(n, m) \right) \exp \left( -\frac{\rho - 1}{\rho} p^{\alpha - 1} \left[ \left\{ \frac{1}{\eta} \alpha \left( p/\rho \right) \right\}^{1/\rho - 1} \right] \right). \]

Now from (2, Lemma 1), we have
\[ \lim_{p \to \infty} \sup \left( \sum_{|m|=p} c(n, m) \right)^{1/p} = n. \]
Hence we have

\[
\lim_{p \to \infty} \sup \left[ \left( \sum_{|m|=p} c(n, m) \right) \exp \left( -\frac{p-1}{p} p \alpha^{-1} \left[ \left\{ \frac{1}{\eta} (p/\rho) \right\}^{1/p} \right] \right) \right]^{1/p}
= n \lim_{p \to \infty} \sup \exp \left( -\frac{p-1}{p} p \alpha^{-1} \left[ \left\{ \frac{1}{\eta} (p/\rho) \right\}^{1/p} \right] \right) = 0.
\]

Hence the series (12) converges to a positive real constant \(B_2\). So from (11), we get

\[
M(r, g) \leq B_1 r^{n_0} + B_2 r^{N(r)} + \sum_{|m| > N(r)} c(n, m) r^{|m|} \exp \left(-|m| \log \{ (n+1)r \} \right)
\]
or

\[
M(r, g) \leq B_1 r^{n_0} + B_2 r^{N(r)} + \sum_{|m| > N(r)} c(n, m) (1/n+1)^{|m|}
\]
or

\[
M(r, g) \leq B_1 r^{n_0} + B_2 r^{N(r)} + \sum_{|m| = 1}^{\infty} c(n, m) \left( \frac{1}{n+1} \right)^{|m|}.
\]  \(\text{(13)}\)

The series in (13) can we rewritten as

\[
\sum_{p=1}^{\infty} \left( \sum_{|m|=p} c(n, m) \right) \left( \frac{1}{n+1} \right)^p.
\]  \(\text{(14)}\)

So we have

\[
\lim_{p \to \infty} \sup \left[ \left( \sum_{|m|=p} c(n, m) \right) \left( \frac{1}{n+1} \right)^p \right]^{1/p} = \frac{n}{n+1} < 1.
\]

Hence the series (14) converges to a positive real constant \(B_1\). Therefore from (13), we get

\[
M(r, g) \leq B_1 r^{n_0} + B_2 r^{N(r)} + B_3.
\]

Since \(N(r) \to \infty\) as \(r \to \infty\) so we can write above inequality as
log $M(r, g) \leq [1 + o(1)] N(r) \log r$

$$\leq [1 + o(1)] \left[ \rho \alpha^{-1} \left\{ \frac{\rho}{(r+1)} \log ((n+1)r) \right\}^{\rho-1} \right] \log r$$

$$\leq [1 + o(1)] \rho \alpha^{-1} \left\{ \frac{\rho}{(r+1)} \log ((n+1)r) \right\}^{\rho-1} \times$$

$$\times \alpha^{-1} \left\{ \left( \frac{\rho}{(r+1)} \log ((n+1)r) \right)^{\rho-1} \right\}$$

$$\leq [1 + o(1)] \rho \alpha^{-1} \left\{ (\eta + 1) \left( \frac{\rho}{(r+1)} \log ((n+1)r) \right) \right\}^{\rho-1}$$

or

$$\alpha [\log M(r, g)] \leq (\eta + 1) \left( \frac{\rho}{(r+1)} \log ((n+1)r) \right)^{\rho-1} [1 + o(1)]$$

or

$$\frac{\alpha [\log M(r, g)]}{(\alpha \left[ \frac{\rho}{(r+1)} \log ((n+1)r) \right])^{\rho-1}} \leq (\eta + 1) [1 + o(1)]$$

or

$$\frac{\alpha [\log M(r, g)]}{(\alpha \left[ \frac{\rho}{(r+1)} \log ((n+1)r) \right])^{\rho-1}} \leq (\eta + 1) [1 + o(1)]$$

or

$$\frac{\alpha [\log M(r, g)]}{\alpha (\log r)^{\rho}} \leq \left( \frac{\alpha [O(1) \log r]}{\alpha (\log r)^{\rho}} \right)^{\rho} (\eta + 1) [1 + o(1)].$$

Proceeding to limits as $r \to \infty$ and using properties of $\alpha(x)$, we get

$$\sigma \leq \eta + 1.$$  

Since $\varepsilon > 0$ is arbitrarily small, we finally get

$$\sigma - 1 < \eta.$$  

Combining (9) and (15), we get (14). Hence Theorem 2.1 is proved.  

REFERENCES

