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CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH CAPUTO'S FRACTIONAL DIFFERENTIATION

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Abstract. In this paper, we define a class $T_{\eta,\lambda}(\alpha,\beta,A,B)$ of analytic functions involving the integral operator given by the authors. Several results like characterization property, distortion theorem and other interesting properties of the same class are provided.

1. Introduction

Due to the extension of the theory of univalent functions, many authors have applied various operators to create new subclasses. Here specifically we look at the fractional calculus operators being used to introduce new subclasses of analytic functions (see [4], [6]) as it's convenient to replace certain fractional integral or differential operator into extensions of well know classes of analytic functions to produce new subclasses. In our case we, are into involving an integral operator presented by [7] to define a subclass of analytic function and then to further investigate the given subclass into its characterization (see [6]), such as the distortion theorem ([3],[5]). Also we can get interesting results by applying Hadamard product ([2]).

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Also denote by T the subclassof A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

In [7], Caputo's definition of the fractional-order derivative has been used to introduce the integral operator $J_{\eta,\lambda}f(z)$ that we will be used as an application to the class $S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$ (see [1]) that has been studied in special cases by many authors like ([3], [4], [5]) by considering different operators. Note also that the modified fractional derivative operator of Saigo (see [1] and [8]) has also been used to study their classes.

Salah and Darus [7] made use of the following Caputo's definition of the fractional-order derivative

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau$$

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where $n-1 < Re(\alpha) \leq n, n \in N$, and the parameter α is allowed to be real or even complex, a is the initial value of the function f, together with the generalization operator of Salagean derivative operator and Libera integral operator

$$\Omega^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)} a_n z^n$$

for any real λ, we introduce the following integral operator

$$J_{\eta,\lambda} f(z) = \frac{\Gamma(2+\eta-\lambda)}{\Gamma(\eta-\lambda)} z^{\lambda-\eta} \int_0^z \frac{\Omega^{\eta} f(\xi)}{(z-\xi)^{\lambda+1-\eta}} d\xi$$

where η(real number) and $(\eta - 1 < \lambda \leq \eta < 2)$.

This operator has the following Taylor series expansion

$$J_{\eta,\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{(\Gamma(n+1))^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} a_n z^n,$$

that satisfies $J_{0,0} = f(z)$, $J_{1,1} = zf'(z)$.
(For further details you may refer to [7]).

Now, depending on the previous contribution we introduce a subclass $S_{\eta,\lambda}(\alpha, \beta, A, B)$ of analytic functions $f \in A$ and satisfying the condition

$$\left| \frac{(J_{\eta,\lambda})' f(z) - 1}{(B + (A-B)(1-\alpha)) - B(J_{\eta,\lambda})' f(z)} \right| < \beta$$

$z \in U$, $0 \leqslant \eta < 2, \eta - 1 < \lambda \leqslant \eta, 0 \leqslant \alpha < 1, 0 < \beta \leqslant 1, -1 \leqslant B < A \leqslant 1, 0 < A \leqslant 1$,

$$J_{\eta,\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{(\Gamma(n+1))^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} a_n z^n$$

Further, we define

$$T_{\eta,\lambda}(\alpha, \beta, A, B) = S_{\eta,\lambda}(\alpha, \beta, A, B) \cap T$$

Next ,we will start our work by studying the characterization properties.

2. Characterization

Theorem 1. *A function f defined by (2) is in the class $T_{\eta,\lambda}(\alpha, \beta, A, B)$ if and only if*

$$\sum_{n=2}^{\infty} n\varphi(n)(1 - B\beta) a_n \leqslant \beta(A-B)(1-\alpha) \tag{3}$$

where

$$\varphi(n) = \frac{(\Gamma(n+1))^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}.$$

Proof. Suppose that (3) holds true, and let $|z| = 1$. Then

$$\left|(J_{\eta,\lambda})' f(z) - 1\right| - \beta\left|(B + (A-B)(1-\alpha)) - B(J_{\eta,\lambda})' f(z)\right|$$

$$= \left| - \sum_{n=2}^{\infty} n\varphi(n) a_n z^{n-1}\right| - \beta\left|((A-B)(1-\alpha)) + B\sum_{n=2}^{\infty} n\varphi(n) a_n z^{n-1}\right|$$

$$\leqslant \sum_{n=2}^{\infty} n\varphi(n)(1-\beta B) a_n - (A-B)(1-\alpha) \leqslant 0 \text{ , by hypothesis.}$$

Therefore, it follows that $f \in T_{\eta,\lambda}(\alpha, \beta, A, B)$

Conversely, suppose that $f \in T_{\eta,\lambda}(\alpha, \beta, A, B)$.

Then

$$\left|\frac{(J_{\eta,\lambda})' f(z) - 1}{(B+(A-B)(1-\alpha)) - B(J_{\eta,\lambda})' f(z)}\right| = \frac{\left|\sum_{n=2}^{\infty} n\varphi(n) a_n z^{n-1}\right|}{\left|((A-B)(1-\alpha)) + B\sum_{n=2}^{\infty} n\varphi(n) a_n z^{n-1}\right|} < \beta, (z \in U).$$

Since $\mathbb{R}(z) \leqslant |z|$, for all z, we get

$$\mathbb{R}\left(\frac{\left|\sum_{n=2}^{\infty} n\varphi(n) a_n z^{n-1}\right|}{\left|((A-B)(1-\alpha)) + B\sum_{n=2}^{\infty} n\varphi(n) a_n z^{n-1}\right|}\right) < \beta.$$

Now choosing the value of z on real axis, simplifying and letting $z \to 1^-$ through the real values, we get

$$\sum_{n=2}^{\infty} n\varphi(n) a_n \leqslant \beta(A-B)(1-\alpha) + B\beta\sum_{n=2}^{\infty} n\varphi(n) a_n,$$

which yields (3).

We also note that the assertion (3) is sharp and extremal function is given by

$$f(z) = z - \frac{\beta(A-B)(1-\alpha)}{n(1-B\beta)\varphi(n)} z^n, (n \geqslant 2).$$

□

Corollary 1. *Let the function f defined by (2) belongs to the class $T_{\eta,\lambda}(\alpha, \beta, A, B)$. Then*

$$a_n \leqslant \frac{\beta(A-B)(1-\alpha)}{n(1-B\beta)\varphi(n)} z^n, (n \geqslant 2).$$

3. Distortion theorem

Theorem 2. *Let the function f defined by (2) belongs to the class $T_{\eta,\lambda}(\alpha, \beta, A, B)$. Then*

$$|f(z)| \geqslant |z| - \frac{\beta(A-B)(1-\alpha)(2-\eta)(2+\eta-\lambda)}{8(1-B\beta)} |z|^2. \quad (4)$$

And

$$|f(z)| \leq |z| + \frac{\beta(A-B)(1-\alpha)(2-\eta)(2+\eta-\lambda)}{8(1-B\beta)} |z|^2 \quad (5)$$

Proof. If $f(z) \in T_{\eta,\lambda}(\alpha,\beta,A,B)$, then by virtue of Theorem 1, we have

$$2\varphi(2)(1-B\beta)\sum_{n=2}^{\infty} a_n \leqslant \sum_{n=2}^{\infty} n\varphi(n)a_n(1-B\beta) \leqslant \beta(A-B)(1-\alpha).$$

This yields

$$\sum_{n=2}^{\infty} a_n \leqslant \frac{\beta(A-B)(1-\alpha)}{2\varphi(2)(1-B\beta)} = \frac{\beta(A-B)(1-\alpha)(2-\eta)(2+\eta-\lambda)}{8(1-B\beta)}.$$

Now,

$$|f(z)| \geqslant |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geqslant |z| - \frac{\beta(A-B)(1-\alpha)(2-\eta)(2+\eta-\lambda)}{8(1-B\beta)}|z|^2.$$

Also

$$|f(z)| \leqslant |z| + |z|^2 \sum_{n=2}^{\infty} a_n \leqslant |z| + \frac{\beta(A-B)(1-\alpha)(2-\eta)(2+\eta-\lambda)}{8(1-B\beta)}|z|^2,$$

which prove the assertion (4) and (5). □

Corollary 2. *Under the hypothesis of Theorem 2, $f(z)$ is included in a disk with its center at the origin and radius r given by*

$$r = 1 + \frac{\beta(A-B)(1-\alpha)(2-\eta)(2+\eta-\lambda)}{8(1-B\beta)}.$$

4. Properties of the class $T_{\eta,\lambda}(\alpha,\beta,A,B)$

Now we study some properties of the class $T_{\eta,\lambda}(\alpha,\beta,A,B)$.

Theorem 3. *Let $0 \leqslant \eta < 2, \eta - 1 < \lambda \leqslant \eta, 0 \leqslant \alpha < 1, 0 < \beta \leqslant 1, -1 \leqslant B < A \leqslant 1, 0 < A \leqslant 1, 0 \leqslant \alpha' < 1, 0 < \beta' \leqslant 1, -1 \leqslant B' < A' \leqslant 1, 0 < A' \leqslant 1$. Then*

$$T_{\eta,\lambda}(\alpha,\beta,A,B) = T_{\eta,\lambda}\left(\alpha',\beta',A',B'\right) \tag{6}$$

if and only if

$$\frac{\beta(A-B)(1-\alpha)}{(1-B\beta)} = \frac{\beta'\left(A'-B'\right)\left(1-\alpha'\right)}{(1-B'\beta')}. \tag{7}$$

Proof. Assume that $f \in T_{\eta,\lambda}(\alpha,\beta,A,B)$, and let the condition (7) holds true. By using Theorem 1 we have

$$\sum_{n=2}^{\infty} n\varphi(n)a_n \leqslant \frac{\beta(A-B)(1-\alpha)}{(1-B\beta)} = \frac{\beta'\left(A'-B'\right)\left(1-\alpha'\right)}{(1-B'\beta')},$$

which shows that $f \in T_{\eta,\lambda}\left(\alpha',\beta',A',B'\right)$ by Theorem 1.

Reversing the above steps, we can establish the other part of the equivalence of (6). Conversely, the assertion (6) can be used to imply the condition (7) and this completes the proof of Theorem 3. □

Theorem 4. *Let* $0 \leqslant \eta < 2, \eta - 1 < \lambda \leqslant \eta, 0 \leqslant \alpha_1 \leqslant \alpha_2 < 1, 0 < \beta \leqslant 1, -1 \leqslant B < A \leqslant 1, 0 < A \leqslant 1$. *Then*

$$T_{\eta,\lambda}(\alpha_2, \beta, A, B) \subset T_{\eta,\lambda}(\alpha_1, \beta, A, B).$$

Proof. The result follows easily from Theorem 1. □

5. Results involving Hadamard Product

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geqslant 0$, $g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geqslant 0$.

Then the modified Hadamard product is given by $(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$.

Theorem 5. *Let the functions* $f_i(z), i = 1, 2, ...r.$ *be defined by*

$$f_i(z) = z - \sum_{n=2}^{\infty} C_{n,i} z^n, (C_{n,i} \geqslant 0) \tag{8}$$

be in the class $T_{\eta,\lambda}(\alpha_i, \beta_i, A_i, B_i)$ *respectively. Then*

$$f_1 * f_2 * \cdots * f_r(z) \in T_{\eta,\lambda}\left(\prod_{i=1}^{r} \alpha_i, \prod_{i=1}^{r} \beta_i, \prod_{i=1}^{r} A_i, \prod_{i=1}^{r} B_i\right).$$

The result is sharp.

Proof. By hypothesis , $f_i(z) \in T_{\eta,\lambda}(\alpha_i, \beta_i, A_i, B_i)$,for all $i = 1, 2, \cdots r$. Therefore by Theorem 1, we have

$$\sum_{n=2}^{\infty} n\varphi(n)(1 - B_i\beta_i)C_{n,i} \leqslant \beta_i(A_i - B_i)(1 - \alpha_i), \tag{9}$$

and

$$\sum_{n=2}^{\infty} C_{n,i} \leqslant \frac{\beta_i(1-\alpha_i)(A_i - B_i)(2-\eta)(2+\eta-\lambda)}{8(1 - B_i\beta_i)}. \tag{10}$$

For β_i satisfying $0 < \beta_i \leqslant 1$, we observe that

$$\sum_{n=2}^{\infty} n\varphi(n)\left[1 - \prod_{i=1}^{r} B_i\beta_i\right] \prod_{i=1}^{r} C_{n,i} \leqslant \sum_{n=2}^{\infty} n\varphi(n)[1 - B_r\beta_r] \prod_{i=1}^{r} C_{n,i}$$
$$= \sum_{n=2}^{\infty} \{n\varphi(n)[1 - B_r\beta_r] C_{n,r}\} \prod_{i=1}^{r-1} C_{n,i}.$$

Now, using (9) for any fixed $i = r$, and (10) for the rest, it follows that

$$\sum_{n=2}^{\infty} n\varphi(n)\left[1 - \prod_{i=1}^{r} B_i\beta_i\right] \prod_{i=1}^{r} C_{n,i} \leqslant \frac{\beta_r(A_r - B_r)(1-\alpha_r) \prod_{i=1}^{r-1} \beta_i(A_i - B_i)(1-\alpha_i)(2-\eta)^{r-1}(2+\eta-\lambda)^{r-1}}{8^{r-1} \prod_{i=1}^{r-1}(1 - B_i\beta_i)}$$

$$\leqslant \prod_{i=1}^{r} \beta_r \prod_{i=1}^{r} (A_i - B_i) \prod_{i=1}^{r} (1-\alpha_i) \left[\frac{(2-\eta)(2+\eta-\lambda)}{8\left(1-\max\limits_{1\leqslant i\leqslant r} B_i\beta_i\right)} \right]^{r-1}$$

$$\leqslant \prod_{i=1}^{r} \beta_i \left[\prod_{i=1}^{r} A_i - \prod_{i=1}^{r} B_i \right] \left[\prod_{i=1}^{r} (1-\alpha_i) \right]$$

because $0 < \left[\frac{(2-\eta)(2+\eta-\lambda)}{8\left(1-\max\limits_{1\leqslant i\leqslant r} B_i\beta_i\right)} \right]^{r-1} < 1$

Hence with the aid of Theorem 3, the proof is complete. □

Corollary 3. *Let the functions $f_1, f_2 \cdots, f_r$ defined by (8) be in the class $T_{\eta,\lambda}(\alpha, \beta, A, B)$. Then*

$$f_1 * f_2 * \cdots * f_r(z) \in T_{\eta,\lambda}(\alpha^r, \beta^r, A^r, B^r).$$

Theorem 6. *Let the functions $f_i(z)\,(i = 1, 2)$, defined by (8) be in the class $T_{\eta,\lambda}(\alpha, \beta, A, B)$. Then*

$$f_1 * f_2(z) \in T_{\eta,\lambda}(\sigma, \beta, A, B), \tag{11}$$

where

$$\sigma = \sigma(\alpha, \beta, A, B) = 1 - \frac{\beta(A-B)(1-\alpha)^2(2+\eta-\lambda)(2-\eta)}{8(1-B\beta)}, \tag{12}$$

the result is sharp.

Proof. We need to find the largest σ such that

$$\sum_{n=2}^{\infty} \frac{n\varphi(n)(1-B\beta)}{\beta(A-B)(1-\sigma)} C_{n,1}C_{n,2} \leqslant 1,$$

where σ is the function given by (12). By Cauchy-Schwarz inequality, it follows from Theorem 1 that

$$\sum_{n=2}^{\infty} \frac{n\varphi(n)(1-B\beta)}{\beta(A-B)(1-\alpha)} \sqrt{C_{n,1}C_{n,2}} \leqslant 1. \tag{13}$$

Let's find σ such that

$$\sum_{n=2}^{\infty} \frac{n\varphi(n)(1-B\beta)}{\beta(A-B)(1-\sigma)} C_{n,1}C_{n,2} \leqslant \sum_{n=2}^{\infty} \frac{n\varphi(n)(1-B\beta)}{\beta(A-B)(1-\alpha)} \sqrt{C_{n,1}C_{n,2}} \leqslant 1,$$

which implies that

$$\sqrt{C_{n,1}C_{n,2}} \leqslant \frac{1-\sigma}{1-\alpha}, n \geqslant 2.$$

In view of (13), it suffices to find the largest σ such that

$$\frac{\beta(A-B)(1-\alpha)}{n\varphi(n)(1-B\beta)} \leqslant \frac{1-\sigma}{1-\alpha},$$

which yields

$$\sigma \leqslant 1 - \frac{\beta (A-B)(1-\alpha)^2}{n\varphi (n)(1-\beta B)}.$$

That is

$$\sigma \leqslant 1 - \frac{\beta (A-B)(1-\alpha)^2}{(1-\beta B)}\theta (n),$$

where

$$\theta (n) = \frac{1}{n\varphi (n)}.$$

Noting that $\theta (n)$ is decreasing function of n. Then

$$\sigma \leqslant \sigma (\alpha, \beta, A, B) = 1 - \frac{\beta (A-B)(1-\alpha)^2}{(1-B\beta)}\theta (2).$$

Hence (11) is proved. The result is sharp by the considering the function

$$f_i (z) = z - \frac{\beta (A-B)(1-\alpha)^2 (2+\eta-\lambda)(2-\eta)}{8(1-B\beta)} z^2, (i=1,2).$$

□

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