TJMM
3 (2011), No. 2, 95-102

GENERALIZED TYPE OF ENTIRE MONOGENIC FUNCTIONS OF SLOW GROWTH

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Abstract. In the present paper we study the generalized growth of entire monogenic functions having slow growth. The characterizations of generalized type of entire monogenic functions have been obtained in terms of their Taylor's series coefficients.

1. Introduction

Clifford analysis offers possibility of generalizing complex function theory to higher dimensions. It considers Clifford algebra valued functions that are defined in open subsets of $\mathbb{R}^n$ for arbitrary finite $n \in \mathbb{N}$ and that are solutions of higher dimensional Cauchy-Riemann systems. These are often called Clifford holomorphic or monogenic functions. In order to make calculations more concise we use following notations, where $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ be $n-$ dimensional multi-index and $\mathbf{x} \in \mathbb{R}^n$:

$$\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \dots x_n^{m_n} \ , \quad \mathbf{m!} = m_1! \dots m_n! \ , \quad |\mathbf{m}| = m_1 + \cdots + m_n.$$

Following Constales, Almeida and Krausshar (see [1] and [2]), we give some definitions and associated properties.

By $\{e_1, e_2, ..., e_n\}$ we denote the canonical basis of the Euclidean vector space $\mathbb{R}^n$. The associated real Clifford algebra Cl_{0n} is the free algebra generated by $\mathbb{R}^n$ modulo $\mathbf{x}^2 = -||\mathbf{x}||^2 e_0$, where e_0 is the neutral element with respect to multiplication of the Clifford algebra Cl_{0n}. In the Clifford algebra Cl_{0n} following multiplication rule holds:

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0 \ , \ \ i, j = 1, 2, \dots, n \, ,$$

where δ_{ij} is Kronecker symbol. A basis for Clifford algebra Cl_{0n} is given by the set $\{e_A : A \subseteq \{1, 2, \dots, n\}\}$ with $e_A = e_{l_1} e_{l_2} \dots e_{l_r}$, where $1 \leq l_1 < l_2 < \cdots < l_r \leq n$, $e_\phi = e_0 = 1$. Each $a \in Cl_{0n}$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. The conjugation in Clifford algebra Cl_{0n} is defined by $\bar{a} = \sum_A a_A \bar{e}_A$, where $e_A = \bar{e}_{l_r} \bar{e}_{l_{r-1}} \dots \bar{e}_{l_1}$ and $\bar{e}_j = -e_j$ for $j = 1, 2, \dots n$, $\bar{e}_0 = e_0 = 1$. The linear subspace $\text{span}_R\{1, e_1, \dots, e_n\} = \mathbb{R} \oplus \mathbb{R}^n \subset Cl_{0n}$ is the so called space of para vectors $z = x_0 + x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ which we simply identify with $\mathbb{R}^{n+1}$. Here $x_0 = \text{Sc}(z)$ is scalar part and $\mathbf{x} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = \text{Vec}(z)$ is vector part of para vector z. The Clifford norm of an arbitrary $a = \sum_A a_A e_A$ is given by

$$||a|| = \left(\sum_A |a_A|^2 \right)^{1/2} .$$

Each para vector $z \in \mathbb{R}^{n+1} \backslash \{0\}$ has an inverse element in $\mathbb{R}^{n+1}$ which can be represented in the form $z^{-1} = \overline{z}/||z||^2$.

2010 *Mathematics Subject Classification.* 30G35, 30D15.
Key words and phrases. Clifford algebra, Clifford analysis, Generalized Cauchy-Riemann system, Entire monogenic function, Generalized type.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is given by

$$D \equiv \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}.$$

If $U \subseteq \mathbb{R}^{n+1}$ is an open set, then a function $g : U \to Cl_{0n}$ is called left (right) monogenic at a point $z \in U$ if $Dg(z) = 0$ ($gD(z) = 0$). The functions which are left (right) monogenic in the whole space are called left (right) entire monogenic functions.

Let A_{n+1} be n-dimensional surface area of $n+1$-dimensional unit ball and $q_{\mathbf{0}}(z) = \overline{z}/||z||^{n+1}$ be Cauchy kernel function. Then every function g which is monogenic in a neighborhood of closure $\overline{G}$ of domain G satisfies the following equation (see [2], p. 766)

$$g(z) = \frac{1}{A_{n+1}} \int_{\partial G} q_{\mathbf{0}}(z-\zeta) d\tau(\zeta) g(\zeta), \text{ for all } z \in G,$$

where

$$d\tau(\zeta) = \sum_{j=0}^{n} (-1)^j e_j \widehat{d\zeta_j}$$

with

$$\widehat{d\zeta_j} = d\zeta_0 \wedge \cdots \wedge d\zeta_{j-1} \wedge d\zeta_{j+1} \wedge \cdots \wedge d\zeta_n$$

is the oriented outer normal surface measure. If g is a left monogenic function in a ball $||z|| < R$, then for all $||z|| < r$ with $0 < r < R$

$$g(z) = \sum_{|\mathbf{m}|=0}^{\infty} V_{\mathbf{m}}(z)\, a_{\mathbf{m}}. \tag{1}$$

In (1) $V_{\mathbf{m}}(z)$ are called Fueter polynomials and are given as

$$V_{\mathbf{m}}(z) = \frac{\mathbf{m!}}{|\mathbf{m}|!} \sum_{\pi \in perm(\mathbf{m})} z_{\pi(m_1)} \dots z_{\pi(m_n)},$$

where $perm(\mathbf{m})$ is the set of all permutations of the sequence $(m_1, m_2 \dots, m_n)$ and $z_i = x_i - x_0 e_i$ for $i = 1, \dots, n$ and $V_{\mathbf{0}}(z) = 1$. Also in (1), $\{a_{\mathbf{m}}\}$ are Clifford numbers which are defined by

$$a_{\mathbf{m}} = \frac{1}{\mathbf{m!}\, A_{n+1}} \int_{||\zeta|| < r} q_{\mathbf{m}}(\zeta) d\tau(\zeta) g(\zeta)$$

and satisfy the inequality

$$||a_{\mathbf{m}}|| \leq c(n, \mathbf{m}) \frac{M(r,g)}{r^{|\mathbf{m}|}}.$$

Here $M(r,g) = \max_{||z||=r} \{||g(z)||\}$ denotes the maximum modulus of the function g in the closed ball of radius r and

$$q_{\mathbf{m}}(z) = \frac{\partial^{m_0 + m_1 + \cdots + m_n}}{\partial x_0^{m_0} \partial x_1^{m_1} \dots \partial x_n^{m_n}} q_{\mathbf{0}}(z), \ \ c(n, \mathbf{m}) = \frac{n(n+1)\dots(n+|\mathbf{m}|-1)}{\mathbf{m!}}.$$

The concept of generalized order and generalized type for entire transcendental functions was given by Seremeta [4], Kapoor and Nautiyal [3]. Hence, let L^0 denote the class of functions $h(x)$ satisfying the following conditions:

(i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii) $\lim\limits_{x\to\infty} \frac{h[\{1+1/\psi(x)\}x]}{h(x)} = 1$, for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions $h(x)$ satisfying conditions (i) and

(iii) $\lim\limits_{x\to\infty} \frac{h(cx)}{h(x)} = 1$, for every $c > 0$, that is $h(x)$ is slowly increasing.

Let Ω be the class of functions $h(x)$ satisfying conditions (i) and

(iv) there exist a function $\delta(x) \in \Lambda$ and constants x_0, K_1 and K_2 such that

$$0<K_1 \le \frac{d\{h(x)\}}{d\{\delta(\log x)\}} \le K_2<\infty,$$

for all $x>x_0$.

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (i) and

(v) $\lim\limits_{x\to\infty} \frac{d\{h(x)\}}{d(\log x)} = K$, $0<K<\infty$.

Kapoor and Nautiyal [3] showed that classes Ω and $\overline{\Omega}$ are contained in Λ and $\Omega \bigcap \overline{\Omega} = \phi$.

For an entire monogenic function $g(z)$ and functions $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$, we define the generalized order $\rho(\alpha, g)$ of $g(z)$ as

$$\rho(\alpha, g) = \lim_{r\to\infty} \sup \frac{\alpha[\log M(r, g)]}{\alpha(\log r)}. \tag{2}$$

Further, for $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$, we define the generalized type $\sigma(\alpha, \rho, g)$ of $g(z)$ as

$$\sigma(\alpha, \rho, g) = \lim_{r\to\infty} \sup \frac{\alpha[\log M(r, g)]}{[\alpha(\log r)]^{\rho}}, \tag{3}$$

where $\rho = \rho(\alpha, g)$ and $1 < \rho < \infty$.

2. Main results

Now we prove

Theorem 1. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} V_{\boldsymbol{m}}(z)a_{\boldsymbol{m}}$. Also if $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$ and $1<\rho<\infty$, then the generalized type $\sigma(\alpha, \rho, g)$ of $g(z)$ is given as*

$$\sigma(\alpha, \rho, g) - 1 = \lim_{|\boldsymbol{m}|\to\infty} \sup \frac{\alpha(|\boldsymbol{m}|/\rho)}{\left(\alpha\left[\frac{\rho}{\rho-1}\log\{||a_{\boldsymbol{m}}/c(n, \boldsymbol{m})||\}^{-1/|\boldsymbol{m}|}\right]\right)^{\rho-1}}. \tag{4}$$

Proof. Write $\sigma = \sigma(\alpha, \rho, g)$ and

$$\eta = \lim_{|\mathbf{m}|\to\infty} \sup \frac{\alpha(|\mathbf{m}|/\rho)}{\left(\alpha\left[\frac{\rho}{\rho-1}\log\{||a_{\mathbf{m}}/c(n, \mathbf{m})||\}^{-1/|\mathbf{m}|}\right]\right)^{\rho-1}}.$$

First we prove that $\eta \le \sigma - 1$. The coefficients of an entire monogenic Taylor's series satisfy Cauchy's inequality, that is

$$||a_{\mathbf{m}}/c(n,\mathbf{m})|| \leq r^{-|\mathbf{m}|}M(r,g). \tag{5}$$

Also from (3), for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$M(r,g) \leq \exp(\alpha^{-1}[\overline{\sigma}\{\alpha(\log r)\}^{\rho}]),$$

where $\overline{\sigma} = \sigma + \varepsilon$ provided r is sufficiently large. So from (5), we have

$$||a_{\mathbf{m}}/c(n,\mathbf{m})|| \leq r^{-|\mathbf{m}|}\exp(\alpha^{-1}[\overline{\sigma}\{\alpha(\log r)\}^{\rho}])$$

or

$$||a_{\mathbf{m}}/c(n,\mathbf{m})|| \leq \exp(-|\mathbf{m}|\log r + \alpha^{-1}[\overline{\sigma}\{\alpha(\log r)\}^{\rho}]). \tag{6}$$

Let $r = r(|\mathbf{m}|)$ be unique root of the equation

$$\alpha\left[\frac{|\mathbf{m}|\log r}{\rho}\right] = (\overline{\sigma}-1)\{\alpha(\log r)\}^{\rho}. \tag{7}$$

Then for all large values of $|\mathbf{m}|$, we have

$$\log r \simeq \alpha^{-1}\left[\left\{\frac{1}{(\overline{\sigma}-1)}\alpha(|\mathbf{m}|/\rho)\right\}^{1/\rho-1}\right] = G\left(|\mathbf{m}|/\rho, 1/\left(\overline{\sigma}-1\right), \rho-1\right). \tag{8}$$

Using (7) and (8) in (6), we get

$$||a_{\mathbf{m}}/c(n,\mathbf{m})|| \leq \exp[-|\mathbf{m}|G + (|\mathbf{m}|/\rho)G]$$

or

$$\frac{\rho}{\rho-1}\log\left[||a_{\mathbf{m}}/c(n,\mathbf{m})||\right]^{-1/|\mathbf{m}|} \geq \alpha^{-1}\left[\left\{\frac{1}{(\overline{\sigma}-1)}\alpha(|\mathbf{m}|/\rho)\right\}^{1/\rho-1}\right]$$

or

$$\frac{\alpha(|\mathbf{m}|/\rho)}{\left[\alpha\left\{\frac{\rho}{\rho-1}\log\left(||a_{\mathbf{m}}/c(n,\mathbf{m})||\right)^{-1/|\mathbf{m}|}\right\}\right]^{\rho-1}} \leq (\overline{\sigma}-1).$$

Proceeding to limits as $|\mathbf{m}| \to \infty$, we get

$$\eta = \lim_{|\mathbf{m}|\to\infty}\sup\frac{\alpha(|\mathbf{m}|/\rho)}{\left[\alpha\left\{\frac{\rho}{\rho-1}\log\left(||a_{\mathbf{m}}/c(n,\mathbf{m})||\right)^{-1/|\mathbf{m}|}\right\}\right]^{\rho-1}} \leq (\overline{\sigma}-1).$$

Since $\varepsilon > 0$ is arbitrarily small, we finally get

$$\eta \leq (\sigma-1). \tag{9}$$

Now we will prove that $\sigma - 1 \leq \eta$. If $\eta = \infty$, then there is nothing to prove. So let us assume that $0 \leq \eta < \infty$. Therefore for all $\varepsilon > 0$ there exist $n_0 \in N$ such that for all multi-indices $\mathbf{m}$ with $|\mathbf{m}| > n_0$, we have

$$0 \leq \frac{\alpha(|\mathbf{m}|/\rho)}{\left[\alpha\left\{\frac{\rho}{\rho-1}\log\left(||a_{\mathbf{m}}/c(n,\mathbf{m})||\right)^{-1/|\mathbf{m}|}\right\}\right]^{\rho-1}} < \eta + \varepsilon = \overline{\eta}$$

or

$$||a_{\mathbf{m}}|| \le c(n,\mathbf{m}) \exp\left(-\frac{\rho-1}{\rho}|\mathbf{m}|\alpha^{-1}\left[\left\{\frac{1}{\overline{\eta}}\alpha\left(|\mathbf{m}|/\rho\right)\right\}^{1/\rho-1}\right]\right).$$

Now from the property of maximum modulus, we have

$$M(r,g) \le \sum_{|\mathbf{m}|=0}^{\infty} ||a_{\mathbf{m}}|| r^{|\mathbf{m}|}$$

or

$$\begin{aligned} M(r,g) &\le \textstyle\sum_{|\mathbf{m}|=0}^{n_0} ||a_{\mathbf{m}}|| r^{|\mathbf{m}|} + \\ &+ \textstyle\sum_{|\mathbf{m}|=n_0+1}^{\infty} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left(-\frac{\rho-1}{\rho}|\mathbf{m}|\alpha^{-1}\left[\left\{\frac{1}{\overline{\eta}}\alpha\left(|\mathbf{m}|/\rho\right)\right\}^{1/\rho-1}\right]\right). \end{aligned}$$

Now for $r>1$, we have

$$\begin{aligned} M(r,g) &\le B_1 r^{n_0} + \\ &+ \textstyle\sum_{|\mathbf{m}|=n_0+1}^{\infty} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left(-\frac{\rho-1}{\rho}|\mathbf{m}|\,\alpha^{-1}\left[\left\{\frac{1}{\overline{\eta}}\alpha\left(|\mathbf{m}|/\rho\right)\right\}^{1/\rho-1}\right]\right), \end{aligned} \tag{10}$$

where B_1 is a positive real constant. We take

$$N(r) = \left[\rho\,\alpha^{-1}\left\{\overline{\eta}\left(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho-1}\right\}\right],$$

where $[x]$ denotes the integer part of $x \ge 0$. Since $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$, the integer $N(r)$ is well defined. Now if r is sufficiently large, then from (10) we have

$$\begin{aligned} M(r,g) &\le B_1 r^{n_0} + r^{N(r)} \times \\ &\times \textstyle\sum_{n_0+1 \le |\mathbf{m}| \le N(r)} c(n,\mathbf{m}) \exp\left(-\frac{\rho-1}{\rho}|\mathbf{m}|\,\alpha^{-1}\left[\left\{\frac{1}{\overline{\eta}}\alpha\left(|\mathbf{m}|/\rho\right)\right\}^{1/\rho-1}\right]\right) + \\ &+ \textstyle\sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left(-\frac{\rho-1}{\rho}|\mathbf{m}|\,\alpha^{-1}\left[\left\{\frac{1}{\overline{\eta}}\alpha\left(|\mathbf{m}|/\rho\right)\right\}^{1/\rho-1}\right]\right) \end{aligned}$$

or

$$\begin{aligned} M(r,g) &\le B_1 r^{n_0} + r^{N(r)} \times \\ &\times \textstyle\sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m}) \exp\left(-\frac{\rho-1}{\rho}|\mathbf{m}|\,\alpha^{-1}\left[\left\{\frac{1}{\overline{\eta}}\alpha\left(|\mathbf{m}|/\rho\right)\right\}^{1/\rho-1}\right]\right) + \\ &+ \textstyle\sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m}) r^{|\mathbf{m}|} \exp\left(-\frac{\rho-1}{\rho}|\mathbf{m}|\,\alpha^{-1}\left[\left\{\frac{1}{\overline{\eta}}\alpha\left(|\mathbf{m}|/\rho\right)\right\}^{1/\rho-1}\right]\right). \end{aligned} \tag{11}$$

Now the first series in (11) can be rewritten as

$$\sum_{p=1}^{\infty}\left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\right) \exp\left(-\frac{\rho-1}{\rho}p\,\alpha^{-1}\left[\left\{\frac{1}{\overline{\eta}}\alpha\left(p/\rho\right)\right\}^{1/\rho-1}\right]\right). \tag{12}$$

Now from ([2], Lemma 1), we have

$$\lim_{p\to\infty}\sup\left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\right)^{1/p} = n.$$

Hence we have

$$\begin{aligned}&\lim_{p\to\infty}\sup\left[\left(\sum\nolimits_{|\mathbf{m}|=p}c(n,\mathbf{m})\right)\exp\left(-\tfrac{\rho-1}{\rho}p\,\alpha^{-1}\left[\left\{\tfrac{1}{\eta}\alpha\,(p/\rho)\right\}^{1/\rho-1}\right]\right)\right]^{1/p}\\&=n\lim_{p\to\infty}\sup\exp\left(-\tfrac{\rho-1}{\rho}p\,\alpha^{-1}\left[\left\{\tfrac{1}{\eta}\alpha\,(p/\rho)\right\}^{1/\rho-1}\right]\right)=0.\end{aligned}$$

Hence the series (12) converges to a positive real constant B_2. So from (11), we get

$$\begin{aligned}M(r,g)&\leq B_1 r^{n_0}+B_2 r^{N(r)}+\\&+\sum\nolimits_{|\mathbf{m}|>N(r)}c(n,\mathbf{m})\,r^{|\mathbf{m}|}\,\exp\left(-\tfrac{\rho-1}{\rho}|\mathbf{m}|\,\alpha^{-1}\left[\left\{\tfrac{1}{\eta}\alpha\,(|\mathbf{m}|/\rho)\right\}^{1/\rho-1}\right]\right)\end{aligned}$$

or

$$M(r,g)\leq B_1 r^{n_0}+B_2 r^{N(r)}+\sum_{|\mathbf{m}|>N(r)}c(n,\mathbf{m})r^{|\mathbf{m}|}\exp\left[-|\mathbf{m}|\log\{(n+1)r\}\right]$$

or

$$M(r,g)\leq B_1 r^{n_0}+B_2 r^{N(r)}+\sum_{|\mathbf{m}|>N(r)}c(n,\mathbf{m})\left(\frac{1}{n+1}\right)^{|\mathbf{m}|}$$

or

$$M(r,g)\leq B_1 r^{n_0}+B_2 r^{N(r)}+\sum_{|\mathbf{m}|=1}^{\infty}c(n,\mathbf{m})\left(\frac{1}{n+1}\right)^{|\mathbf{m}|}. \tag{13}$$

The series in (13) can we rewritten as

$$\sum_{p=1}^{\infty}\left(\sum_{|\mathbf{m}|=p}c(n,\mathbf{m})\right)\left(\frac{1}{n+1}\right)^{p}. \tag{14}$$

So we have

$$\lim_{p\to\infty}\sup\left[\left(\sum_{|\mathbf{m}|=p}c(n,\mathbf{m})\right)\left(\frac{1}{n+1}\right)^{p}\right]^{1/p}=\frac{n}{n+1}<1.$$

Hence the series (14) converges to a positive real constant B_3. Therefore from (13), we get

$$M(r,g)\leq B_1 r^{n_0}+B_2 r^{N(r)}+B_3.$$

Since $N(r)\to\infty$ as $r\to\infty$ so we can write above inequality as

$$\begin{aligned}\log M(r,g) &\le [1+o(1)]N(r)\log r\\ &\le [1+o(1)]\left[\rho\alpha^{-1}\left\{\overline{\eta}\left(\alpha\left[\tfrac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho-1}\right\}\right]\log r\\ &\le [1+o(1)]\rho\alpha^{-1}\left\{\overline{\eta}\left(\alpha\left[\tfrac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho-1}\right\}\times\\ &\quad\times\alpha^{-1}\left\{\left(\alpha\left[\tfrac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho-1}\right\}\\ &\le [1+o(1)]\rho\alpha^{-1}\left\{(\overline{\eta}+1)\left(\alpha\left[\tfrac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho-1}\right\}\end{aligned}$$

or

$$\alpha[\log M(r,g)] \le (\overline{\eta}+1)\left(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho-1}[1+o(1)]$$

or

$$\frac{\alpha[\log M(r,g)]}{\left(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho-1}} \le (\overline{\eta}+1)\,[1+o(1)]$$

or

$$\frac{\alpha[\log M(r,g)]}{\left(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho}} \le \frac{\alpha[\log M(r,g)]}{\left(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho-1}} \le (\overline{\eta}+1)\,[1+o(1)]$$

or

$$\frac{\alpha[\log M(r,g)]}{[\alpha(\log r)]^{\rho}} \le \left(\frac{\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right]}{[\alpha(\log r)]}\right)^{\rho}(\overline{\eta}+1)\;[1+o(1)]$$

or

$$\frac{\alpha[\log M(r,g)]}{\left(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right]\right)^{\rho}} \le \left(\frac{\alpha\,[O(1)\log r\}]}{[\alpha(\log r)]}\right)^{\rho}(\overline{\eta}+1)\,[1+o(1)].$$

Proceeding to limits as $r\to\infty$ and using properties of $\alpha(x)$, we get

$$\sigma \le \overline{\eta}+1.$$

Since $\varepsilon > 0$ is arbitrarily small, we finally get

$$\sigma - 1 < \eta. \tag{15}$$

Combining (9) and (15), we get (4). Hence Theorem 2.1 is proved. □

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