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A GENERALIZED GOLDBACH (DESCARTES) BINARY PROBLEM

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Abstract. In this paper the authors will introduce a Diophantine Equation which can be considered as the generalized Descartes (misnamed Goldbach) binary problem. The authors do not solve the problem but give some important hints which could lead to the solution of this problem. Also we should not be surprised if it will take again more than 360 years until a complete solution will be given to our generalized conjecture.

1. Introduction

On June 17th, 1742, in a letter to Euler, Goldbach stated:

"That every even integer is the sum of two integers, p and q, where each of p and q are either one or odd primes".

In the other words, that every integer n greater than five is the sum of three primes. Euler replied to Goldbach:

"There is little doubt that this result is true, but that every even number is a sum of two primes, I consider an entirely certain theorem in spite of that I am not able to demonstrate it".

Therefore we have the Goldbach's conjecture:

Conjecture 1. *(Goldbach) Every even integer n greater than two is the sum of two primes.*

Descartes was the first who knew this two primes decomposition version of this conjecture before Goldbach did. Erdos said that the conjecture is misnamed and it "is better that the conjecture be named after Goldbach because mathematically speaking, Descartes was infinitely rich and Goldbach was poor".

In this paper we deal with the quantity of solutions of the Diophantine equation

$$t = p_1 + p_2^k, \tag{1}$$

where, $k >> 2$, t is an even natural number, and p_1, p_2 are odd primes.

The case $k = 1$ is of course the classical binary Goldbach problem, solved in [1], [2] and [3].

Needless to say, it is commonly considered as a more intractable problem that was the Goldbach problem in its moment.

We do not solve this problem, but establish the firm grounds on which it could be solved.

At difference with the binary Goldbach decomposition not every small even number is decomposable in the form (1), and it seems at first sight that it is highly doubtful if the majority of even numbers admit such decomposition.

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2. The starting formula

Denote with $N(t)$ the quantity of solutions of equation (1) then we have the exact relation

$$N(t) = \int_0^t \Delta\pi(x+1)\Delta\pi_k(t-x)dx \tag{2}$$

of ref [4], where

$$\Delta\pi(x) = \pi(x) - \pi(x-1),$$
$$\Delta\pi_k(x) = \pi(\sqrt[k]{x}) - \pi(\sqrt[k]{x-1}).$$

Furthermore, we have the asymptotic relations

$$\Delta\pi(x) \sim \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \cdot \frac{C_q(u)}{\log u} \tag{3}$$

obtained in [4], and

$$\Delta\pi_k(x) \sim \frac{x^{\frac{1}{k}-1}}{\log x} \sum_{q=1}^{\infty} \sum_{\substack{h=0 \\ (h,q)=1}}^{q-1} \frac{W(k,q,h)}{\varphi(q)} \cdot e^{-2\pi i \frac{h}{q} x}. \tag{4}$$

(to be published in a forth coming book Diophantine equations by A. Peretti).

Replacing the appropriate values of (3) and (4) in (2) we get:

$$N(t) \sim \int_0^t \left\{ \sum_{q_1=1}^{\infty} \frac{\mu(q_1)}{\varphi(q_1)} \cdot \frac{C_{q_1}(x+1)}{\log(x+1)} \right\} \cdot \tag{5}$$
$$\cdot \left\{ \frac{(t-x)^{\frac{1}{k}-1}}{\log(t-x)} \sum_{q_2=1}^{\infty} \sum_h \frac{W(k,q_2,h)}{\varphi(q_2)} \cdot e^{-2\pi i \frac{h}{q_2}(t-x)} \right\} dx =$$
$$= \int_0^t \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \cdot \frac{C_q(x+1)}{\log(x+1)} \cdot \frac{(t-x)^{\frac{1}{k}-1}}{\log(t-x)} dx +$$
$$+ \int_0^t \left\{ \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \cdot \frac{C_q(x+1)}{\log(x+1)} \right\} \cdot$$
$$\cdot \left\{ \frac{(t-x)^{\frac{1}{k}-1}}{\log(t-x)} \sum_{q=2}^{\infty} \sum_h \frac{W(k,q,h)}{\varphi(q)} \cdot e^{-2\pi i \frac{h}{q}(t-x)} \right\} dx.$$

3. The singular series of the problem

According to what occurs in the binary Goldbach problem, we know that the dominant term in (5) is obtained when we select there only the terms with $q_1 = q_2$. Hence we can put

$$N(t) \sim \int_0^t \sum \frac{\mu(q)}{\varphi^2(q)} C_q(x) \left(\sum W(k,q,h) \cdot e^{-2\pi i \frac{h}{q}(t-x)} \right) \cdot \tag{6}$$
$$\cdot \frac{(t-x)^{\frac{1}{k}-1}}{\log(x+1) \cdot \log(t-x)} dx.$$

Performing integration by parts we obtain:

$$N(t) \sim \sum \frac{\mu(q)}{\varphi^2(q)} C_q(t) \sum W(k,q,h) \int_2^t \frac{(t-x)^{\frac{1}{k}-1}}{\log(x+1)\cdot\log(t-x)} dx + T_0 \tag{7}$$

$$\sim \sum \frac{\mu(q)}{\varphi^2(q)} C_q(t) \sum W(k,q,h) \frac{kt^{\frac{1}{k}-1}}{\log^2 t} + T_0,$$

where, T_0 is the second term of the integration by parts.

If we assume that T_0 is a negligible term with respect to the first one, then the series in (7) is the singular series to the problem.

If in change we apply the first mean value theorem of the integral calculus:

$$\int_p^q f(x)g(x)dx = f(\xi)\int_p^q g(x)dx, \qquad p \le \xi \le q.$$

In (6) we obtain:

$$N(t) \sim \sum_q \frac{\mu(q)}{\varphi^2(q)} C_q(\xi) \left(\sum_h W(k,q,h)\cdot e^{-2\pi i \frac{h}{q}(t-\xi)}\right) \cdot \tag{8}$$

$$\cdot \int_0^t \frac{(t-x)^{\frac{1}{k}-1}}{\log(x+1)\cdot\log(t-x)} dx$$

$$\sim \sum_q \frac{\mu(q)}{\varphi^2(q)} C_q(\xi) \left(\sum_h W(k,q,h)\cdot e^{-2\pi i \frac{h}{q}(t-\xi)}\right) \frac{kt^{\frac{1}{k}-1}}{\log^2 t}.$$

From (8) follows the theorem:

Theorem 1. *The equation (1), for arbitrary fixed k, has solution only for those values of n such that the singular series is positive.*

4. Comparison with known results

L. K. Hua has proved, by the circle method, in ref [4] that the equation has solution for almost all even numbers.

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