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# INEQUALITIES OF OSTROWSKI AND SIMPSON TYPE FOR MAPPINGS OF TWO VARIABLES WITH BOUNDED VARIATION AND APPLICATIONS

Y. JAWARNEH AND M.S.M. NOORANI

Abstract. In this paper two inequalities of Ostrowski type and Simpson type for mappings of two independent variables with bounded variation are established. General inequality of trapezoid type is obtained. Applications to some special means and to a quadrature formula are given.

## 1. Introduction

In 2001, Dragomir et al., pointed out an inequality of Ostrowski type for mapping of two independent variables, as follows:

**Theorem 1.** *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be such that the partial derivatives* $\frac{\partial f(t,s)}{\partial t}$, $\frac{\partial f(t,s)}{\partial s}$, $\frac{\partial^2 f(t,s)}{\partial t\partial s}$ *exist and are continuous on* $[a,b]\times[c,d]$. *Then for all* $(x,y)\in[a,b]\times[c,d]$, *we have*

$$\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(t,s)\,dtds\right|\le M_1(x)+M_2(y)+M_3(x,y),\qquad(1)$$

*where,*

$$M_1(x)=\begin{cases}\frac{\left[\frac14(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]}{b-a}\left\|\frac{\partial f}{\partial t}\right\|_\infty, & \frac{\partial f(t,s)}{\partial t}\in L_\infty([a,b]\times[c,d]);\\[2ex] \frac{\left[\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}}}{(b-a)(d-c)^{\frac{1}{p_1}}}\left\|\frac{\partial f}{\partial t}\right\|_{p_1}, & \frac{\partial f(t,s)}{\partial t}\in L_{p_1}([a,b]\times[c,d]),\\ & p_1>1,\frac{1}{p_1}+\frac{1}{q_1}=1;\\[2ex] \frac{\left[\frac12(b-a)+\left|x-\frac{a+b}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f}{\partial t}\right\|_1, & \frac{\partial f(t,s)}{\partial t}\in L_1([a,b]\times[c,d]).\end{cases}$$

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$$M_2(y) = \begin{cases} \frac{\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]}{d-c}\left\|\frac{\partial f}{\partial t}\right\|_\infty, & \frac{\partial f(t,s)}{\partial s} \in L_\infty([a,b]\times[c,d]); \\ \frac{\left[\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right]^{\frac{1}{q_2}}}{(d-c)(b-a)^{\frac{1}{p_2}}}\left\|\frac{\partial f}{\partial s}\right\|_{p_2}, & \frac{\partial f(t,s)}{\partial s} \in L_{p_2}([a,b]\times[c,d]), \\ & p_2>1, \frac{1}{p_2}+\frac{1}{q_2}=1; \\ \frac{\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f}{\partial s}\right\|_1, & \frac{\partial f(t,s)}{\partial s} \in L_1([a,b]\times[c,d]). \end{cases}$$

*and*

$$M_3(x,y) = \begin{cases} \frac{\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\cdot\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty, \\ \qquad \frac{\partial^2 f(t,s)}{\partial t\partial s} \in L_\infty([a,b]\times[c,d]); \\ \frac{\left[\frac{(b-x)^{q_3+1}+(x-a)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}\cdot\left[\frac{(d-y)^{q_3+1}+(y-c)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3} \\ \qquad \frac{\partial^2 f(t,s)}{\partial t\partial s} \in L_{p_3}([a,b]\times[c,d]),\ p_3>1, \frac{1}{p_3}+\frac{1}{q_3}=1; \\ \frac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1, \\ \qquad \frac{\partial^2 f(t,s)}{\partial t\partial s} \in L_1([a,b]\times[c,d]). \end{cases}$$

In a recent years, several authors studied the well–known Ostrowski inequality in one variable for variant types of mappings such as, absolutely continuous, Lipschitzian and $n$-differentiable mappings as well as the mappings of bounded variation. However, a small attention and a few works have been considered for mappings of two variables. Among others, S.S. Dragomir and his group studied a very interesting inequalities for mapping of two independent variables see [2]–[5] and [8]–[20]. For recent, inequalities of Ostrowski type for mappings of one and two variables we refer the reader to the recent study [18] and to the comprehensive book [16].

The aim of this paper is to introduce two inequalities of Ostrowski and Simpson type for mappings of two independent variables with bounded variation. General inequality of trapezoid type is provided. Applications to some special means and to quadrature formula are given.

## 2. Preliminaries and lemmas

In 1910, Fréchet [17] has given the following characterization for the double Riemann–Stieltjes integral. Assume that $f(x,y)$ and $\alpha(x,y)$ are defined over the rectangle

$$Q : (a \le x \le b\ ;\ c \le y \le d);$$

let $R$ be the divided into rectangular subdivisions, or cells, by the net of straight lines $x = x_i$, $y = y_j$,

$$a = x_0 < x_1 < \cdots < x_n = b, \quad and \quad c = y_0 < y_1 < \cdots < y_m = d;$$

let $\zeta_i, \eta_j$ be any numbers satisfying the inequalities $x_{i-1} \le \zeta_i \le x_i$, $y_{j-1} \le \eta_j \le y_j$, $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$; and for all $i, j$ let

$$\Delta_{11}\alpha(x_i, y_j) = \alpha(x_{i-1}, y_{j-1}) - \alpha(x_{i-1}, y_j) - \alpha(x_i, y_{j-1}) + \alpha(x_i, y_j).$$

Then if the sum

$$S=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\zeta_i,\eta_j\right)\Delta_{11}\alpha\left(x_i,y_j\right)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of $f$ with respect to $\alpha$ is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f\left(x,y\right)d_xd_y\alpha\left(x,y\right). \tag{2}$$

If in the above formulation $S$ is replaced by the sum

$$S^*=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\zeta_{ij},\eta_{ij}\right)\Delta_{11}\alpha\left(x_i,y_j\right),$$

where $\zeta_{ij},\eta_{ij}$ are any numbers satisfying the inequalities $x_{i-1}\le\zeta_{ij}\le x_i$, $y_{j-1}\le\eta_{ij}\le y_j$, we call the limit, when it exists, the unrestricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f\left(x,y\right)d_xd_y\alpha\left(x,y\right). \tag{3}$$

The existence of (3) implies both the existence of (2) and its equality to (3). On the other hand, Clarkson [6] has shown that the existence of (2) does not imply the existence of (3) (see [6]).

**Definition 1.** *([7]) Let $f$ be of bounded variation on $Q$. If*

$$P:=\left\{(x_i,y_j):x_{i-1}\le x\le x_i\,;\,y_{j-1}\le y\le y_j\,;\,i=1,\dots,n\,;\,j=1,\dots,m\right\}$$

*is a partition of $Q$, write*

$$\Delta_{11}f\left(x_i,y_j\right)=f\left(x_{i-1},y_{j-1}\right)-f\left(x_{i-1},y_j\right)-f\left(x_i,y_{j-1}\right)+f\left(x_i,y_j\right)$$

*for $i=1,2,\cdots,n$ and $j=1,2,\cdots,m$. The function $f(x,y)$ is said to be of* bounded variation in the Vitali sense *if there exists a positive quantity $M$ such that for every partition on $Q$ we have $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left|\Delta_{11}f\left(x_i,y_j\right)\right|\le M$.*

Therefore, one can define the concept of total variation of a function of two variables, as follows:

Let $f$ be of bounded variation on $Q$, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left|\Delta_{11}f\left(x_i,y_j\right)\right|$ corresponding to the partition $P$ of $Q$. The number

$$\bigvee_Q(f):=\bigvee_c^d\bigvee_a^b(f):=\sup\left\{\sum P:P\in\mathcal{P}\left(Q\right)\right\},$$

is called the total variation of $f$ on $Q$.

The aim of this paper is to deduce some basic theory of double Riemann–Stieltjes integral and then use it to prove inequalities of Ostrowski and Simpson type for mappings of bounded variation in two independent variables.

For $a,b,c,d\in\mathbb{R}$, we consider the subset $Q=\{(x,y):a\le x\le b,c\le y\le d\}$ of $\mathbb{R}^2$.

**Lemma 1.** *(Integration by parts) If $f \in \mathcal{RS}(\alpha)$ on $Q$, then $\alpha \in \mathcal{RS}(f)$ on $Q$, and we have*

$$\int_c^d \int_a^b f(t,s)\, d_t d_s \alpha(t,s) + \int_c^d \int_a^b \alpha(t,s)\, d_t d_s f(t,s)$$
$$= f(b,d)\,\alpha(b,d) - f(b,c)\,\alpha(b,c) - f(a,d)\,\alpha(a,d) + f(a,c)\,\alpha(a,c). \tag{4}$$

*Proof.* Let $\epsilon > 0$ be given. Since $\int_c^d \int_a^b f(t,s)\, d_t d_s \alpha(t,s)$ exists, there is a partition $P_\epsilon$ of $Q$ such that for every $P'$ finer than $P_\epsilon$, we have

$$\left| S(P', f, \alpha) - \int_c^d \int_a^b f(t,s)\, d_t d_s \alpha(t,s) \right| < \epsilon. \tag{5}$$

Consider an arbitrary Riemann–Stieltjes sum for the integral $\alpha(t,s)\, d_t d_s f(t,s)$, say

$$\begin{aligned} S(P,f,\alpha) &= \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i,s_j)\, \Delta_{11} f(x_i,y_j) \\ &= \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i,s_j)\, f(x_{i-1},y_{j-1}) - \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i,s_j)\, f(x_{i-1},y_j) \\ &\quad - \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i,s_j)\, f(x_i,y_{j-1}) + \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i,s_j)\, f(x_i,y_j), \end{aligned}$$

where $P$ finer than $P_\epsilon$. Writing

$$A = f(b,d)\,\alpha(b,d) - f(b,c)\,\alpha(b,c) - f(a,d)\,\alpha(a,d) + f(a,c)\,\alpha(a,c),$$

we have the identity

$$\begin{aligned} A &= \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{i-1},y_{j-1})\,\alpha(x_{i-1},y_{j-1}) - \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{i-1},y_j)\,\alpha(x_{i-1},y_j) \\ &\quad - \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i,y_{j-1})\,\alpha(x_i,y_{j-1}) + \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i,y_j)\,\alpha(x_i,y_j). \end{aligned}$$

Subtracting the last two displayed equations, we find

$$\begin{aligned} A - S(P,f,\alpha) &= \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{i-1},y_{j-1}) \left[\alpha(x_{i-1},y_{j-1}) - \alpha(t_i,s_j)\right] \\ &\quad + \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{i-1},y_j) \left[\alpha(t_i,s_j) - \alpha(x_{i-1},y_j)\right] \\ &\quad + \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i,y_{j-1}) \left[\alpha(t_i,s_j) - \alpha(x_i,y_{j-1})\right] \\ &\quad + \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i,y_j) \left[\alpha(x_i,y_j) - \alpha(t_i,s_j)\right]. \end{aligned}$$

The sums on the right can be combined into a single sum of the form $S(P', f, \alpha)$, where $P'$ is that partition of $Q$ obtained by taking the points $(t_i, s_j)$, $(x_i, y_j)$ together. Then $P'$ is finer than $P$ and hence finer than $P_\epsilon$. Therefore the inequality (5) is valid and this means that we have

$$\left| A - S(P, f, \alpha) - \int_c^d \int_a^b f(t, s)\, d_t d_s \alpha(t, s) \right| < \epsilon,$$

whenever $P$ is finer than $P_\epsilon$. But this is exactly the statement $\int_c^d \int_a^b \alpha(t, s)\, d_t d_s f(t, s)$ exists and equals $A - \int_c^d \int_a^b f(t, s)\, d_t d_s \alpha(t, s)$. □

**Lemma 2.** *If $f$ is continuous on $Q$ and if $\alpha$ is of bounded variation on $Q$, then $f \in \mathcal{RS}(\alpha)$.*

*Proof.* First of all, we note that, by Lemma 1, a second sufficient condition can be obtained by interchanging $f$ and $\alpha$ in the hypothesis. It suffices to prove the theorem when $\alpha$ is monotonically increasing with $\alpha(a, \cdot) \le \alpha(b, \cdot)$, $\alpha(\cdot, c) \le \alpha(\cdot, d)$ and $\alpha(t, s) \le \alpha(x, y)$, for all $t < x$ and $s < y$ in $Q$. Since $f$ is continuous on $Q$ then $f$ is uniformly continuous on $Q$, i.e., $\forall \epsilon > 0$ there exists $\delta > 0$, such that

$$|f(x, y) - f(t, s)| < \frac{\epsilon}{A} \qquad \textit{whenever} \qquad \|(x - t, y - s)\| < \delta,$$

where $A = 4[\alpha(b, d) - \alpha(b, c) - \alpha(a, d) + \alpha(a, c)]$. If $P_\epsilon$ is a partition of $Q$ with $\|P_\epsilon\| < \delta$, then for $P$ finer than $P_\epsilon$ we must have $M_{ij}(f) - m_{ij}(f) \le \epsilon/A$, where

$$M_{ij}(f) - m_{ij}(f) = \sup\{f(x, y) - f(x, s) - f(t, y) + f(t, s) : (x, y), (t, s) \in Q\}.$$

Multiplying the inequality by $\Delta\alpha_{11}$ and summing, we find

$$U(P, f, \alpha) - L(P, f, \alpha) \le \frac{\epsilon}{A} \sum_{j=1}^{m} \sum_{i=1}^{n} \Delta_{11}\alpha = \frac{\epsilon}{4} < \epsilon.$$

Hence, $f \in \mathcal{RS}(\alpha)$ on $Q$. □

**Lemma 3.** *Assume that $g \in \mathcal{RS}(\alpha)$ on $Q$ and $\alpha$ is of bounded variation on $Q$, then*

$$\left| \int_c^d \int_a^b g(x, y)\, d_x d_y \alpha(x, y) \right| \le \sup_{(x,y) \in Q} |g(x, y)| \cdot \bigvee_Q (\alpha). \tag{6}$$

*Proof.* The existence of $\int_c^d \int_a^b g(x, y)\, d_x d_y \alpha(x, y)$ follows from Lemma 2. Let $\Delta_n := a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta_m := c = y_0 < y_1 < \cdots < y_m = d$ be a partitions of $[a, b]$ and $[c, d]$; respectively. Let

$$\Delta_{n,m} := \Delta_n \times \Delta_m =$$
$$\{\ (x_0, y_0), \cdots, (x_0, y_m), (x_1, y_0), \cdots, (x_1, y_m), \cdots, (x_n, y_0), \cdots, (x_n, y_m)\ \}$$

be a partition of $Q$ and $l(\Delta_{n,m}) := \max_{i,j}\{x_{i+1}-x_i, y_{j+1}-y_j\}$ be the length of $Q$, therefore,

$$\begin{aligned}\left|\int_c^d\int_a^b g(x,y)\,d_xd_y\alpha(x,y)\right| &= \left|\lim_{l(\Delta_{n,m})\to 0}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1} g\left(\xi_i^{(n)},\eta_j^{(m)}\right)\Delta_{11}\alpha(x_i,y_j)\right| \\ &\le \lim_{l(\Delta_{n,m})\to 0}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left|g\left(\xi_i^{(n)},\eta_j^{(m)}\right)\right||\Delta_{11}\alpha(x_i,y_j)| \\ &\le \sup_{(x,y)\in Q}|g(x,y)|\cdot\sup_{\Delta_{n,m}}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}|\Delta_{11}\alpha(x_i,y_j)| \\ &= \sup_{(x,y)\in Q}|g(x,y)|\cdot\bigvee_Q(\alpha),\end{aligned}$$

which is required. □

## 3. Ostrowski inequality for mappings bounded variation

**Theorem 2.** *Let $f : Q \to \mathbb{R}$ be a mapping of bounded variation on $Q$. Then for all $(x,y)\in Q$, we have the inequality*

$$\begin{aligned}&\left|(b-a)(d-c)f(x,y) - \int_c^d\int_a^b f(t,s)\,dtds\right| \\ &\qquad\le \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]\cdot\bigvee_Q(f),\end{aligned} \tag{7}$$

*where $\bigvee_Q(f)$ denotes the total (double) variation of $f$ on $Q$.*

*Proof.* From Lemma 1, we have

$$\int_c^y\int_a^x (t-a)(s-c)\,d_td_sf(t,s) = (x-a)(y-c)f(x,y) - \int_c^y\int_a^x f(t,s)\,dtds,$$

$$\int_y^d\int_a^x (t-a)(s-d)\,d_td_sf(t,s) = (x-a)(d-y)f(x,y) - \int_y^d\int_a^x f(t,s)\,dtds,$$

$$\int_c^y\int_x^b (t-b)(s-c)\,d_td_sf(t,s) = (b-x)(y-c)f(x,y) - \int_c^y\int_x^b f(t,s)\,dtds,$$

and

$$\int_y^d\int_x^b (t-b)(s-d)\,d_td_sf(t,s) = (x-b)(y-d)f(x,y) - \int_y^d\int_x^b f(t,s)\,dtds.$$

Adding the above equalities, we get

$$\int_c^d\int_a^b P(x,t;y,s)\,d_td_sf(t,s) = (b-a)(d-c)f(x,y) - \int_c^d\int_a^b f(t,s)\,dtds$$

where,

$$P(x,t;y,s)=\begin{cases}(t-a)(s-c), & (x,y)\in[a,x]\times[c,y]\\ (t-a)(s-d), & (x,y)\in[a,x]\times(y,d]\\ (t-b)(s-c), & (x,y)\in(x,b]\times[c,y]\\ (t-b)(s-d), & (x,y)\in(x,b]\times(y,d]\end{cases}$$

for all $(t,s)\in Q$.

Now, applying Lemma 3, by letting $g=P$ and $\alpha=f$, we get

$$\begin{aligned}&\left|\int_c^d\int_a^b P(x,t;y,s)\,d_td_sf(t,s)\right|\\ &\le \sup_{(x,y)\in Q}|P(x,t;y,s)|\cdot\bigvee_Q(f)\\ &=\max_{x,y}\{(x-a)(y-c),(x-a)(d-y),(b-x)(y-c),(b-x)(d-y)\}\cdot\bigvee_Q(f),\end{aligned}$$

but,

$$\begin{aligned}M&=\max_{x,y}\{(x-a)(y-c),(x-a)(d-y),(b-x)(y-c),(b-x)(d-y)\}\\ &=\max_x\left\{\max_y\{(x-a)(y-c),(x-a)(d-y),(b-x)(y-c),(b-x)(d-y)\}\right\},\end{aligned}$$

and since $\max\limits_y$ is independent of $x$, we have

$$\begin{aligned}M&=\max_x\left\{(x-a)\cdot\max_y\{(y-c),(d-y)\},(b-x)\cdot\max_y\{(y-c),(d-y)\}\right\}\\ &\le\max_x\{(x-a),(b-x)\}\cdot\max_y\{(y-c),(d-y)\}\\ &=\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right],\end{aligned}$$

it follows that,

$$\left|\int_c^d\int_a^b P(x,t;y,s)\,d_td_sf(t,s)\right|\le\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]\cdot\bigvee_Q(f),$$

which completes the proof. □

**Corollary 1.** *In Theorem 2. Let* $x=\frac{a+b}{2}$ *and* $y=\frac{c+d}{2}$, *then we have*

$$\left|(b-a)(d-c)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_c^d\int_a^b f(t,s)\,dtds\right|\le\frac{(b-a)(d-c)}{4}\cdot\bigvee_Q(f),$$

**Remark 1.** *Similar inequalities can be found if we assume that u is monotonous on Q, we left the details to the interested reader.*

**Corollary 2.** *In Theorem 2. Assume* $[a,b]=[c,d]$*, we get*

$$\left|(b-a)^2 f(x,y)-\int_a^b\int_a^b f(t,s)\,dtds\right|\leqslant$$
$$\leqslant\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{b-a}{2}+\left|y-\frac{a+b}{2}\right|\right]\cdot\bigvee_Q(f).$$

A trapezoid inequality for mappings of two variables may be stated as follows:

**Theorem 3.** *Let* $f:Q\to\mathbb{R}$ *be a mapping of bounded variation on* $Q$*. Then for all* $(x,y)\in Q$*, we have the inequality*

$$\left|\frac{(b-a)(d-c)}{4}\cdot[f(b,d)-f(b,c)-f(a,d)+f(a,c)]-\int_c^d\int_a^b f(t,s)\,dtds\right|$$
$$\leq\frac{(b-a)(d-c)}{4}\cdot\bigvee_Q(f),\quad(8)$$

*The constant* $\frac{1}{4}$ *is best possible value.*

*Proof.* From Lemma 1, we have

$$\int_c^d\int_a^b R(t,s)\,d_t d_s f(t,s)$$
$$=\frac{(b-a)(d-c)}{4}\cdot[f(b,d)-f(b,c)-f(a,d)+f(a,c)]-\int_c^d\int_a^b f(t,s)\,dtds,$$

where, $R(t,s)=\left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right),\quad a\leq t\leq b\ ;\ c\leq s\leq d.$

Now, applying Lemma 3, by letting $g=R$ and $\alpha=f$, we get

$$\left|\int_c^d\int_a^b R(t,s)\,d_t d_s f(t,s)\right|\leq\sup_{(t,s)\in Q}|R(t,s)|\cdot\bigvee_c^d\bigvee_a^b(f)=\frac{(b-a)(d-c)}{4}\cdot\bigvee_Q(f),$$

which is required. □

The following theorem generalize the inequality (8).

**Theorem 4.** *Let* $f:Q\to\mathbb{R}$ *be a mapping of bounded variation on* $Q$*. Then for all* $(x,y)\in Q$*, we have the inequality*

$$|f(b,d)\alpha(b,d)-f(b,c)\alpha(b,c)-f(a,d)\alpha(a,d)+f(a,c)\alpha(a,c)$$
$$\left.-\int_c^d\int_a^b f(t,s)\,d_t d_s\alpha(t,s)\right|$$
$$\leq\sup_{(t,s)\in Q}|\alpha(t,s)|\cdot\bigvee_Q(f).\quad(9)$$

*Proof.* From Lemma 1, we have

$$\int_c^d\int_a^b \alpha(t,s)\,d_t d_s f(t,s)=$$
$$=f(b,d)\alpha(b,d)-f(b,c)\alpha(b,c)-f(a,d)\alpha(a,d)+f(a,c)\alpha(a,c)-$$
$$-\int_c^d\int_a^b f(t,s)\,d_t d_s\alpha(t,s).$$

Now, applying Lemma 3, by letting $g = R$ and $\alpha = f$, we get

$$\left| \int_c^d \int_a^b \alpha(t,s)\, d_t d_s f(t,s) \right| \le \sup_{(t,s)\in Q} |\alpha(t,s)| \cdot \bigvee_Q (f),$$

which is required. □

## 4. Simpson's inequality for mappings of bounded variation

In [22], L. Zhongxue proved the following Simpson type inequality for mappings of two independent variables:

**Theorem 5.** *Let $f : [a,c] \times [b,d] \to R$ be an absolutely continuous function, whose partial derivative of order 2 is $f'' \in L^2([a,c] \times [c,d])$. Then*

$$\begin{aligned} &\Bigg| \frac{f\left(a, \frac{b+d}{2}\right) + f\left(\frac{a+c}{2}, b\right) + f\left(\frac{a+c}{2}, d\right) + f\left(c, \frac{b+d}{2}\right) + 4f\left(\frac{a+c}{2}, \frac{b+d}{2}\right)}{9} \\ &\quad + \frac{f(a,b) + f(a,d) + f(c,b) + f(c,d)}{36} - \frac{\int_a^c \left[f(s,b) + 4f\left(s, \frac{b+d}{2}\right) + f(s,d)\right] ds}{6(c-a)} \\ &\quad - \frac{\int_b^d \left[f(a,t) + 4f\left(\frac{a+c}{2}, t\right) + f(c,t)\right] dt}{6(d-b)} + \frac{\int_a^c \int_b^d f(s,t)\, ds dt}{(c-a)(d-b)} \Bigg| \\ &\le \frac{[(c-a)(d-b)]^{1/2}}{36} \sqrt{\sigma(f'')}, \end{aligned} \tag{10}$$

*where $\sigma(\cdot)$ is defined by*

$$\sigma(f) = \|f\|_2^2 - \frac{1}{(c-a)(d-b)} \left( \int_a^c \int_b^d f(s,t)\, ds dt \right)^2,$$

*and*

$$\|f\|_2^2 = \left( \int_a^c \int_b^d |f(s,t)|^2\, ds dt \right)^{1/2}.$$

*The inequality (10) is sharp in the sense that the constant $1/36$ cannot be replaced by a smaller one.*

A generalization of (10) is considered recently by Z. Liu [21].

In the following we obtain an inequality of Simpson type for mappings of bounded variation:

**Theorem 6.** *Let $f : Q \to \mathbb{R}$ be a mapping of bounded variation on $Q$. Then for all $(x, y) \in Q$, we have the inequality*

$$\left| \frac{(b-a)(d-c)}{36}\left[f(b,d) - f(b,c) - f(a,d) + f(a,c)\right] + \frac{(b-a)(d-c)}{9}\left[f\left(\frac{a+b}{2}, d\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - f\left(\frac{a+b}{2}, c\right) - f\left(a, \frac{c+d}{2}\right)\right] - \int_a^b \int_c^d f(s,t)\, dtds \right| \le \frac{(b-a)(d-c)}{9} \cdot \bigvee_Q (f), \quad (11)$$

*where $\bigvee_Q (f)$ denotes the total (double) variation of $f$ on $Q$.*

*Proof.* From Lemma 1, we have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left(s - \frac{5a+b}{6}\right)\left(t - \frac{5c+d}{6}\right) d_t d_s f(t,s) \\
&= \frac{(b-a)(d-c)}{9} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad - \frac{(b-a)(d-c)}{18}\left[f\left(\frac{a+b}{2}, c\right) + f\left(a, \frac{c+d}{2}\right)\right] \\
&\quad + \frac{(b-a)(d-c)}{36} f(a,c) - \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(s,t)\, dtds
\end{aligned}$$

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^{d} \left(s - \frac{5a+c}{6}\right)\left(t - \frac{c+5d}{6}\right) d_t d_s f(t,s) \\
&= \frac{(b-a)(d-c)}{9} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad - \frac{(b-a)(d-c)}{18}\left[f\left(a, \frac{c+d}{2}\right) - f\left(\frac{a+b}{2}, d\right)\right] \\
&\quad - \frac{(b-a)(d-c)}{36} f(a,d) - \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^{d} f(s,t)\, dtds
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{b} \int_c^{\frac{c+d}{2}} \left(s - \frac{a+5b}{6}\right)\left(t - \frac{5c+d}{6}\right) d_t d_s f(t,s) \\
&= \frac{(b-a)(d-c)}{9} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad + \frac{(b-a)(d-c)}{18}\left[f\left(b, \frac{c+d}{2}\right) - f\left(\frac{a+b}{2}, c\right)\right] \\
&\quad - \frac{(b-a)(d-c)}{36} f(b,c) - \int_{\frac{a+b}{2}}^{b} \int_c^{\frac{c+d}{2}} f(s,t)\, dtds
\end{aligned}$$

and

$$\begin{aligned}&\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left(s-\frac{a+5b}{6}\right)\left(t-\frac{c+5d}{6}\right)d_t d_s f(t,s)\\&=\frac{(b-a)(d-c)}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\quad+\frac{(b-a)(d-c)}{18}\left[f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)\right]\\&\quad+\frac{(b-a)(d-c)}{36}f(b,d)-\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}f(s,t)\,dtds\end{aligned}$$

Adding the above equalities, we get

$$\begin{aligned}&\int_a^b\int_c^d K(s,t)\,d_t d_s f(s,t)=\frac{(b-a)(d-c)}{36}\left[f(b,d)-f(b,c)-f(a,d)+f(a,c)\right]+\\&+\frac{(b-a)(d-c)}{9}\left[f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\right.\\&\left.-f\left(\frac{a+b}{2},c\right)-f\left(a,\frac{c+d}{2}\right)\right]-\int_a^b\int_c^d f(s,t)\,dtds\end{aligned}$$

where,

$$K(s,t)=\begin{cases}\left(s-\frac{5a+b}{6}\right)\left(t-\frac{5c+d}{6}\right), & a\le s\le\frac{a+b}{2}, c\le t\le\frac{c+d}{2},\\[2mm]\left(s-\frac{5a+b}{6}\right)\left(t-\frac{c+5d}{6}\right), & a\le s\le\frac{a+b}{2}, \frac{c+d}{2}<t\le d,\\[2mm]\left(s-\frac{a+5b}{6}\right)\left(t-\frac{5c+d}{6}\right), & \frac{a+b}{2}<s\le b, c\le t\le\frac{c+d}{2},\\[2mm]\left(s-\frac{a+5b}{6}\right)\left(t-\frac{c+5d}{6}\right), & \frac{a+b}{2}<s\le b, \frac{c+d}{2}<t\le d.\end{cases}$$

for all $(t,s)\in Q$.

Now, applying Lemma 3, by letting $g=P$ and $\alpha=f$, we get

$$\left|\int_c^d\int_a^b K(s,t)\,d_t d_s f(t,s)\right|\le\sup_{(x,y)\in Q}|K(s,t)|\cdot\bigvee_c^d\bigvee_a^b(f)=\frac{(b-a)(d-c)}{9}\cdot\bigvee_c^d\bigvee_a^b(f),$$

which completes the proof. □

## 5. Applications to special means

A function $M:\mathbb{R}_+^2\to\mathbb{R}_+$, is called a Mean function if it has the following properties [4]:

(1) Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
(2) Symmetry : $M(x,y)=M(y,x)$,
(3) Reflexivity : $M(x,x)=x$,
(4) Monotonicity: If $x\le x'$ and $y\le y'$, then $M(x,y)\le M(x',y')$,
(5) Internality: $\min\{x,y\}\le M(x,y)\le\max\{x,y\}$.

We shall consider some means for arbitrary positive real numbers $\alpha,\beta$ $(\alpha\neq\beta)$ [4].

(1) The arithmetic mean :

$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}.$$

(2) The geometric mean :

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

(3) The harmonic mean :

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

(4) The power mean :

$$P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

(5) The identric mean:

$$I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta. \end{cases}$$

(6) The logarithmic mean :

$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \ |\alpha| \neq |\beta|.$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}, \ p \in \mathbb{R}\backslash\{-1, 0\}.$$

It is well known that $L_p$ is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Let $Q_+ := [a, b] \times [c, d]$, where $a, b, c, d$ are positive real numbers such that $a < b$ and $c < d$.

**Proposition 1.** *Let $M(t, s)$ be a mean function defined on $Q_+$. In Theorem 2 set $M(t, s) = f(t, s)$, $t, s \in Q_+$, then we have*

$$\left|(b-a)(d-c)M(x, y) - \int_c^d \int_a^b M(t, s)\, dtds\right| \leqslant$$
$$\leqslant \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \cdot \left[\frac{d-c}{2} + \left|y - \frac{c+d}{2}\right|\right] \cdot \bigvee_c^d \bigvee_a^b (M),$$

*where, $M(\cdot, \cdot)$ is one of the above mentioned means and we shall left the details to the interested reader.*

A general result may be obtained by using Theorem 4, as follows:

**Proposition 2.** *Let $N : Q_+ \to \mathbb{R}$ be a continuous mean function on $Q_+$ and $M(t, s)$ be a mean function which is of bounded variation on $Q_+$, then we have*

$$|M(b, d)N(b, d) - M(b, c)N(b, c) - M(a, d)N(a, d) + M(a, c)N(a, c) -$$
$$\left.\int_c^d \int_a^b M(t, s)\, d_t d_s N(t, s)\right| \leqslant \sup_{(t,s)\in Q} |N(t, s)| \cdot \bigvee_c^d \bigvee_a^b (M).$$

## 6. Applications to quadrature formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and $J_m : c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$, where $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$ and $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, 1, \cdots, m-1)$ are intermediate points. Consider the Riemann sum

$$R(f, I_n, J_m, \xi, \eta) = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (x_{i+1} - x_i)(y_{j+1} - y_j) f(\xi_i, \eta_j) \tag{12}$$

Using Theorem 2, we can state the following theorem

**Theorem 7.** *Let $f$ as in Theorem 2. Then we have*

$$\int_c^d \int_a^b f(t, s)\, dtds = R(f, I_n, J_m, \xi, \eta) + E(f, I_n, J_m, \xi, \eta), \tag{13}$$

*where $R(f, I_n, J_m, \xi, \eta)$ is the Riemann sum defined in (12) and the remainder the through the approximation $E(f, I_n, J_m, \xi, \eta)$ satisfies the bound*

$$E(f, I_n, J_m, \xi, \eta) \leqslant$$
$$\leqslant \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{x_{i+1} - x_i}{2} + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \cdot \left[\frac{y_{j+1} - y_j}{2} + \left|\eta_j - \frac{y_j + y_{j+1}}{2}\right|\right] \cdot \bigvee_{y_j}^{y_{j+1}} \bigvee_{x_i}^{x_{i+1}} (f).$$

*Proof.* Applying Theorem 2 on the bidimentional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we get the required result. □

Similarly, we can give the following estimation for the Simpson's rule for mappings of bounded variation in two independent variables:

**Theorem 8.** *Let $f$ as in Theorem 6. Then we have*

$$\int_c^d \int_a^b f(t, s)\, dtds = R_S(f, I_n, J_m, \xi, \eta) + E_S(f, I_n, J_m, \xi, \eta), \tag{14}$$

*where $R_S(f, I_n, J_m, \xi, \eta)$ is the Riemann sum defined such as*

$$R_S(f, I_n, J_m, \xi, \eta) =$$
$$\frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{36} [f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j) +$$
$$+4f\left(\frac{x_i + x_{i+1}}{2}, y_{j+1}\right) + 4f\left(x_{i+1}, \frac{y_j + y_{j+1}}{2}\right) - 4f\left(\frac{x_i + x_{i+1}}{2}, y_j\right) +$$
$$+4f\left(x_i, \frac{y_j + y_{j+1}}{2}\right) + 16f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right) - \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(t, s)\, dtds,$$

*and the remainder the through the approximation $E_S(f, I_n, J_m, \xi, \eta)$ satisfies the bound*

$$|E_S(f, I_n, J_m, \xi, \eta)| \le \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{9} \cdot \bigvee_{y_j}^{y_{j+1}} \bigvee_{x_i}^{x_{i+1}} (f) \tag{15}$$

*Proof.* Applying Theorem 6 on the bidimentional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we get the required result. □

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Department of Mathematics
School of Mathematical Sciences
Faculty of Science and Technology
43600 Bangi, Selangor, Malaysia
*E-mail address*: `y.jawarneh82@gmail.com`

Department of Mathematics
School of Mathematical Sciences
Faculty of Science and Technology
43600 Bangi, Selangor, Malaysia
*E-mail address*: `msm@ukm.my`