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OSCILLATION OF A CLASS OF TWO-VARIABLES FUNCTIONAL EQUATIONS WITH MIX NONLINEAR TYPE

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Abstract. In this study we will establish some sufficient conditions of oscillation of a class of two-variables functional equations with mix nonlinear type. As applications, the two-variables functional equations with several nonlinear terms are also considered.

1. Introduction

The study of oscillation of functional equations is a relatively new field, but the qualitative theory of oscillation of functional equations has attracted many experts and mathematical workers. The proliferation of this area has been witnessed by several hundreds of research papers and a number of research monographs, i.e., [1-8].

In 2001, Zhang and Zhou [1] and Zhou [2] considered a series of sufficient conditions of the following two-variables equation of the form

$$a\mathbf{A}(x,\sigma(y)) + b\mathbf{A}(\tau(x),y) - c\mathbf{A}(\tau(x),\sigma(y))$$
$$+ \mathbf{p}(x,y)\mathbf{A}(\tau^{k+1}(x),\sigma^{l+1}(y)) = 0, \quad x,y \in I,$$

where I is an unbounded subset of $\mathbb{R}^+$, a,b,c are positive constants, $k \geq 1$ and $l \geq 1$ are positive integers, function $\mathbf{p} : I \times I \to \mathbb{R}^+$, $\tau,\sigma : I \to I$, $\tau(x) \neq x$, $\sigma(y) \neq y$, and $\lim_{x\to\infty} \tau(x) = \infty$, $\lim_{y\to\infty} \sigma(y) = \infty$, $x,y \in I$. By τ^i and σ^i we denote the $i-$th iterate of the function τ and σ, respectively, i.e.

$$\tau^0(x) = x, \quad \tau^{i+1}(x) = \tau(\tau^i(x)), \quad i = 0,1,\cdots, \quad x \in I,$$

and

$$\sigma^0(y) = y, \quad \sigma^{i+1}(y) = \sigma(\sigma^i(y)), \quad i = 0,1,\cdots, \quad y \in I.$$

The main aim of this paper is to establish some sufficient conditions of oscillation of two-variables functional equation with with mix nonlinear type as follows:

$$\mathbf{A}(x,\sigma(y)) + \mathbf{A}(\tau(x),y) - \mathbf{A}(\tau(x),\sigma(y))$$
$$+ \mathbf{p}(x,y)|\mathbf{A}(\tau^{k_1+1}(x),\sigma^{l_1+1}(y))|^{\alpha}\mathrm{sgn}\mathbf{A}(\tau^{k_1+1}(x),\sigma^{l_1+1}(y))$$
$$+ \mathbf{q}(x,y)|\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y))|^{\beta}\mathrm{sgn}\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y)) = 0, \quad x,y \in I, \tag{1}$$

where I is an unbounded subset of $\mathbb{R}^+$, $k_1 \geq k_2 \geq 1$ and $l_1 \geq l_2 \geq 1$ are positive integers, functions $\mathbf{p},\mathbf{q} : I \times I \to \mathbb{R}^+$, $\alpha \in (0,1)$ and $\beta \in (1,\infty)$. $\mathbb{R}^+$ denotes the set of positive real numbers. τ and σ are defined by the above.

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As applications, consider the two-variables functional equations with several nonlinear terms

$$\mathbf{A}(x,\sigma(y))+\mathbf{A}(\tau(x),y)-\mathbf{A}(\tau(x),\sigma(y))+\sum_{i=1}^{u}\big[\mathbf{p}_i(x,y)|$$
$$\times\mathbf{A}(\tau^{k_i+1}(x),\sigma^{l_i+1}(y))|^{\alpha_i}\operatorname{sgn}\mathbf{A}(\tau^{k_i+1}(x),\sigma^{l_i+1}(y))\big]=0,\quad x,y\in I, \tag{2}$$

where $\alpha_u>\alpha_{u-1}>\cdots>\alpha_k>1>\alpha_{k-1}>\cdots>\alpha_1>0$, functions $\mathbf{p}_i: I\times I\to\mathbb{R}^+$, $k_i,l_i\in\mathbb{N}^+$, $i=1,2,\cdots,u$. $\mathbb{N}^+$ denotes the set of positive integers. The other conditions are given by the above.

Our results extend the recent results in the paper [1].

Definition 1. *A solution $\mathbf{A}(x,y)$ of equation (1) or (2) is said to be eventually positive if $\mathbf{A}(x,y)>0$ for all large x and y, and eventually negative if $\mathbf{A}(x,y)<0$ for all large x and y. It is said to oscillatory if it is neither eventually positive nor eventually negative.*

2. Main results

In this section, we give our main results. For the sake of convenience, we set $p_1=(\beta-1)/(\beta-\alpha)$ and $p_2=(1-\alpha)/(\beta-\alpha)$. And we assume that the following is satisfied without furthermore mention:

(H_1) $\limsup_{I\ni x,y\to\infty}[\mathbf{p}(x,y)]^{p_1}[\mathbf{q}(x,y)]^{p_2}>0.$

Define a set E and I_α by

$$E=\{\lambda>0|1-\lambda[\mathbf{p}(x,y)]^{p_1}[\mathbf{q}(x,y)]^{p_2}>0\ \text{ eventually}\}$$

and

$$I_\alpha=[\alpha,\infty)\cap I\ \text{ for }\ \alpha\in\mathbb{R}^+$$

respectively.

The following inequality will be used to prove the main of this section.

Lemma 1. *(see[3,p.169]) Let $x,y\ge 0$, $m,n>1$, and $1/m+1/n=1$. Then*

$$\frac{x}{m}+\frac{y}{n}\ge x^{\frac{1}{m}}y^{\frac{1}{n}}. \tag{3}$$

Theorem 1. *Assume that (H_1) holds and there exist $X,Y\in I$ such that*

$$\sup_{\lambda\in E,x\in I_X,y\in I_Y}\lambda\theta\left\{\prod_{i=1}^{k_2}\prod_{j=1}^{l_2}\left(1-\lambda[\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2}\right)\right\}^{\frac{1}{\eta}}<1. \tag{4}$$

where $\eta=\min\{k_2,l_2\}$, $\theta=\min\{p_1,p_2\}$. Then every solution of equation (1) oscillates.

Proof. Suppose to the contrary, we let $\mathbf{A}(x,y)$ be an eventually positive solution of (1). We define the set $S(A)$ of positive numbers as follows:

$$S(A)=\{\lambda>0|\mathbf{A}(x,\sigma(y))+\mathbf{A}(\tau(x),y)-\mathbf{A}(\tau(x),\sigma(y))$$
$$\times(1-\lambda[\mathbf{p}(x,y)]^{p_1}[\mathbf{q}(x,y)]^{p_2})\le 0\ \text{ eventually}\}. \tag{5}$$

From (1), we have

$$\mathbf{A}(x,\sigma(y))+\mathbf{A}(\tau(x),y)\le\mathbf{A}(\tau(x),\sigma(y)). \tag{6}$$

Due to (6), we have

$$\mathbf{A}(\tau(x),\sigma(y)) \leq \mathbf{A}(\tau^2(x),\sigma(y)), \quad \mathbf{A}(\tau(x),\sigma(y)) \leq \mathbf{A}(\tau(x),\sigma^2(y)). \tag{7}$$

For $k_1 \geq k_2 \geq 1$ and $l_1 \geq l_2 \geq 1$, we obtain

$$\mathbf{A}(\tau^{k_2}(x),\sigma^{l_2}(y)) \leq \mathbf{A}(\tau^{k_1}(x),\sigma^{l_1}(y)),$$

$$\begin{aligned}&\mathbf{A}(x,\sigma(y)) + \mathbf{A}(\tau(x),y) - \mathbf{A}(\tau(x),\sigma(y)) + \mathbf{p}(x,y)\\ &\times [\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y))]^{\alpha} + \mathbf{q}(x,y)[\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y))]^{\beta} \leq 0.\end{aligned} \tag{8}$$

By Lemma 1, we have

$$\begin{aligned}&\mathbf{p}(x,y)[\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y))]^{\alpha} + \mathbf{q}(x,y)[\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y))]^{\beta}\\ &\geq \theta[\mathbf{p}(x,y)]^{p_1}[\mathbf{q}(x,y)]^{p_2}\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y)).\end{aligned} \tag{9}$$

From (8) and (9), we obtain

$$\begin{aligned}&\mathbf{A}(x,\sigma(y)) + \mathbf{A}(\tau(x),y) - \mathbf{A}(\tau(x),\sigma(y))\\ &+ \theta[\mathbf{p}(x,y)]^{p_1}[\mathbf{q}(x,y)]^{p_2}\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y)) \leq 0.\end{aligned} \tag{10}$$

Thus, we have

$$\mathbf{A}(x,\sigma(y)) + \mathbf{A}(\tau(x),y) - (1 - \theta[\mathbf{p}(x,y)]^{p_1}[\mathbf{q}(x,y)]^{p_2})\mathbf{A}(\tau(x),\sigma(y)) \leq 0,$$

which implies that $S(A)$ is nonempty.

For any $\lambda \in S(A)$, then we have $1 - \lambda[\mathbf{p}(x,y)]^{p_1}[\mathbf{q}(x,y)]^{p_2} > 0$ eventually, which show that $S(A) \subset E$. Due to the condition (H_1), then the set E is bounded, hence $S(A)$ is bounded. Let $\mu \in S(A)$, then we obtain

$$\mathbf{A}(\tau(x),\sigma(y)) \leq (1 - \mu[\mathbf{p}(\tau(x),y)]^{p_1}[\mathbf{q}(\tau(x),y)]^{p_2})\,\mathbf{A}(\tau^2(x),\sigma(y))$$

and so

$$\begin{aligned}\mathbf{A}(\tau(x),\sigma(y)) \leq &\left\{\prod_{i=1}^{k_2}\left(1 - \mu[\mathbf{p}(\tau^i(x),y)]^{p_1}[\mathbf{q}(\tau^i(x),y)]^{p_2}\right)\right\}\\ &\times \mathbf{A}(\tau^{k_2+1}(x),\sigma(y)).\end{aligned} \tag{11}$$

Similarly, we have

$$\mathbf{A}(\tau(x),\sigma(y)) \leq (1 - \mu[\mathbf{p}(x,\sigma(y))]^{p_1}[\mathbf{q}(x,\sigma(y))]^{p_2})\,\mathbf{A}(\tau(x),\sigma^2(y))$$

and so

$$\begin{aligned}\mathbf{A}(\tau(x),\sigma(y)) \leq &\left\{\prod_{j=1}^{l_2}\left(1 - \mu[\mathbf{p}(x,\sigma^j(y))]^{p_1}[\mathbf{q}(x,\sigma^j(y))]^{p_2}\right)\right\}\\ &\times \mathbf{A}(\tau(x),\sigma^{l_2+1}(y)).\end{aligned} \tag{12}$$

Combination (7) and (11), we obtain

$$\begin{aligned}[\mathbf{A}(\tau(x),\sigma(y))]^{k_2} &\leq \mathbf{A}(\tau(x),\sigma^2(y))\mathbf{A}(\tau(x),\sigma^3(y))\cdots\mathbf{A}(\tau(x),\sigma^{l_2+1}(y))\\ &\leq \left\{\prod_{j=1}^{l_2}\prod_{i=1}^{k_2}\left(1 - \mu[\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2}\right)\right\}\\ &\times [\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y))]^{k_2}.\end{aligned} \tag{13}$$

Similarly, we have

$$[\mathbf{A}(\tau(x),\sigma(y))]^{l_2} \le \left\{ \prod_{i=1}^{k_2}\prod_{j=1}^{l_2} \left(1-\mu[\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2}\right) \right\} \times [\mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y))]^{l_2}. \quad (14)$$

Combination (13) and (14), we obtain

$$\mathbf{A}(\tau(x),\sigma(y)) \le \left\{ \prod_{i=1}^{k_2}\prod_{j=1}^{l_2} \left(1-\mu[\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2}\right) \right\}^{\frac{1}{\eta}} \times \mathbf{A}(\tau^{k_2+1}(x),\sigma^{l_2+1}(y)). \quad (15)$$

Substituting (15) into (10), we obtain

$$\mathbf{A}(x,\sigma(y)) + \mathbf{A}(\tau(x),y) + \mathbf{A}(\tau(x),\sigma(y)) \times \left\{ 1-\theta\left[\prod_{i=1}^{k_2}\prod_{j=1}^{l_2} \left(1-\mu[\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2}\right) \right]^{-\frac{1}{\eta}} \right\} \le 0, \quad (16)$$

which implies that

$$\theta\left\{ \sup_{x\in I_X, y\in I_Y} \left[\prod_{i=1}^{k_2}\prod_{j=1}^{l_2} \left(1-\mu[\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2}\right) \right]^{\frac{1}{\eta}} \right\}^{-1} \in S(A). \quad (17)$$

From (4), there exists $\gamma \in (0,1)$ such that

$$\sup_{\lambda\in E, x\in I_X, y\in I_Y} \lambda\theta \left\{ \prod_{i=1}^{k_2}\prod_{j=1}^{l_2} \left(1-\lambda[\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2}\right) \right\}^{\frac{1}{\eta}} \le \gamma < 1. \quad (18)$$

Hence

$$\sup_{\lambda\in E, x\in I_X, y\in I_Y} \theta \left\{ \prod_{i=1}^{k_2}\prod_{j=1}^{l_2} \left(1-\mu[\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2}\right) \right\}^{\frac{1}{\eta}} \le \frac{\gamma}{\mu}, \quad (19)$$

so that $\mu/\gamma \in S(A)$. By induction, $\mu/\gamma^r \in S(A), r = 1, 2, \cdots$. This contradicts the boundedness of $S(A)$. The proof is complete. □

From Theorem 1, we can derive an explicit oscillation criterion.

Corollary 1. *Assume that* (H_1) *holds and for* $k_2, l_2 > 1$

$$\liminf_{I\ni x,y\to\infty} \frac{1}{k_2 l_2} \sum_{i=1}^{k_2}\sum_{j=1}^{l_2} [\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2} > \frac{\theta\delta^\delta}{(\delta+1)^{\delta+1}},$$

where $\delta = \max\{k_2, l_2\}$ *and* $\theta = \min\{p_1, p_2\}$. *Then every solution of (1) is oscillatory.*

Proof. Let $g(\lambda) = \lambda(1 - \lambda M)^{\delta}$ for $\lambda > 0, M > 0$. Then

$$\max_{\lambda > 0} g(\lambda) = \frac{\delta^{\delta}}{M(\delta + 1)^{\delta+1}}.$$

Set

$$M = \frac{1}{k_2 l_2} \sum_{i=1}^{k_2} \sum_{j=1}^{l_2} [\mathbf{p}(\tau^i(x), \sigma^j(y))]^{p_1} [\mathbf{q}(\tau^i(x), \sigma^j(y))]^{p_2}.$$

Since

$$\left\{ 1 - \frac{\lambda}{k_2 l_2} \sum_{i=1}^{k_2} \sum_{j=1}^{l_2} [\mathbf{p}(\tau^i(x), \sigma^j(y))]^{p_1} [\mathbf{q}(\tau^i(x), \sigma^j(y))]^{p_2} \right\}^{\delta}$$

$$\geq \left\{ 1 - \lambda \prod_{i=1}^{k_2} \prod_{j=1}^{l_2} [\mathbf{p}(\tau^i(x), \sigma^j(y))]^{p_1} [\mathbf{q}(\tau^i(x), \sigma^j(y))]^{p_2} \right\}^{\frac{1}{\eta}},$$

we have obtain

$$1 > \theta \frac{\delta^{\delta}}{(\delta + 1)^{\delta+1}} \left\{ \frac{1}{k_2 l_2} \sum_{i=1}^{k_2} \sum_{j=1}^{l_2} [\mathbf{p}(\tau^i(x), \sigma^j(y))]^{p_1} [\mathbf{q}(\tau^i(x), \sigma^j(y))]^{p_2} \right\}^{-1}$$

$$\geq \lambda\theta \left\{ 1 - \frac{\lambda}{k_2 l_2} \sum_{i=1}^{k_2} \sum_{j=1}^{l_2} [\mathbf{p}(\tau^i(x), \sigma^j(y))]^{p_1} [\mathbf{q}(\tau^i(x), \sigma^j(y))]^{p_2} \right\}^{a}$$

$$\geq \lambda\theta \left\{ 1 - \lambda \prod_{i=1}^{k_2} \prod_{j=1}^{l_2} [\mathbf{p}(\tau^i(x), \sigma^j(y))]^{p_1} [\mathbf{q}(\tau^i(x), \sigma^j(y))]^{p_2} \right\}^{\frac{1}{\eta}}.$$

Then the conclusion follows from Theorem 1. □

Remark 1. *If $k_1 \geq k_2 \geq 1$ and $l_2 \geq l_1 \geq 1$, then (10) becomes*

$$\begin{aligned} &\mathbf{A}(x, \sigma(y)) + \mathbf{A}(\tau(x), y) - \mathbf{A}(\tau(x), \sigma(y)) \\ &+\theta[\mathbf{p}(x, y)]^{p_1} [\mathbf{q}(x, y)]^{p_2} \mathbf{A}(\tau^{k_2+1}(x), \sigma^{l_1+1}(y)) \leq 0, \end{aligned}$$

and (4) becomes

$$\sup_{\lambda \in E, x \in I_X, y \in I_Y} \lambda\theta \left\{ \prod_{i=1}^{k_2} \prod_{j=1}^{l_1} \left(1 - \lambda[\mathbf{p}(\tau^i(x), \sigma^j(y))]^{p_1} [\mathbf{q}(\tau^i(x), \sigma^j(y))]^{p_2}\right) \right\}^{\frac{1}{\eta}} < 1.$$

where $\eta = \min\{k_2, l_1\}$. Then Theorem 1 is also true. Similarly, we can easily derive the form of Theorem 1 for the case that $k_2 \geq k_1 \geq 1$ and $l_2 \geq l_1 \geq 1$. If $k_2 \geq k_1 \geq 1$ and $l_2 \geq l_1 \geq 1$, then (10) becomes

$$\begin{aligned} &\mathbf{A}(x, \sigma(y)) + \mathbf{A}(\tau(x), y) - \mathbf{A}(\tau(x), \sigma(y)) \\ &+\theta[\mathbf{p}(x, y)]^{p_1} [\mathbf{q}(x, y)]^{p_2} \mathbf{A}(\tau^{k_1+1}(x), \sigma^{l_1+1}(y)) \leq 0, \end{aligned}$$

and (4) becomes

$$\sup_{\lambda\in E,x\in I_X,y\in I_Y}\lambda\theta\left\{\prod_{i=1}^{k_1}\prod_{j=1}^{l_1}\left(1-\lambda[\mathbf{p}(\tau^i(x),\sigma^j(y))]^{p_1}[\mathbf{q}(\tau^i(x),\sigma^j(y))]^{p_2}\right)\right\}^{\frac{1}{\eta}}<1.$$

where $\eta=\min\{k_1,l_1\}$. *Then Theorem 1 is also true.*

3. Applications

In this section we consider the two-variables functional equations with several nonlinear terms (2).

By using the inequality (see[3,p.178])

$$\sum_{i=1}^{u}\alpha_i x_i\geq\prod_{i=1}^{u}x_i^{\alpha_i}, \tag{20}$$

where $\alpha_i>0$, $\sum_{i=1}^{u}\alpha_i=1$, $x_i\geq 0$, $i=1,2,\cdots,u$. We give our result in the following.

Theorem 2. *Assume that there exist* a_i $(i=1,2,\cdots,u)$ *such that*

$$\sum_{i=1}^{u}a_i=\sum_{i=1}^{u}a_i\alpha_i=1,$$

$$\limsup_{I\ni x,y\to\infty}\prod_{i=1}^{u}[\mathbf{p}_i(x,y)]^{a_i}>0, \tag{21}$$

and there exist $X,Y\in I$ *such that*

$$\sup_{\lambda\in E,x\in I_X,y\in I_Y}\lambda\theta\left\{\prod_{i=1}^{\overline{k}}\prod_{j=1}^{\overline{l}}\left(1-\lambda\prod_{h=1}^{u}[\mathbf{p}_h(x,y)]^{a_h}\right)\right\}^{\frac{1}{\eta}}<1, \tag{22}$$

where

$$\overline{k}=\min_{1\leq i\leq u}\{k_i\},\quad \overline{l}=\min_{1\leq i\leq u}\{l_i\},\quad \eta=\min\{\overline{k},\overline{l}\},\quad \theta=\min\left\{\frac{1}{a_1},\cdots,\frac{1}{a_u}\right\}.$$

Then every solution of equation (2) oscillates.

Proof. Suppose to the contrary, we let $\mathbf{A}(x,y)$ be an eventually positive solution of (1). We define the set $S(A)$ of positive numbers as follows:

$$E=\left\{\lambda>0|1-\lambda\prod_{h=1}^{u}[\mathbf{p}_h(x,y)]^{a_h}>0\ \text{ eventually}\right\}. \tag{23}$$

Due to the condition (3.2), then the set E is bounded. From (7) we obtain

$$\mathbf{A}(\tau^{\overline{k}}(x),\sigma^{\overline{l}}(y))\leq\mathbf{A}(\tau^{k_i}(x),\sigma^{l_i}(y)),\quad i=1,2,\cdots,u. \tag{24}$$

From (2) and (24), we have

$$\begin{aligned}&\mathbf{A}(x,\sigma(y))+\mathbf{A}(\tau(x),y)-\mathbf{A}(\tau(x),\sigma(y))\\&+\sum_{i=1}^{u}\mathbf{p}_i(x,y)\left[\mathbf{A}(\tau^{\overline{k}}(x),\sigma^{\overline{l}}(y))\right]^{\alpha_i}\leq 0.\end{aligned} \tag{25}$$

Using (20) and (25), we have

$$\mathbf{A}(x,\sigma(y))+\mathbf{A}(\tau(x),y)-\mathbf{A}(\tau(x),\sigma(y)) + \theta\prod_{i=1}^{u}[\mathbf{p}_i(x,y)]^{a_i}\,\mathbf{A}(\tau^{\overline{k}}(x),\sigma^{\overline{l}}(y))\le 0. \tag{26}$$

The rest of the proof is similar to theorem 1. We omit it in detail. The proof is complete. □

For example, we consider the case $u=3$, $\alpha_3>1>\alpha_2>\alpha_1>0$. Let

$$a_1=\frac{\alpha_3-1}{2(\alpha_3-\alpha_1)},\quad a_2=\frac{\alpha_3-1}{2(\alpha_3-\alpha_2)},$$

$$a_3=\frac{2(1-\alpha_2)(\alpha_3-\alpha_1)+(\alpha_3-1)(\alpha_2-\alpha_1)}{2(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}.$$

Then $a_i>0, i=1,2,3$, $\sum_{i=1}^{3}a_i=\sum_{i=1}^{3}a_i\alpha_i=1$.

Corollary 2. *Assume that*

$$\limsup_{I\ni x,y\to\infty}\prod_{i=1}^{3}[\mathbf{p}_i(x,y)]^{a_i}>0,$$

and there exist $X,Y\in I$ such that

$$\sup_{\lambda\in E,x\in I_X,y\in I_Y}\lambda\theta\left\{\prod_{i=1}^{\overline{k}}\prod_{j=1}^{\overline{l}}\left(1-\lambda\prod_{h=1}^{3}[\mathbf{p}_h(x,y)]^{a_h}\right)\right\}^{\frac{1}{\eta}}<1,$$

where

$$\overline{k}=\min_{1\le i\le 3}\{k_i\},\quad \overline{l}=\min_{1\le i\le 3}\{l_i\},\quad \eta=\min\{\overline{k},\overline{l}\},\quad \theta=\min\left\{\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3}\right\}.$$

Then every solution of equation

$$\mathbf{A}(x,\sigma(y))+\mathbf{A}(\tau(x),y)-\mathbf{A}(\tau(x),\sigma(y))+\sum_{i=1}^{3}\big[\mathbf{p}_i(x,y)| \times\mathbf{A}(\tau^{k_i+1}(x),\sigma^{l_i+1}(y))|^{\alpha_i}\mathrm{sgn}\mathbf{A}(\tau^{k_i+1}(x),\sigma^{l_i+1}(y))\big]=0,\quad x,y\in I \tag{27}$$

is oscillatory.

From the above, we obtain the following explicit oscillation criterion.

Corollary 3. *Assume that (27) holds and for $k_2,l_2>1$*

$$\liminf_{I\ni x,y\to\infty}\frac{1}{\overline{k}\overline{l}}\sum_{i=1}^{\overline{k}}\sum_{j=1}^{\overline{l}}[\mathbf{p}_1(\tau^i(x),\sigma^j(y))]^{a_1}[\mathbf{p}_2(\tau^i(x),\sigma^j(y))]^{a_2}[\mathbf{p}_3(\tau^i(x),\sigma^j(y))]^{a_3} > \frac{\theta\delta^{\delta}}{(\delta+1)^{\delta+1}},$$

where $\overline{k}=\min_{1\le i\le 3}\{k_i\}$, $\overline{l}=\min_{1\le i\le 3}\{l_i\}$ and $\delta=\max\{\overline{k},\overline{l}\}$. Then every solution of (27) is oscillatory.

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