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ON UNIFORM EXPONENTIAL STABILITY OF VARIATIONAL DIFFERENCE EQUATIONS IN BANACH SPACES

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Abstract. In this paper we study the uniform exponential stability property for variational difference equations in Banach spaces. Characterizations of this concept are given. The obtained results can be considered as variants of the classical results due to E.Barbashin ([1]) and R.Datko ([2]) for variational difference equations.

1. Introduction

We start with some notations. Let $\mathbb{N}$ be the set of all positive integer and let Δ respectively T be the sets defined by

$$\Delta = \left\{(m,n) \in \mathbb{N}^2, \text{ with}\, m \geq n\right\}$$

respectively

$$T = \left\{(m,n,p) \in \mathbb{N}^3, \text{ with}\, m \geq n \geq p\right\}.$$

Let (X,d) be a metric space and V a real or complex Banach space. The norm on V, V^* (the dual space of V) and on $\mathcal{B}(V)$ (the Banach algebra of all bounded linear operators on V) will be denoted by $\|\cdot\|$.

Definition 1. *A mapping $\varphi : \Delta \times X \to X$ is called a discrete evolution semiflow on X if the following conditions hold:*
$s_1)\varphi(n,n,x) = x$, for all $(n,x) \in \mathbb{N} \times X$;
$s_2)\varphi(m,n,\varphi(n,p,x)) = \varphi(m,p,x)$, for all $(m,n,p,x) \in T \times X$.

Example 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function and for $s \in \mathbb{R}$ we denote $f_s(t) = f(t+s)$ for all $t \in \mathbb{R}$. Then $X = \{f_s,\, s \in \mathbb{R}\}$ is a metric space with the metric $d(x_1,x_2) = \sup\limits_{t\in\mathbb{R}} |x_1(t) - x_2(t)|$. The mapping $\varphi : \Delta \times X \to X$ defined by $\varphi(m,n,x) = x_{m-n}$ is a discrete evolution semiflow.*

Let $\varphi : \Delta \times X \to X$ be a discrete evolution semiflow on X and let $A : X \to \mathcal{B}(V)$.

In this paper we suppose that there exists $f : \Delta \times X \to V$ such that:

$$f(m+1,n,x) = A(\varphi(m,n,x))\, f(m,n,x)$$

for all $(m,n,x) \in \Delta \times X$.

If we denote

$$v_m(n,x) = f(m,n,x)$$

then the preceding equality can be rewritten as

$$(A,\varphi) \quad v_{m+1}(n,x) = A(\varphi(m,n,x))\, v_m(n,x)$$

and it is called the *discrete variational system associated to the pair* (A,φ).

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For $(m,n) \in \Delta$ we define the mapping $\Phi_m^n : X \to \mathcal{B}(V)$ by

$$\Phi_m^n(x)v = \begin{cases} A(\varphi(m-1,n,x)) \dots A(\varphi(n+1,n,x))\, A(x)v, & \text{if } m > n \\ v, & \text{if } m = n. \end{cases}$$

Remark 1. *From the definitions of v_m and Φ_m^n it follows that:*
c_1) $\Phi_m^m(x)v = v$, for all $(m,x,v) \in \mathbb{N} \times X \times V$;
c_2) $\Phi_m^p(x) = \Phi_m^n(\varphi(n,p,x))\, \Phi_n^p(x)$, for all $(m,n,p,x) \in T \times X$;
c_3) $v_m(n,x) = \Phi_m^n(x) v_n(n,x)$, for all $(m,n,x) \in \Delta \times X$.

The properties (c_1) and (c_2) shows that the mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$ defined by

$$\Phi(m,n,x)v = \Phi_m^n(x)v$$

for all $(m,n,x,v) \in \Delta \times X \times V$ is a discrete evolution cocycle over discrete evolution semiflow φ.

The general concept of evolution cocycle was introduced by M.Megan and C.Stoica in [5]. It generalizes the classical notions of C_0-semigroups, evolution operators and linear skew-product semiflows.

The goal of this paper is to give characterizations for the uniform exponential stability for discrete variational systems in Banach spaces. Some results in the continuous case has been obtained by M.Megan and C.Stoica in [6] and [7]. Thus we obtain generalizations of the well-known results due to E.Barbasin ([1]) and R.Datko ([2]) for discrete variational systems in Banach spaces. We remark that our proofs are not discretizations of the proofs from [1] and [2].

2. Uniform exponential stability

Let (A,φ) be a discrete variational system associated to the discrete evolution semiflow $\varphi : \Delta \times X \to X$ and to mapping $A : X \to \mathcal{B}(V)$.

Definition 2. *The system (A,φ) is said to be uniformly exponentially stable (and denote u.e.s.) if there are the constants $N \geq 1$ and $\alpha > 0$ such that:*

$$e^{\alpha(m-n)} \|\Phi_m^n(x)v\| \leq N \|v\|$$

for all $(m,n,x,v) \in \Delta \times X \times V$.

Remark 2. *It is easy to see that (A,φ) is uniformly exponentially stable if and only if there are $N \geq 1$ and $\alpha > 0$ with*

$$e^{\alpha(m-n)} \|\Phi_m^p(x)v\| \leq N \|\Phi_n^p(x)v\|$$

for all $(m,n,p,x,v) \in T \times X \times V$.

Example 2. *Let $\mathcal{C} = \mathcal{C}(\mathbb{R},\mathbb{R})$ be the metric space of all continuous functions $x : \mathbb{R} \to \mathbb{R}$, with the topology of uniform convergence on compact subsets of $\mathbb{R}$. $\mathcal{C}$ is metrizable relative to the metric given in Example 1.*

Let $f : \mathbb{R}_+ \to (0,\infty)$ be a decreasing function with the property that there exists $\lim\limits_{t\to\infty} f(t) = \alpha > 0$. We denote by X the closure in $\mathcal{C}$ of the set $\{f_t, t \in \mathbb{R}\}$, where $f_t(s) = f(t+s)$ for all $s \in \mathbb{R}_+$. The mapping $\varphi : \Delta \times X \to X$ defined by $\varphi(m,n,x) = x_{m-n}$ is a discrete evolution semiflow.

Let us consider the Banach space $V = \mathbb{R}$ and let $A : X \to \mathcal{B}(V)$ defined by

$$A(x)v = e^{-\int_0^1 x(\tau)d\tau} v$$

for all $(x,v) \in X \times V$.

Hence, if we calculate

$$A\left(\varphi(n+1,n,x)\right)v=A(x_1)v=e^{-\int_0^1 x_1(\tau)d\tau}v=$$
$$=e^{-\int_0^1 x(\tau+1)d\tau}v=e^{-\int_1^2 x(\tau)d\tau}v$$
$$A\left(\varphi(n+2,n,x)\right)v=A(x_2)v=e^{-\int_0^1 x_2(\tau)d\tau}v=$$
$$=e^{-\int_0^1 x(\tau+2)d\tau}v=e^{-\int_2^3 x(\tau)d\tau}v$$

and, further

$$A\left(\varphi(m-1,n,x)\right)v=A(x_{m-n-1})v=e^{-\int_0^1 x_{m-n-1}(\tau)d\tau}v=$$
$$=e^{-\int_0^1 x(m-n-1+\tau)d\tau}v=e^{-\int_{m-n-1}^{m-n} x(\tau)d\tau}v$$

for all $(m,n,x,v)\in\Delta\times X\times V$*, then we obtain that*

$$\Phi_m^n(x)v=\begin{cases} e^{-\int_0^{m-n} x(\tau)d\tau}v, & \text{if } m>n\\ v, & \text{if } m=n.\end{cases}$$

According to the properties of function x, it follows that

$$\left\|\Phi_m^n(x)v\right\|\le e^{-\alpha(m-n)}\left\|v\right\|$$

for all $(m,n,x,v)\in\Delta\times X\times V$*, showing that* (A,φ) *is u.e.s.*

Lemma 1. *The following assertions are equivalent:*
(i) the system (A,φ) *is uniformly exponentially stable;*
(ii) there exists a decreasing sequence of real numbers (a_n) *with* $a_n\to 0$ *such that:*

$$\left\|\Phi_m^p(x)v\right\|\le a_{m-n}\left\|\Phi_n^p(x)v\right\|$$

for all $(m,n,p,x,v)\in T\times X\times V$*;*
(iii) there is a decreasing sequence of real numbers (a_n) *with* $a_n\to 0$ *such that:*

$$\left\|\Phi_m^n(x)v\right\|\le a_{m-n}\left\|v\right\|$$

for all $(m,n,x,v)\in\Delta\times X\times V$*;*
(iv) there exist a constant $a\in(0,1)$ *and* $k\in\mathbb{N}^*$ *such that:*

$$\left\|\Phi_m^n(x)v\right\|\le a\left\|v\right\|$$

for all $(m,n,x,v)\in\Delta\times X\times V$ *with* $m\ge n+k$*;*
(v) there exist a constant $a\in(0,1)$ *and* $k\in\mathbb{N}^*$ *such that:*

$$\left\|\Phi_m^p(x)v\right\|\le a\left\|\Phi_n^p(x)v\right\|$$

for all $(m,n,p,x,v)\in T\times X\times V$ *with* $m\ge n+k$*.*

Proof. It is easy to see that the implications $(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(iv)\Rightarrow(v)$ are true.
$(v)\Rightarrow(iv)$ It is immediate for $p=n$.
$(iv)\Rightarrow(i)$ For all $(m,n)\in\Delta$ with $m-n>k$ there exist $p\le k$ with $(m,n,p)\in T$, $q\in\mathbb{N}$ and $r\in[0,p)$ such that $m=n+pq+r$.

From hypothesis and Remark 1, we obtain that

$$\left\|\Phi_m^n(x)v\right\|=\left\|\Phi_{n+pq+r}^n(x)v\right\|=$$
$$=\left\|\Phi_{n+pq+r}^{n+pq}\left(\varphi(n+pq,n,x)\right)\Phi_{n+pq}^n(x)v\right\|\le a\left\|\Phi_{n+pq}^n(x)v\right\|=$$

$$= a \left\| \Phi_{n+pq}^{n+(q-1)p} \left(\varphi(n+(q-1)p, n, x)\right) \Phi_{n+(q-1)p}^{n}(x)v \right\| \leq$$

$$\leq a^2 \left\| \Phi_{n+(q-1)p}^{n}(x)v \right\| =$$

$$= a^2 \left\| \Phi_{n+(q-1)p}^{n+(q-2)p} \left(\varphi(n+(q-2)p, n, x)\right) \Phi_{n+(q-2)p}^{n}(x)v \right\| \leq$$

$$\leq a^3 \left\| \Phi_{n+(q-2)p}^{n}(x)v \right\| \leq \ldots \leq a^{q+1} \|v\| =$$

$$= a e^{q \ln a} \|v\| = a e^{\frac{m-n-r}{p} \ln a} \|v\| = a e^{\alpha r} e^{-\alpha(m-n)} \|v\| \leq$$

$$\leq e^{-\alpha(m-n)} \|v\| = e^{-\alpha(m-n)} \|v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$, where $\alpha = -\frac{\ln a}{p}$ and the proof is complete. □

An important result for uniform exponential stability of discrete variational systems is given by

Theorem 1. *The following assertions are equivalent:*
(i) the system (A, φ) is uniformly exponentially stable;
(ii) there are $d > 0$ and $D \geq 1$ such that:

$$\sum_{k=m}^{\infty} e^{d(k-m)} \|\Phi_k^n(x)v\| \leq D \|\Phi_m^n(x)v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$;
(iii) there exists $D \geq 1$ such that:

$$\sum_{k=m}^{\infty} \|\Phi_k^n(x)v\| \leq D \|\Phi_m^n(x)v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate.
For $(iii) \Rightarrow (i)$ we observe that (iii) implies that

$$\|\Phi_m^n(x)v\| \leq D \|v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$.
Then

$$(m-n+1) \|\Phi_m^n(x)v\| = \sum_{k=n}^{m} \|\Phi_m^n(x)v\| =$$

$$= \sum_{k=n}^{m} \left\| \Phi_m^k \left(\varphi(k, n, x)\right) \Phi_k^n(x)v \right\| \leq$$

$$\leq D \sum_{k=n}^{m} \|\Phi_k^n(x)v\| \leq D^2 \|v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$ and from Lemma 1 we have that (A, φ) is u.e.s. □

Remark 3. *Theorem 1 is a generalization of a theorem due to Datko ([2]) for the case of uniform exponential stability of evolution operators.*

Theorem 2. *The following assertions are equivalent:*
(i) the system (A, φ) *is uniformly exponentially stable;*
(ii) there are $b > 0$ *and* $B \geq 1$ *such that:*

$$\sum_{k=n}^{m} e^{b(m-k)} \left\| \left(\Phi_m^k(x)\right)^* v^* \right\| \leq B \left\| \left(\Phi_m^n(x)\right)^* v^* \right\|$$

for all $(m, n, x, v^*) \in \Delta \times X \times V^*$;
(iii) there exist $b > 0$ *and* $B > 1$ *such that:*

$$\sum_{k=n}^{m} \left\| \left(\Phi_m^k(x)\right)^* v^* \right\| \leq B \left\| \left(\Phi_m^n(x)\right)^* v^* \right\|$$

for all $(m, n, x, v^*) \in \Delta \times X \times V^*$.

Proof. The implication $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate.
$(iii) \Rightarrow (i)$ The hypothesis implies that

$$\left\| \left(\Phi_m^n(x)\right)^* v^* \right\| \leq B \left\| v^* \right\|$$

for all $(m, n, x, v^*) \in \Delta \times X \times V^*$.

Then, for $v^* = \left(\Phi_m^k\left(\varphi(k, n, x)\right)\right)^* u^*$ we obtain

$$\left\| \left(\Phi_m^n(x)\right)^* u^* \right\| = \left\| \left(\Phi_k^n(x)\right)^* \left(\Phi_m^k\left(\varphi(k, n, x)\right)^* u^* \right\| \leq$$

$$\leq B \left\| \left(\Phi_m^k\left(\varphi(k, n, x)\right)^* u^* \right\|.$$

Hence,

$$(m - n + 1) \left\| \left(\Phi_m^n(x)\right)^* u^* \right\| = \sum_{k=n}^{m} \left\| \left(\Phi_m^n(x)\right)^* u^* \right\| \leq$$

$$\leq B \sum_{k=n}^{m} \left\| \left(\Phi_m^k\left(\varphi(k, n, x)\right)^* u^* \right\|$$

for all $(m, n, x, v^*) \in \Delta \times X \times V^*$. As a conclusion, it follows from Lemma 1 that (A, φ) is u.e.s. □

Remark 4. *Theorem 2 is a generalization of a theorem due to Barbashin ([1]) for the case of uniform exponential stability of evolution operators.*

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