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ON SOME NEW FEJÉR-TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper some new mappings associated with the Fejér inequality for double integrals are defined and as a consequence some new Fejér type inequalities for co-ordinated convex functions are established as well.

1. Introduction

It is well known in literature that a function $f : [a, b] \to \mathbb{R}$ is convex on $[a, b]$ if the inequality:

$$f(\lambda x + (1-\lambda)\, y) \leq \lambda f(x) + (1-\lambda)\, f(y),$$

holds for all $x,\ y \in [a, b]$ and $\lambda \in [0, 1]$.

Many inequalities have been established for convex functions in past few years but the most famous is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications [7, 8]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The following inequality gives the weighted version of (1), called Fejér's inequality [5]:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \leq \frac{1}{b-a}\int_a^b f(x)p(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)dx, \tag{2}$$

for convex function $f : [a, b] \to \mathbb{R}$ and $p : [a, b] \to [0, \infty)$ an integrable and symmetric about $x = \frac{a+b}{2}$.

The inequalities (1) and (2) have been generalized, extended and refined in a number of ways (for instance see [2, 3, 4, 6, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein).

Consider now a bi-dimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if

$$f(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)w) \leq \alpha f(x, y) + (1-\alpha)f(z, w),$$

holds for all $(x, y), (z, w) \in \Delta$ and $\alpha \in [0, 1]$.

In [2, 4], S. S. Dragomir introduced a new concept of convexity, called the co-ordinated convexity as:

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial functions $f_y : [a, b] \to \mathbb{R}$ and $f_x : [c, d] \to \mathbb{R}$ defined by $f_y(u) = f(u, y)$ and $f_x(v) = f(x, v)$ are convex for $(x, y) \in [a, b] \times [c, d]$.

A formal definition for co-ordinated convex functions may be stated in

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Definition 1. *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if*

$$\begin{aligned} &f(tx + (1-t)y, su + (1-s)w) \\ &\leq tsf(x,u) + t(1-s)f(x,w) + s(1-t)f(y,u) + (1-t)(1-s)f(y,w) \end{aligned}$$

holds for all $t, s \in [0,1]$ and $(x,u), (y,w) \in \Delta$.

Clearly, every convex function $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists a co-ordinated convex function which is not convex [2, 4].

In [4], an inequality of Hermite-Hadamard type for co-ordinated convex functions on a rectangle from the plane was established and some properties of functions associated to it were also proved. In [16], D. Y. Hwang et al. considered a monotonic nondecreasing function connected with Hadamard type inequalities in two variables and established some Hadamard type inequalities for Lipschizian function as well. Recently M. Alomari et al. [1], proved a Fejér inequality for double integrals and considered some functions associated to it to establish some inequalities for Lipschitzian functions.

The aim of this paper is to establish some new Fejér-type inequalities for co-ordinated convex functions on rectangle from the plane.

2. Main results

Throughout the section let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a co-ordinated convex function, $p : [a,b] \times [c,d] \to [0,\infty)$ be integrable and symmetric about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. We now define the following functions on $[0,1]^2$ associated with Fejér inequality for double integrals proved in [1]:

$$\begin{aligned} G(t,s) = \frac{1}{4}\Bigg[&f\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) + \\ &f\left(ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right) \\ &+ f\left(tb + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) \\ &+ f\left(tb + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right)\Bigg], \end{aligned}$$

$$H(t,s) = \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dydx,$$

$$H_p(t,s) = \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) p(x,y)dydx,$$

$$\begin{aligned} L(t,s) = \frac{1}{4(b-a)(d-c)} \int_a^b \int_c^d &[f(ta + (1-t)x, sc + (1-s)y) \\ &+ f(ta + (1-t)x, sd + (1-s)y) + f(tb + (1-t)x, sc + (1-s)y) \\ &+ f(tb + (1-t)x, sd + (1-s)y)], \end{aligned}$$

and

$$\begin{aligned} L_p(t,s) = \frac{1}{4}\int_a^b \int_c^d &[f(ta + (1-t)x, sc + (1-s)y) \\ &+ f(ta + (1-t)x, sd + (1-s)y) + f(tb + (1-t)x, sc + (1-s)y) \\ &+ f(tb + (1-t)x, sd + (1-s)y)]\, p(x,y)dydx. \end{aligned}$$

To prove our results we need:

Lemma 1. [1] *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a co-ordinated convex function and let*

$$a \leq y_1 \leq x_1 \leq x_2 \leq y_2 \leq b \text{ with } x_1 + x_2 = y_1 + y_2,$$

$$a \leq w_1 \leq v_1 \leq v_2 \leq w_2 \leq b \text{ with } v_1 + v_2 = w_1 + w_2.$$

Then, for the convex partial functions $f_y : [a,b] \to \mathbb{R}$ and $f_x : [c,d] \to \mathbb{R}$ defined by $f_y(u) = f(u,y)$ and $f_x(v) = f(x,v)$ for $(x,y) \in [a,b] \times [c,d]$, the followings hold:

$$f(x_1, v) + f(x_2, v) \leq f(y_1, v) + f(y_2, v), \text{ for all } v \in [c,d] \tag{3}$$

and

$$f(u, v_1) + f(u, v_2) \leq f(u, w_1) + f(u, w_2), \text{ for all } u \in [a,b]. \tag{4}$$

We begin with following results:

Theorem 1. *Let f, p, H_p be defined as above, then*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y)dydx \\
& \leq 4 \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \int_{\frac{3c+d}{4}}^{\frac{c+3d}{4}} f(x,y) p\left(2x - \frac{a+b}{2}, 2y - \frac{c+d}{2}\right) dydx \leq \int_0^1 \int_0^1 H_p(t,s) dsdt \\
& \qquad \leq \frac{1}{4} \left[f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y)dydx + \right. \\
& \left. \int_a^b \int_c^d \left\{ f(x,y) + f\left(\frac{a+b}{2}, y\right) + f\left(x, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right\} p(x,y)dydx \right].
\end{aligned} \tag{5}$$

Proof. By using the simple techniques of integration, under the assumptions on p, the following identities hold:

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y)dydx \\
& \qquad = 16 \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) p(x,y) dsdtdydx,
\end{aligned} \tag{6}$$

$$\begin{aligned}
& \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \int_{\frac{3c+d}{4}}^{\frac{c+3d}{4}} f(x,y) p\left(2x - \frac{a+b}{2}, 2y - \frac{c+d}{2}\right) dydx \\
& \quad = \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left[f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{y}{2} + \frac{c+d}{4}\right) \right. \\
& + f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{3(c+d)}{4} - \frac{y}{2}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}, \frac{y}{2} + \frac{c+d}{4}\right) \\
& \qquad \left. + f\left(\frac{3(a+b)}{4} - \frac{x}{2}, \frac{3(c+d)}{4} - \frac{y}{2}\right) \right] p(x,y) dsdtdydx,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\int_0^1\int_0^1 H_p(s,t)dsdt \\
&= \int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f\left(t\frac{a+b}{2}+(1-t)x, s\frac{c+d}{2}+(1-s)y\right)\right. \\
&\quad + f\left(t\frac{a+b}{2}+(1-t)x, sy+(1-s)\frac{c+d}{2}\right) \\
&\quad + f\left(tx+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)y\right) \\
&\quad \left. +f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dsdtdydx \\
&+ \int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)(c+d-y)\right)\right. \\
&\quad + f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\
&\quad + f\left(t\frac{a+b}{2}+(1-t)(a+b-x), s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\
&\quad f\left(t\frac{a+b}{2}+(1-t)(a+b-x), s\frac{c+d}{2}+(1-s)(c+d-y)\right) \\
&+ \int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[\left(t\frac{a+b}{2}+(1-t)x, s\frac{c+d}{2}+(1-s)(c+d-y)\right)\right. \\
&\quad + f\left(t\frac{a+b}{2}+(1-t)x, s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\
&\quad + f\left(tx+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)(c+d-y)\right) \\
&\quad \left. +f\left(tx+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dsdtdydx \\
&+ \int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right. \\
&\quad + f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)y\right) \\
&\quad + f\left(t\frac{a+b}{2}+(1-t)(a+b-x), sy+(1-s)\frac{c+d}{2}\right) \\
&\quad \left. +f\left(t\frac{a+b}{2}+(1-t)(a+b-x), s\frac{c+d}{2}+(1-s)y\right)\right]p(x,y)dsdtdydx \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{4}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx+\int_a^b\int_c^d\left\{f(x,y)+f\left(\frac{a+b}{2},y\right)\right.\right.\\
&\qquad\left.\left.+f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}p(x,y)dydx\right]\\
&=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left\{f(x,y)+f\left(\frac{a+b}{2},y\right)+f\left(x,\frac{c+d}{2}\right)\right.\\
&\left.+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}p(x,y)dsdtdydx+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left\{f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\qquad+f(a+b-x,c+d-y)+f\left(\frac{a+b}{2},c+d-y\right)\\
&\qquad\left.+f\left(a+b-x,\frac{c+d}{2}\right)\right\}p(x,y)dsdtdydx\\
&+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left\{f(x,c+d-y)+f\left(\frac{a+b}{2},c+d-y\right)+f\left(x,\frac{c+d}{2}\right)\right.\\
&\left.+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}p(x,y)dsdtdydx+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\{f(a+b-x,y)\\
&f\left(\frac{a+b}{2},y\right)+f\left(a+b-x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Big\}p(x,y)dsdtdydx. \quad (9)
\end{aligned}$$

By Lemma 1, the following inequalities hold for all $(t,s)\in\left[0,\frac{1}{2}\right]^2$, $(x,y)\in[a,\frac{a+b}{2}]\times[c,\frac{c+d}{2}]$:
By setting $y_1=\frac{x}{2}+\frac{a+b}{4}$, $x_1=x_2=\frac{a+b}{2}$, $y_2=\frac{3(a+b)}{4}-\frac{x}{2}$ in (3), for all $v\in[c,\frac{c+d}{2}]$, we observe that

$$2f\left(\frac{a+b}{2},v\right)\le f\left(\frac{x}{2}+\frac{a+b}{4},v\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2},v\right), \quad (10)$$

holds.
Multiplying both sides of the inequality (10) by 2, replacing $v=\frac{c+d}{2}$ and then applying (4) for $w_1=\frac{y}{2}+\frac{c+d}{4}$, $v_1=v_2=\frac{c+d}{2}$, $w_2=\frac{3(c+d)}{4}-\frac{x}{2}$ to both of the expressions on right-side of (10), we have that the following inequality:

$$\begin{aligned}
&4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\qquad\le\left[f\left(\frac{x}{2}+\frac{a+b}{4},\frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{x}{2}+\frac{a+b}{4},\frac{3(c+d)}{4}-\frac{y}{2}\right)\right.\\
&\qquad\left.+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{3(c+d)}{4}-\frac{y}{2}\right)\right], \quad (11)
\end{aligned}$$

holds.
Now by choosing $y_1=t\frac{a+b}{2}+(1-t)x$, $x_1=x_2=\frac{x}{2}+\frac{a+b}{4}$, $y_2=tx+(1-t)\frac{a+b}{2}$ in (3),

the following inequality holds:

$$\begin{aligned} 2f\left(\frac{x}{2}+\frac{a+b}{4},v\right) \\ \leq f\left(t\frac{a+b}{2}+(1-t)\,x,v\right)+f\left(tx+(1-t)\,\frac{a+b}{2},v\right), \quad (12) \end{aligned}$$

for all $v\in[c,\frac{c+d}{2}]$.
By replacing $y_1=t\,(a+b-x)+(1-t)\,\frac{a+b}{2}$, $x_1=x_2=\frac{3(a+b)}{4}-\frac{x}{2}$, $y_2=t\frac{a+b}{2}+(1-t)\,(a+b-x)$ in (3), we notice that

$$\begin{aligned} 2f\left(\frac{3\,(a+b)}{4}-\frac{x}{2},v\right) \\ \leq f\left(t\,(a+b-x)+(1-t)\,\frac{a+b}{2},v\right)+f\left(t\frac{a+b}{2}+(1-t)\,(a+b-x)\,,v\right), \quad (13) \end{aligned}$$

holds for all $v\in[c,\frac{c+d}{2}]$.
Multiplying both sides of (12) and (13) by 2, setting $v=\frac{y}{2}+\frac{c+d}{2}$ and $v=\frac{3(c+d)}{4}-\frac{y}{2}$, then using (4) for the particular choices of w_1, w_2, v_1 and v_2, we have that the following inequalities hold:

$$\begin{aligned} 4f\left(\frac{x}{2}+\frac{a+b}{4},\frac{y}{2}+\frac{c+d}{4}\right) \\ \leq f\left(t\frac{a+b}{2}+(1-t)\,x,s\frac{c+d}{2}+(1-s)\,y\right) \\ +f\left(t\frac{a+b}{2}+(1-t)\,x,sy+(1-s)\,\frac{c+d}{2}\right) \\ +f\left(tx+(1-t)\,\frac{a+b}{2},s\frac{c+d}{2}+(1-s)\,y\right) \\ +f\left(tx+(1-t)\,\frac{a+b}{2},sy+(1-s)\,\frac{c+d}{2}\right), \quad (14) \end{aligned}$$

$$\begin{aligned} 4f\left(\frac{x}{2}+\frac{a+b}{4},\frac{3\,(c+d)}{4}-\frac{y}{2}\right) \\ \leq f\left(t\frac{a+b}{2}+(1-t)\,x,s\frac{c+d}{2}+(1-s)\,(c+d-y)\,y\right) \\ +f\left(t\frac{a+b}{2}+(1-t)\,x,s\,(c+d-y)+(1-s)\,\frac{c+d}{2}\right) \\ +f\left(tx+(1-t)\,\frac{a+b}{2},s\frac{c+d}{2}+(1-s)\,(c+d-y)\right) \\ +f\left(tx+(1-t)\,\frac{a+b}{2},s\,(c+d-y)+(1-s)\,\frac{c+d}{2}\right), \quad (15) \end{aligned}$$

$$\begin{aligned}
4f&\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{3(c+d)}{4}-\frac{y}{2}\right)\\
&\leq f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)(c+d-y)\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s\frac{c+d}{2}+(1-s)(c+d-y)\right)
\end{aligned}\tag{16}$$

and

$$\begin{aligned}
4f&\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{y}{2}+\frac{c+d}{4}\right)\\
&\leq f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)y\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),sy+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s\frac{c+d}{2}+(1-s)y\right).
\end{aligned}\tag{17}$$

By setting $y_1=x$, $x_1=t\frac{a+b}{2}+(1-t)x$, $x_2=tx+(1-t)\frac{a+b}{2}$, $y_2=\frac{a+b}{2}$ in (3), the following holds:

$$\begin{aligned}
f&\left(t\frac{a+b}{2}+(1-t)x,v\right)+f\left(tx+(1-t)\frac{a+b}{2},v\right)\\
&\leq f(x,v)+f\left(\frac{a+b}{2},v\right),
\end{aligned}\tag{18}$$

for all $v\in[c,\frac{c+d}{2}]$.
By the choice of $y_1=\frac{a+b}{2}$, $x_1=t(a+b-x)+(1-t)\frac{a+b}{2}$, $x_2=t\frac{a+b}{2}+(1-t)(a+b-x)$, $y_2=a+b-x$ in (3), we have that the following inequality:

$$\begin{aligned}
f&\left(t(a+b-x)+(1-t)\frac{a+b}{2},v\right)+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),v\right)\\
&\leq f(a+b-x,v)+f\left(\frac{a+b}{2},v\right),
\end{aligned}\tag{19}$$

holds for all $v\in[c,\frac{c+d}{2}]$.

By respective settings $v=s\frac{c+d}{2}+(1-s)y$, $v=sy+(1-s)\frac{c+d}{2}$, $v=s\frac{c+d}{2}+(1-s)(c+d-y)$ and $v=s(c+d-y)+(1-s)\frac{c+d}{2}$ in (18) and (19), using (4) for the particular choices of w_1, w_2, v_1 and v_2 and then summing up the resulting inequalities we obtain

$$\begin{aligned}
& f\left(t\frac{a+b}{2}+(1-t)\,x, s\frac{c+d}{2}+(1-s)\,y\right) \\
&\qquad + f\left(t\frac{a+b}{2}+(1-t)\,x, sy+(1-s)\,\frac{c+d}{2}\right) \\
&\qquad + f\left(tx+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,y\right) \\
&\qquad + f\left(tx+(1-t)\,\frac{a+b}{2}, sy+(1-s)\,\frac{c+d}{2}\right) \\
&\qquad\qquad \leq f\,(x,y)+f\left(x,\frac{c+d}{2}\right) \\
&\qquad\qquad\qquad + f\left(\frac{a+b}{2},y\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right), \qquad (20)
\end{aligned}$$

$$\begin{aligned}
& f\left(t\frac{a+b}{2}+(1-t)\,x, s\frac{c+d}{2}+(1-s)\,(c+d-y)\right) \\
&\qquad + f\left(t\frac{a+b}{2}+(1-t)\,x, s\,(c+d-y)+(1-s)\,\frac{c+d}{2}\right) \\
&\qquad + f\left(tx+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,(c+d-y)\right) \\
&\qquad + f\left(tx+(1-t)\,\frac{a+b}{2}, s\,(c+d-y)+(1-s)\,\frac{c+d}{2}\right) \\
&\qquad\qquad \leq f\,(x,c+d-y)+f\left(x,\frac{c+d}{2}\right) \\
&\qquad\qquad\qquad + f\left(\frac{a+b}{2},c+d-y\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right), \qquad (21)
\end{aligned}$$

$$\begin{aligned}
& f\left(t\frac{a+b}{2}+(1-t)\,(a+b-x)\,, s\frac{c+d}{2}+(1-s)\,y\right) \\
&\qquad + f\left(t\frac{a+b}{2}+(1-t)\,(a+b-x)\,, sy+(1-s)\,\frac{c+d}{2}\right) \\
&\qquad + f\left(t\,(a+b-x)+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,y\right) \\
&\qquad + f\left(t\,(a+b-x)+(1-t)\,\frac{a+b}{2}, sy+(1-s)\,\frac{c+d}{2}\right) \\
&\qquad\qquad \leq f\,(a+b-x,y)+f\left(a+b-x,\frac{c+d}{2}\right) \\
&\qquad\qquad\qquad + f\left(\frac{a+b}{2},y\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
& f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)(c+d-y)\right) \\
&\qquad + f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\
&\qquad + f\left(t\frac{a+b}{2}+(1-t)(a+b-x), s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\
&\qquad + f\left(t\frac{a+b}{2}+(1-t)(a+b-x), s\frac{c+d}{2}+(1-s)(c+d-y)\right) \\
&\qquad\qquad \le f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f(a+b-x, c+d-y) \\
&\qquad\qquad\qquad + f\left(\frac{a+b}{2}, c+d-y\right) + f\left(a+b-x, \frac{c+d}{2}\right). \qquad (23)
\end{aligned}$$

Multiplying the inequalities (11)-(23) by $p(x,y)$ and integrating respectively over t on $[0,\frac{1}{2}]$, over s on $[0,\frac{1}{2}]$, over x on $[a,\frac{a+b}{2}]$ and over y on $[c,\frac{c+d}{2}]$ and using the identities (6)-(9), we derive (5). □

Theorem 2. *Let f, p, H_p be defined as above. Let f be twice differentiable on $[a,b]\times[c,d]$ such that the second order partial derivatives are continuous. If the first order partial derivatives of f are co-ordinated convex and p is bounded on $[a,b]\times[c,d]$, then*

$$\begin{aligned}
0 &\ge H_p(t,s) - \int_a^b\int_c^d f(x,y)p(x,y)dydx \\
&\le (1-t)(1-s)\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right. \\
&\qquad\qquad \left. - \int_a^b\int_c^d f(x,y)\,dydx\right]\|p\|_\infty, \qquad (24)
\end{aligned}$$

for $(s,t)\in[0,1]^2$ and $\|p\|_\infty = \sup_{(x,y)\in[a,b]\times[c,d]}|p(x,y)|$. And

$$\begin{aligned}
0 &\ge 2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx - H_p(t,s) \\
&\qquad - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx \\
&\le \frac{(b-a)(d-c)}{16}\left[\frac{\partial^2 f(a,d)}{\partial y\partial x}+\frac{\partial^2 f(b,c)}{\partial y\partial x}-\frac{\partial^2 f(a,c)}{\partial y\partial x}-\frac{\partial^2 f(b,d)}{\partial y\partial x}\right] \\
&\qquad\qquad \times\int_a^b\int_c^d p(x,y)dydx. \qquad (25)
\end{aligned}$$

Proof. Using the substitution rules for integration, under the assumptions on p, the following identities hold:

$$\int_a^b\int_c^d f(x,y)p(x,y)dydx$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)p(x,y)dydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(a+b-x,y)p(x,y)dydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,c+d-y)p(x,y)dydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(a+b-x,c+d-y)p(x,y)dydx \quad (26)$$

and

$$H_p(t,s)$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right.$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$\left.+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx. \quad (27)$$

By integrating by parts, we also have the following identity:

$$\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\left[\frac{\partial^2 f(a+b-x,c+d-y)}{\partial x\partial y}\right.$$
$$\left.-\frac{\partial^2 f(a+b-x,y)}{\partial x\partial y}-\frac{\partial^2 f(x,c+d-y)}{\partial x\partial y}+\frac{\partial^2 f(x,y)}{\partial x\partial y}\right]dydx$$
$$=\int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x\partial y}dydx$$
$$=\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right.$$
$$\left.+\int_a^b\int_c^d f(x,y)\,dydx\right]$$
$$-\frac{d-c}{2}\int_a^b[f(x,c)+f(x,d)]\,dx$$
$$-\frac{b-a}{2}\int_c^d[f(a,y)+f(b,y)]\,dy. \quad (28)$$

Now using the co-ordinated convexity of the first order partial derivatives and that of f, under the assumptions on p, the inequality

$$\begin{aligned}
&\Bigg[f(a+b-x,c+d-y)-f\left(a+b-x,s\left(c+d-y\right)+\left(1-s\right)\frac{c+d}{2}\right)\\
&\quad-f\left(t\left(a+b-x\right)+\left(1-t\right)\frac{a+b}{2},c+d-y\right)\\
&\quad+f\left(t\left(a+b-x\right)+\left(1-t\right)\frac{a+b}{2},s\left(c+d-y\right)+\left(1-s\right)\frac{c+d}{2}\right)\Bigg]p\left(x,y\right)\\
&\quad+\Bigg[f\left(a+b-x,y\right)-f\left(a+b-x,sy+\left(1-s\right)\frac{c+d}{2}\right)\\
&\quad-f\left(t\left(a+b-x\right)+\left(1-t\right)\frac{a+b}{2},y\right)\\
&\quad+f\left(t\left(a+b-x\right)+\left(1-t\right)\frac{a+b}{2},sy+\left(1-s\right)\frac{c+d}{2}\right)\Bigg]p(x,y)\\
&\quad+\Bigg[f(x,c+d-y)-f\left(x,s\left(c+d-y\right)+\left(1-s\right)\frac{c+d}{2}\right)\\
&\quad-f\left(tx+\left(1-t\right)\frac{a+b}{2},c+d-y\right)
\end{aligned}$$

$$\begin{aligned}
&+f\left(tx+\left(1-t\right)\frac{a+b}{2},s\left(c+d-y\right)+\left(1-s\right)\frac{c+d}{2}\right)\Bigg]p(x,y)\\
&\quad+\Bigg[f\left(x,y\right)-f\left(tx+\left(1-t\right)\frac{a+b}{2},y\right)-f\left(x,sy+\left(1-s\right)\frac{c+d}{2}\right)\\
&\quad-f\left(tx+\left(1-t\right)\frac{a+b}{2},sy+\left(1-s\right)\frac{c+d}{2}\right)\Bigg]p(x,y)\\
&\leq\left(1-t\right)\left(1-s\right)\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\Bigg[\frac{\partial^2 f\left(a+b-x,c+d-y\right)}{\partial x\partial y}\\
&\quad-\frac{\partial^2 f\left(x,c+d-y\right)}{\partial x\partial y}-\frac{\partial^2 f\left(a+b-x,y\right)}{\partial x\partial y}+\frac{\partial^2 f\left(x,y\right)}{\partial x\partial y}\Bigg]\left\|p\right\|_{\infty}, \quad (29)
\end{aligned}$$

holds for all $(s,t)\in[0,1]^2$ and $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$.
Integrating (29) over $x$on $\left[a,\frac{a+b}{2}\right]$ and over y on $\left[c,\frac{c+d}{2}\right]$, using (26)-(28), the inequality [4, Theorem 1] and the facts $H_p(t,1)\leq H_p(1,1)$, $H_p(1,s)\leq H_p(1,1)$, the second inequality in (24) holds. By [1, Theorem 2.2] the first inequality in (24) does hold.
By the co-ordinated convexity of first order partial derivatives and that of f, the following inequalities hold:

$$\frac{-f(a,c)+f\left(\frac{a+b}{2},c\right)+f\left(a,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{4}\leq-\frac{\left(b-a\right)\left(d-c\right)}{16}\frac{\partial^2 f\left(a,c\right)}{\partial y\partial x},$$

$$\frac{-f(a,d)+f\left(\frac{a+b}{2},d\right)+f\left(a,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{4}\leq\frac{\left(b-a\right)\left(d-c\right)}{16}\frac{\partial^2 f\left(a,d\right)}{\partial y\partial x},$$

$$\frac{-f(b,c)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{4}\leq\frac{\left(b-a\right)\left(d-c\right)}{16}\frac{\partial^2 f\left(b,c\right)}{\partial y\partial x},$$

and

$$\frac{-f(b,d)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{4}\leq-\frac{(b-a)(d-c)}{16}\frac{\partial^2 f(b,d)}{\partial y\partial x}.$$

Adding these results, multiplying the resulting inequality by $p(x,y)$ then integrating over $(x,y)\in[a,b]\times[c,d]$, the following holds:

$$\begin{aligned}&-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\times\\&\int_a^b\int_c^d p(x,y)dydx+\frac{1}{2}\left[f\left(\frac{a+b}{2},c\right)+f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)\right]\\&\times\int_a^b\int_c^d p(x,y)dydx\leq\frac{(b-a)(d-c)}{16}\times\\&\left[\frac{\partial^2 f(a,d)}{\partial y\partial x}+\frac{\partial^2 f(b,c)}{\partial y\partial x}-\frac{\partial^2 f(b,d)}{\partial y\partial x}-\frac{\partial^2 f(a,c)}{\partial y\partial x}\right]\int_a^b\int_c^d p(x,y)dydx.\end{aligned}\tag{30}$$

Since

$$\begin{aligned}&\frac{1}{2}\left[f\left(\frac{a+b}{2},c\right)+f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)\right]\int_a^b\int_c^d p(x,y)dydx\\&\geq 2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx\end{aligned}\tag{31}$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx\leq H(t,s)\leq\int_a^b\int_c^d f(x,y)p(x,y)dydx.\tag{32}$$

From (30)-(32), (25) does hold and this completes the proof. □

Theorem 3. *Let f, p, H_p, G be defined as above, then*

$$H_p(t,s)\leq G(t,s)\int_a^b\int_c^d p(x,y)dydx \quad \text{for all} \quad (t,s)\in[0,1]^2.\tag{33}$$

$$\begin{aligned}&4\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\int_{\frac{3c+d}{4}}^{\frac{c+3d}{4}} f(x,y)p\left(2x-\frac{a+b}{2},2y-\frac{c+d}{2}\right)dydx\leq\frac{1}{4}\left[f\left(\frac{3a+b}{4},\frac{3c+d}{4}\right)\right.\\&\left.+f\left(\frac{3a+b}{4},\frac{c+3d}{4}\right)+f\left(\frac{a+3b}{4},\frac{3c+d}{4}\right)+f\left(\frac{a+3b}{4},\frac{c+3d}{4}\right)\right]\int_a^b\int_c^d p(x,y)dydx\\&\leq(b-a)(d-c)\int_0^1\int_0^1 G(t,s)p\left((1-t)a+tb,(1-s)c+sd\right)dsdt\\&\leq\frac{1}{4}\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{2}\left\{f\left(a,\frac{c+d}{2}\right)\right.\right.\\&\left.\left.+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right\}\right]\int_a^b\int_c^d p(x,y)dydx.\end{aligned}\tag{34}$$

Let the first order partial derivatives of f be co-ordinated convex and let p be bounded on $[a,b]\times[c,d]$. If f has continuous second order partial derivatives on $[a,b]\times[c,d]$ then

$$0\geq f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx-H(t,s)\leq(b-a)(d-c)\left[H(t,s)\right.$$
$$\left.+G(t,s)-2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\|p\|_\infty, \quad (35)$$

where

$$\|p\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}|p(x,y)|.$$

Proof. Using the simple techniques of integration, under the assumptions on p, the following does hold:

$$G(t,s)\int_a^b\int_c^d p(x,y)dydx=\left[f\left(ta+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\right.$$
$$+f\left(ta+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(tb+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)$$
$$\left.+f\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right]\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}p(x,y)dydx, \quad (36)$$

for $(s,t)\in[0,1]^2$.
By setting, $y_1=ta+(1-t)\frac{a+b}{2}$, $x_1=tx+(1-t)\frac{a+b}{2}$, $x_2=t(a+b-x)+(1-t)\frac{a+b}{2}$, $y_2=tb+(1-t)\frac{a+b}{2}$ in (3) for respective choices of $v=sy+(1-s)\frac{c+d}{2}$ and $v=s(c+d-y)+(1-s)\frac{c+d}{2}$, the followings hold true:

$$f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)$$
$$\leq f\left(ta+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(tb+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right) \quad (37)$$

and

$$f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$\leq f\left(ta+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(tb+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right), \quad (38)$$

for $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$.

Adding (37) and (38), applying (4) for the particular choices of w_1, w_2, v_1 and v_2, multiplying the resulting inequality by $p(x,y)$, integrating over $(x,y) \in \left[a, \frac{a+b}{2}\right] \times \left[c, \frac{c+d}{2}\right]$ and using (36), we get (33).

By simple techniques of integration, under the assumptions on p, the following identities hold:

$$4\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \int_{\frac{3c+d}{4}}^{\frac{c+3d}{4}} f(x,y) p\left(2x - \frac{a+b}{2}, 2y - \frac{c+d}{2}\right) dydx$$
$$= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{y}{2} + \frac{c+d}{4}\right) + f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{3(c+d)}{4} - \frac{y}{2}\right) + \right.$$
$$\left. f\left(\frac{3(a+b)}{4} - \frac{x}{2}, \frac{y}{2} + \frac{c+d}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}, \frac{3(c+d)}{4} - \frac{y}{2}\right)\right] p(x,y) dydx, \quad (39)$$

$$\left[f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right) + f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right) + f\left(\frac{a+3b}{4}, \frac{3c+d}{4}\right) \right.$$
$$\left. + f\left(\frac{a+3b}{4}, \frac{c+3d}{4}\right)\right] \int_a^b \int_c^d p(x,y) dydx$$
$$= 4\left[f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right) + f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right) \right.$$
$$\left. + f\left(\frac{a+3b}{4}, \frac{3c+d}{4}\right) + f\left(\frac{a+3b}{4}, \frac{c+3d}{4}\right)\right] \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} p(x,y) dydx, \quad (40)$$

$$(b-a)(d-c) \int_0^1 \int_0^1 G(t,s) p((1-t)a + tb, (1-s)c + sd) dsdt = \frac{(b-a)(d-c)}{4} \times$$
$$\left[\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 f\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) p(ta + (1-t)b, sc + (1-s)d) dsdt \right.$$
$$+ \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) p((1-t)a + tb, sc + (1-s)d) dsdt$$
$$+ \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} f\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) p(ta + (1-t)b, (1-s)c + sd) dsdt$$
$$+ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) p((1-t)a + tb, (1-s)c + sd) dsdt$$
$$+ \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 f\left(ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right) p(ta + (1-t)b, sc + (1-s)d) dsdt$$
$$+ \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f\left(ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right) p((1-t)a + tb, sc + (1-s)d) dsdt$$
$$+ \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} f\left(ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right) p(ta + (1-t)b, (1-s)c + sd) dsdt$$
$$+ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f\left(ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right) p((1-t)a + tb, (1-s)c + sd) dsdt$$

$$+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) p\left(ta+(1-t)b, sc+(1-s)d\right) dsdt$$
$$+\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) p\left((1-t)a+tb, sc+(1-s)d\right) dsdt$$
$$+\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) p\left(ta+(1-t)b, (1-s)c+sd\right) dsdt$$
$$+\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) p\left((1-t)a+tb, (1-s)c+sd\right) dsdt$$
$$+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) p\left(ta+(1-t)b, sc+(1-s)d\right) dsdt$$
$$+\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) p\left((1-t)a+tb, sc+(1-s)d\right) dsdt$$
$$+\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) p\left(ta+(1-t)b, (1-s)c+sd\right) dsdt$$
$$\left.+\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) p\left((1-t)a+tb, (1-s)c+sd\right) dsdt\right]$$
$$=\frac{1}{4}\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left[f\left(\frac{b+x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{c+y}{2}\right)\right.$$
$$+f\left(\frac{a+x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{b+x}{2},\frac{d+y}{2}\right)$$
$$+f\left(\frac{2a+b-x}{2},\frac{d+y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{a+x}{2},\frac{d+y}{2}\right)$$
$$+f\left(\frac{a+x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{a+x}{2},\frac{c+y}{2}\right)$$
$$+f\left(\frac{a+2b-x}{2},\frac{d+y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+y}{2}\right)+f\left(\frac{b+x}{2},\frac{c+y}{2}\right)$$
$$\left.+f\left(\frac{b+x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{2c+d-y}{2}\right)\right]p(x,y)dydx$$

and

$$\frac{1}{4}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{2}\left\{f\left(a,\frac{c+d}{2}\right)\right.\right.$$
$$\left.\left.+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right\}\right]\int_{a}^{b}\int_{c}^{d}p(x,y)dydx$$
$$=\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{2}\left\{f\left(a,\frac{c+d}{2}\right)\right.\right.$$
$$\left.\left.+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right\}\right]\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}p(x,y)dydx. \quad (41)$$

By choosing, $y_1 = \frac{3a+b}{4}$, $x_1 = \frac{x}{2} + \frac{a+b}{4}$, $x_2 = \frac{3(a+b)}{4} - \frac{x}{2}$, $y_2 = \frac{a+3b}{4}$ in (3) for respective settings $v = \frac{y}{2} + \frac{c+d}{4}$ and $v = \frac{3(c+d)}{4} - \frac{y}{2}$, the following hold true:

$$\begin{aligned} & f\left(\frac{x}{2}+\frac{a+b}{4}, \frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}, \frac{y}{2}+\frac{c+d}{4}\right) \\ & \leq f\left(\frac{3a+b}{4}, \frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{a+3b}{4}, \frac{y}{2}+\frac{c+d}{4}\right) \end{aligned} \tag{42}$$

and

$$\begin{aligned} & f\left(\frac{x}{2}+\frac{a+b}{4}, \frac{3(c+d)}{4}-\frac{y}{2}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}, \frac{3(c+d)}{4}-\frac{y}{2}\right) \\ & \leq f\left(\frac{3a+b}{4}, \frac{3(c+d)}{4}-\frac{y}{2}\right)+f\left(\frac{a+3b}{4}, \frac{3(c+d)}{4}-\frac{y}{2}\right), \end{aligned} \tag{43}$$

for $(x, y) \in \left[a, \frac{a+b}{2}\right] \times \left[c, \frac{c+d}{2}\right]$.

Adding (42) and (43), applying (4) for the appropriate choices of w_1, w_2, v_1 and v_2, the following holds:

$$\begin{aligned} & f\left(\frac{x}{2}+\frac{a+b}{4}, \frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}, \frac{y}{2}+\frac{c+d}{4}\right) \\ & \quad +f\left(\frac{x}{2}+\frac{a+b}{4}, \frac{3(c+d)}{4}-\frac{y}{2}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}, \frac{3(c+d)}{4}-\frac{y}{2}\right) \\ & \leq f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right)+f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right)+f\left(\frac{a+3b}{4}, \frac{3c+d}{4}\right)+f\left(\frac{a+3b}{4}, \frac{c+3d}{4}\right) \end{aligned} \tag{44}$$

Choosing $y_1 = \frac{x+a}{2}$, $x_1 = x_2 = \frac{3a+b}{4}$, $y_2 = \frac{2a+b-x}{2}$ in (3) for respective settings $v = \frac{3c+d}{4}$ and $v = \frac{c+3d}{4}$, the following hold true:

$$f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right) \leq \frac{1}{2}\left[f\left(\frac{x+a}{2}, \frac{3c+d}{4}\right)+f\left(\frac{2a+b-x}{2}, \frac{3c+d}{4}\right)\right] \tag{45}$$

and

$$f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right) \leq \frac{1}{2}\left[f\left(\frac{x+a}{2}, \frac{c+3d}{4}\right)+f\left(\frac{2a+b-x}{2}, \frac{c+3d}{4}\right)\right]. \tag{46}$$

Adding (45) and (46), applying (4) for the suitable choices of w_1, w_2, v_1 and v_2 the following holds:

$$\begin{aligned} & f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right)+f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right) \leq \frac{1}{4}\left[f\left(\frac{2a+b-x}{2}, \frac{2c+d-y}{2}\right)\right. \\ & \quad +f\left(\frac{x+a}{2}, \frac{2c+d-y}{2}\right)+f\left(\frac{2a+b-x}{2}, \frac{y+c}{2}\right)+f\left(\frac{2a+b-x}{2}, \frac{c+2d-y}{2}\right) \\ & \quad +f\left(\frac{x+a}{2}, \frac{y+d}{2}\right)+f\left(\frac{2a+b-x}{2}, \frac{y+d}{2}\right)+ \\ & \quad \left. f\left(\frac{x+a}{2}, \frac{y+c}{2}\right)+f\left(\frac{x+a}{2}, \frac{c+2d-y}{2}\right)\right]. \end{aligned} \tag{47}$$

Analogously

$$\begin{aligned} f\left(\frac{a+3b}{4},\frac{3c+d}{4}\right)+f\left(\frac{a+3b}{4},\frac{c+3d}{4}\right) \\ \leq \frac{1}{4}\left[f\left(\frac{a+2b-x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{y+d}{2}\right)\right. \\ +f\left(\frac{x+b}{2},\frac{y+d}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{c+2d-y}{2}\right) \\ \left.+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+2d-y}{2}\right)\right]. \end{aligned} \tag{48}$$

Again by replacing $y_1=a$, $x_1=\frac{x+a}{2}$, $x_2=\frac{2a+b-x}{4}$, $y_2=\frac{a+b}{2}$ in (3) for respective settings $v=\frac{2c+d-y}{2}$, $v=\frac{y+c}{2}$, $v=\frac{y+d}{2}$ and $v=\frac{c+2d-y}{2}$, the followings hold:

$$\begin{aligned} f\left(\frac{x+a}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{2c+d-y}{2}\right) \\ \leq f\left(a,\frac{2c+d-y}{2}\right)+f\left(\frac{a+b}{2},\frac{2c+d-y}{2}\right), \end{aligned} \tag{49}$$

$$\begin{aligned} f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{y+c}{2}\right) \\ \leq f\left(a,\frac{c+y}{2}\right)+f\left(\frac{a+b}{2},\frac{c+y}{2}\right), \end{aligned} \tag{50}$$

$$\begin{aligned} f\left(\frac{x+a}{2},\frac{y+d}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{y+d}{2}\right) \\ \leq f\left(a,\frac{y+d}{2}\right)+f\left(\frac{a+b}{2},\frac{y+d}{2}\right) \end{aligned} \tag{51}$$

and

$$\begin{aligned} f\left(\frac{x+a}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{c+2d-y}{2}\right) \\ \leq f\left(a,\frac{c+2d-y}{2}\right)+f\left(\frac{a+b}{2},\frac{c+2d-y}{2}\right). \end{aligned} \tag{52}$$

Adding (49)-(52), applying (4) for the particular choices of w_1, w_2, v_1 and v_2 to obtain:

$$\begin{aligned} f\left(\frac{x+a}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right) \\ +f\left(\frac{2a+b-x}{2},\frac{y+d}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+c}{2}\right) \\ +f\left(\frac{x+a}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{c+2d-y}{2}\right)\leq 2f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ +f(a,c)+f(a,d)+2f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right). \end{aligned} \tag{53}$$

Analogously

$$
\begin{aligned}
&f\left(\frac{a+2b-x}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{2c+d-y}{2}\right)\\
&\quad+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+2d-y}{2}\right)\\
&\quad+f\left(\frac{a+2b-x}{2},\frac{y+d}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\\
&\quad+f\left(\frac{a+2b-x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{x+b}{2},\frac{c+2d-y}{2}\right)\\
&\leq 2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(b,c\right)+f\left(b,d\right)\\
&\quad+2f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right). \qquad (54)
\end{aligned}
$$

From the inequalities (44), (47), (48), (53), (54), under the assumptions on p and the identities (39)-(41), we drive (34).

By integration by parts, we have

$$
\begin{aligned}
&st\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\\
&\quad\times\left[\frac{\partial^2}{\partial x\partial y}f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right.\\
&\quad-\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\\
&\quad-\frac{\partial^2}{\partial x\partial y}f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&\quad\left.+\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right]dydx\\
&=st\int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\\
&\quad\times\frac{\partial^2}{\partial x\partial y}f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dydx\\
&=(b-a)(d-c)\left[G(t,s)+H(t,s)\right]\\
&\quad-\frac{b-a}{2}\int_c^d\left\{f\left(ta+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right.\\
&\quad\left.+f\left(tb+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right\}dy\\
&\quad-\frac{d-c}{2}\int_a^b\left\{f\left(tx+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\right.\\
&\quad\left.+f\left(tx+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right\}dx. \qquad (55)
\end{aligned}
$$

By using the co-ordinated convexity of the first order partial derivatives and that of f, under the assumptions on p, the following inequality holds:

$$
\begin{aligned}
&\left[f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.-f\left(tx+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y)\\
&\quad-\left[f\left(\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y)\\
&\quad+\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.-f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y)\\
&\quad-\left[f\left(\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y)\\
&\quad+\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.-f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y)\\
&\quad-\left[f\left(\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y)\\
&\quad+\left[f\left(tx+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.-f\left(tx+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y)\\
&\quad-\left[f\left(\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y)\\
&\leq st\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\left[\frac{\partial^2}{\partial x\partial y}f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right.\\
&\quad-\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\\
&\quad-\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&\quad\left.+\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right]\|p\|_\infty,
\end{aligned}
$$

for all $(t,s)\in[0,1]^2$ and $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$.

Integrating the above inequality over $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$, under the assumptions on p, using the facts $H(0,s)\leq H(t,s)$, $H(t,0)\leq H(t,s)$ and [1, Theorem 2.2],we get

$$\begin{aligned}0\geq f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\int_a^b\int_c^d p(x,y)dydx-H(t,s)\\&\leq(b-a)(d-c)\left[H(t,s)+G(t,s)\right]\|p\|_\infty\\-\Bigg[\frac{b-a}{2}\int_c^d\Bigg\{&f\left(ta+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\\&+f\left(tb+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dy\\+\frac{d-c}{2}\int_c^d&f\left(tx+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\\&+f\left(tx+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\Bigg\}dx\Bigg]\|p\|_\infty.\end{aligned}\tag{56}$$

Since f is convex on the co-ordinates, by Jensen's inequality for integrals the followings hold:

$$\begin{aligned}&\frac{b-a}{2}\int_c^d f\left(ta+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dy\\&+\frac{b-a}{2}\int_c^d f\left(tb+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dy\\&\geq(b-a)(d-c)G(t,0)\geq(b-a)(d-c)G(0,0),\end{aligned}\tag{57}$$

and

$$\begin{aligned}&\frac{d-c}{2}\int_c^d f\left(tx+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)dx\\&+\frac{d-c}{2}\int_c^d f\left(tx+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)dx\\&\geq(b-a)(d-c)G(0,s)\geq(b-a)(d-c)G(0,0).\end{aligned}\tag{58}$$

By (56)-(58) we get (35).
This completes the proof of the theorem. □

Theorem 4. *Let f, p, G, H_p, L_p be defined as above, then L_p is co-ordinated convex on $[0,1]^2$ and we have the following inequalities:*

$$\begin{aligned}G(t,s)\int_a^b\int_c^d p(x,y)dydx\leq L_p(t,s)\leq(1-t)(1-s)\int_a^b\int_c^d f(x,y)p(x,y)dydx\\+ts\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx+\frac{1}{2}t(1-s)\times\\\int_a^b\int_c^d\left[f(a,y)+f(b,y)\right]p(x,y)dydx+\frac{s(1-t)}{2}\int_a^b\int_c^d\left[f(x,c)+f(x,d)\right]p(x,y)\times\\dydx\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4},\end{aligned}\tag{59}$$

$$H_p(1-t,1-s)\leq L_p(t,s)\tag{60}$$

and

$$\frac{H_p(1-t,1-s)+H_p(t,s)}{2} \leq L_p(t,s). \tag{61}$$

Moreover, the following bound is true:

$$\sup_{(t,s)\in[0,1]^2} L_p(t,s) = \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \tag{62}$$

Proof. Co-ordinated convexity of L_p directly follows from co-ordinated convexity of f. By simple techniques of integration, under the assumption on p, the following does hold:

$$\begin{aligned} L_p(t,s) = \frac{1}{4}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} &[f(ta+(1-t)x, sc+(1-s)y) \\ &+ f(ta+(1-t)x, sc+(1-s)(c+d-y)) \\ &+ f(ta+(1-t)(a+b-x), sc+(1-s)y) \\ &f(ta+(1-t)(a+b-x), sc+(1-s)(c+d-y)) \\ &+ f(ta+(1-t)(a+b-x), sd+(1-s)y) \\ &+ f(ta+(1-t)(a+b-x), sd+(1-s)(c+d-y)) \\ &+ f(ta+(1-t)x, sd+(1-s)y) + f(ta+(1-t)x, sd+(1-s)(c+d-y)) \\ &+ f(tb+(1-t)x, sc+(1-s)y) + f(tb+(1-t)x, sc+(1-s)(c+d-y)) \\ &+ f(tb+(1-t)(a+b-x), sc+(1-s)y) \\ &+ f(tb+(1-t)(a+b-x), sc+(1-s)(c+d-y)) \\ &+ f(tb+(1-t)(a+b-x), sd+(1-s)y) \\ &+ f(tb+(1-t)(a+b-x), sd+(1-s)(c+d-y)) \\ &+ f(tb+(1-t)x, sd+(1-s)(c+d-y)) \\ &+ f(tb+(1-t)x, sd+(1-s)y)]\, p(x,y)dydx, \end{aligned} \tag{63}$$

for $(s,t)\in[0,1]^2$.
By setting, $y_1 = ta+(1-t)x$, $x_1 = x_2 = ta+(1-t)\frac{a+b}{2}$, $y_2 = ta+(1-t)(a+b-x)$ in (3) for respective settings $v = sc+(1-s)\frac{c+d}{2}$ and $v = sd+(1-s)\frac{c+d}{2}$, the following hold true:

$$\begin{aligned} 2f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-c)\frac{c+d}{2}\right) &\leq f\left(ta+(1-t)x, sc+(1-c)\frac{c+d}{2}\right) \\ &+ f\left(ta+(1-t)(a+b-x), sc+(1-c)\frac{c+d}{2}\right) \end{aligned} \tag{64}$$

and

$$\begin{aligned} 2f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-c)\frac{c+d}{2}\right) &\leq f\left(ta+(1-t)x, sd+(1-c)\frac{c+d}{2}\right) \\ &+ f\left(ta+(1-t)(a+b-x), sd+(1-c)\frac{c+d}{2}\right), \end{aligned} \tag{65}$$

for $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$. □

Multiplying (64) and (65) by 2, then adding and using (4) for the particular choices of w_1, w_2, v_1 and v_2, we get

$$\begin{aligned}
&4\left[f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.+f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right]\\
&\leq f\left(ta+(1-t)x, sc+(1-s)y\right)+f\left(ta+(1-t)x, sc+(1-s)(c+d-y)\right)\\
&\quad+f\left(ta+(1-t)(a+b-x), sc+(1-s)y\right)\\
&\quad+f\left(ta+(1-t)(a+b-x), sc+(1-s)(c+d-y)\right)\\
&\quad+f\left(ta+(1-t)x, sd+(1-s)y\right)+f\left(ta+(1-t)x, sd+(1-s)(c+d-y)\right)\\
&\quad+f\left(ta+(1-t)(a+b-x), sd+(1-s)y\right)\\
&\quad+f\left(ta+(1-t)(a+b-x), sd+(1-s)(c+d-y)\right). \qquad (66)
\end{aligned}$$

Analogously

$$\begin{aligned}
&4\left[f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.+f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right]\\
&\leq f\left(tb+(1-t)x, sc+(1-s)y\right)+f\left(tb+(1-t)x, sc+(1-s)(c+d-y)\right)\\
&\quad+f\left(tb+(1-t)(a+b-x), sc+(1-s)y\right)\\
&\quad+f\left(tb+(1-t)(a+b-x), sc+(1-s)(c+d-y)\right)\\
&\quad+f\left(tb+(1-t)x, sd+(1-s)y\right)+f\left(tb+(1-t)x, sd+(1-s)(c+d-y)\right)\\
&\quad+f\left(tb+(1-t)(a+b-x), sd+(1-s)y\right)\\
&\quad+f\left(tb+(1-t)(a+b-x), sd+(1-s)(c+d-y)\right). \qquad (67)
\end{aligned}$$

Multiplying the inequalities (66) and (67) by $p(x,y)$, integrating the resulting over $(x,y) \in \left[a, \frac{a+b}{2}\right] \times \left[c, \frac{c+d}{2}\right]$ and making use of identities (36) and (63), the first inequality of (59) holds true.

By using the co-ordinated convexity of f and the inequalities:

$$\int_a^b\int_c^d f(a,y)p(x,y)dydx \leq \frac{f(a,c)+f(a,d)}{2}\int_a^b\int_c^d p(x,y)dydx,$$

$$\int_a^b\int_c^d f(b,y)p(x,y)dydx \leq \frac{f(b,c)+f(b,d)}{2}\int_a^b\int_c^d p(x,y)dydx,$$

$$\int_a^b\int_c^d f(x,c)p(x,y)dydx \leq \frac{f(a,c)+f(b,c)}{2}\int_a^b\int_c^d p(x,y)dydx,$$

$$\int_a^b\int_c^d f(x,d)p(x,y)dydx \leq \frac{f(a,d)+f(b,d)}{2}\int_a^b\int_c^d p(x,y)dydx$$

and

$$\int_a^b\int_c^d f(x,y)p(x,y)dydx \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx,$$

we get second inequalities of (59).
Again using the co-ordinated convexity of f

$$H_p(1-t,1-s)=\int_a^b\int_c^d f\left((1-t)\,x+t\frac{a+b}{2},(1-s)\,y+s\frac{c+d}{2}\right)p(x,y)dydx$$
$$=\int_a^b\int_c^d f\left(\frac{ta+(1-t)\,x}{2}+\frac{tb+(1-t)\,x}{2},\frac{sc+(1-s)\,y}{2}+\frac{sd+(1-s)\,y}{2}\right)\times$$
$$p(x,y)dydx\le L_p(t,s).$$

This proves 60.
From (33), (59) and (60), we get (61). Using (59), we get (62). This completes the proof.

Remark 1. *If $p\,(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$, then $H_p\,(t,s)=H\,(t,s)$ and $L_p\,(t,s)=L\,(t,s)$ for all $(t,s)\in[0,1]^2$ and hence from all the above Theorems we get the inequalities related to the mappings H, G and L.*

References

[1] Alomari, M., Darus, M., *Fejér inequality for double integrals*, Facta Universitatis (NIŠ), Ser. Math. Inform. **24** (2009), 15.28.

[2] Dragomir, S. S., Pearce, C. E. M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. Online: `http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html`.

[3] Dragomir, S. S., Cho, Y. J., Kim, S. S., *Inequalities of Hadamard's type for Lipschitzian mappings and their applications*, J. Math. Anal. Appl. **245** (2000), 489-501.

[4] Dragomir, S. S., *On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese Journal of Mathematics **4** (2001), 775-788.

[5] Fejér, L., *Über die Fourierreihen*, II, Math. Naturwiss. Anz Ungar. Akad. Wiss. **24** (1906), 369–390.(In Hungarian).

[6] Ho, Ming-In, *Fejer inequalities for Wright-convex functions*, JIPAM. J. Inequal. Pure Appl. Math. **8** (1) (2007), article 9.

[7] Hadamard, J., *Étude sur les propriétés des fonctions entières en particulier d'une function considérée par Riemann*, J. Math. Pures and Appl. **58** (1983), 171-215.

[8] Hermite, Ch., *Sur deux limites d'une intégrale définie*, Mathesis **3** (1883) 82.

[9] Latif, M. A., *On some Fejér-type inequalities for double integrals*, to appear.

[10] Tseng, K. L., Hwang, S. R., Dragomir, S. S., *On some new inequalities of Hermite-Hadamard-Fejér type involving convex functions*, Demonstratio Math. **XL**(1) (2007), 51–64.

[11] Tseng, K. L., Hwang, S. R., Dragomir, S. S., *Fejér-type Inequalities (I)*, (Submitted) Preprint RGMIA *Res. Rep. Coll.* **12** (2009), Supplement, (4) Article 5. Online `http://www.staff.vu.edu.au/RGMIA/v12n4.asp`.

[12] Tseng, K. L., Hwang, S. R., Dragomir, S. S., *Fejér-type Inequalities (II)*, (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12**(2009), Supplement, Article 16, pp.1-12. Online `http://www.staff.vu.edu.au/RGMIA/v12(E).asp`.

[13] Tseng, K. L., Hwang, S. R., Dragomir, S. S., *Some companions of Fejér's inequality for convex functions*, (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12** (2009), Supplement, Article 19, pp.1-12. Online `http://www.staff.vu.edu.au/RGMIA/v12(E).asp`.

[14] Tseng, K. L., Hwang, S. R., Dragomir, S. S., *Refinements of Fejér's inequality for convex functions*, (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12**(2009), Supplement, Article 20, pp.1-11. Online `http://www.staff.vu.edu.au/RGMIA/v12(E).asp`.

[15] Tseng, K. L., Hwang, S. R., Dragomir, S. S., *On some weighted integral inequalities for convex functions related to Fejér's results*, (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12** (2009), Supplement, Article 21, pp.1-20. Online `http://www.staff.vu.edu.au/RGMIA/v12(E).asp`.

[16] Hwang, D. Y., Tseng, K. L., Yang, G. S., *Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane*, Taiwanese Journal ofMathematics **11** (2007), 63–73.

[17] Yang, G. S., Tseng, K. L., *Inequalities of Hermite-Hadamard-Fejér type for convex functions and Lipschitzian functions*, Taiwanese J. Math. **7** (3) (2003), 433–440.

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