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# GROWTH AND WEIGHTED POLYNOMIAL APPROXIMATION OF ANALYTIC FUNCTIONS

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Abstract. Let $H_R$ be the class of functions analytic in $G_R$ but not in $G_{R'}$ if $R < R'$, $G_{R_o} = \text{int } S_{R_o}$, $0 < R_o < R < 1$ and $S_{R_o} = \{z \in \mathbb{C} : |ze^{1-z}| = R_o,\ |z| \leq 1\}$. This paper deals with the characterization of rate of decay of weighted approximation error on $S_{R_o}$, in terms of order and type of $f \in H_R$.

## 1. Introduction

Szegö [11] showed that the normalized partial sum $s_n(nz)$ satisfies the following equation:

$$e^{-nz} s_n(nz) = 1 - \frac{\sqrt{n}}{\tau_n \sqrt{2\pi}} \int_0^z \left(\zeta e^{1-\zeta}\right)^n d\zeta, \quad n \geq 1, \quad z \in \mathbb{C}, \tag{1}$$

where $s_n(z) = \sum_{k=0}^n (z^k/k!)$ and from Sterling's asymptotic series formula, see Henrici [2]

$$\tau_n = \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} \simeq 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots,\ n \to \infty$$

so that $\tau_n \to 1$ as $n \to \infty$.

The curve $\{z \in \mathbb{C} : |\phi(z)| = 1\}$, $\phi(z) = ze^{1-z}$ introduced by Szegö divides the complex plane $\mathbb{C}$ into three domains: One of them is the bounded domain $G$ contained in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ whose boundary consists of that part of the Szegö curve, *i.e.*,

$$S = \{z \in \mathbb{C} : |ze^{1-z}| = 1,\ |z| \leq 1\}$$

which is contained in the closed unit disc. $S$ is a piecewise analytic Jordan curve with one corner point at $z = 1$. Szegö [11] proved that $G$, in the $z$-plane is mapped conformally onto the unit disc $D$, in the $w$-plane, by the function $w = \phi(z)$, and that the unbounded domains, also determined from the Szegö curve, are given by $\Omega_o = \{z : |\phi(z)| < 1,\ |z| > 1\}$ and $\Omega_\infty = \{z : |\phi(z)| > 1\}$.

For each $R_o$ with $0 < R_o \leq 1$, the set

$$S_{R_o} = \{z \in \mathbb{C} : |\phi(z)| = R_o,\ |z| \leq 1,\ 0 < R_o \leq 1\}$$

is an associated level curve of the mapping $\phi$. Clearly, $S_{R_o} \subset G$ for any $R_o$ with $0 < R_o < 1$, and $S_1 = S$. Also, we have $G_{R_o} = int\ S_{R_o}$, where $int\ S_{R_o}$ means the interior points of the curve $S_{R_o}$ and $G_1 = G$.

Let the error of the best weighted approximation on $S_{R_o}$, or equivalently, $\overline{G}_{R_o}$ for a function $f$ analytic in $G_R$, where $0 < R_o < R < 1$ is defined by

$$E_n^w\left(f, \overline{G}_{R_o}\right) = \inf_{P_n \in \pi_n} \left\| e^{-nz} P_n(z) - f(z) \right\|_{S_{R_o}}, \tag{2}$$

where $\pi_n$ is the set of all polynomials of degree $\leq n$.

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For classifying analytic functions by their growth, the concept of order was introduced. If the order is a (finite) positive number, then the concept of type permits a subclassification. For the cases of order $\rho = 0$ and $\rho = \infty$ no subclassification is possible. For this particular subclassification the type of $f$ can be defined by using the concept of index-pair $(p, q)$ introduced by Juneja et.all [1].

Let $H_R$ be the class of functions analytic in $G_R$ but not in $G_{R'}$ if $R < R'$. A function $f \in H_R$, $0 < R < 1$ said to be of order $\rho$, if

$$\rho = \limsup_{r \to R} \frac{\ln^+ \ln^+ M(r)}{\ln(R/(R-r))}. \tag{3}$$

If $0 < \rho < \infty$, then type $T$ is defined by

$$T = \limsup_{r \to R} \frac{\ln^+ M(r)}{(R/(R-r))^\rho}, \tag{4}$$

where $\ln^+ x = \max(0, \ln x)$, $x > 0$ and $M(r) = \max_{z \in G_r} |f(z)|$.

Pritsker and Varga [9] have given a necessary and sufficient condition for the validity of the locally uniform approximation of any function $f(z)$ which is analytic in an open bounded set $G^*$ in the complex plane by weighted polynomials of the form $\{w_*^n(z)P_n(z)\}_{n=0}^\infty$, where $w_*(z)$ is analytic and different from zero in $G^*$. Also, they have generalized the Theorems 3.8 and 4.3 of [8].

In this paper, we obtain the characterization of rate of decay of approximation error (defined by (2)) in terms of order $\rho$ and type $T$ of $f \in H_R$.

It is significant to mention that our main results are different from those of Pritsker and Varga ([8], [9]).

For the weighted normalized partial sums $e^{-nz}s_n(z)$, the following inequality is valid

$$\left|e^{-nz}s_n(z) - 1\right| \le \frac{4}{\sqrt{2n\pi}|z-1|}, \quad z \in \overline{G} \setminus \{1\}, \quad n \ge 1 \tag{5}$$

The detailed proof of (5) is available in [8].

In view of (5), a consequence of (1) is that $e^{-nz}s_n(z)$ converges to $f(z) \equiv 1$, locally uniformly in $G$, (*i.e.*, uniformly on every compact subset of $G$). This raises the question of possibility of uniform approximation of any function analytic in $G$ by weighted polynomials $\{e^{-nz}P_n(z)\}$, where $P_n$ is a complex polynomial of degree $\le n$, for each $n \ge 0$. This type of problem evolved from Lorentz approximation by "incomplete-polynomials" [5] on the real line, and has been developed into the general theory of approximation with varying weights. Equation (1) opens the door to a special weighted approximation of analytic functions in the complex plane. This has been noticed that weighted approximation in complex plane has not been studied so far as in the case of real line.

The harmonic measure at the point $z = 0$ with respect to $G$ is defined as the pre-image of the normalized are-length measure on $\Gamma = \{w \in \mathbb{C} : |w| = 1\}$ under the mapping $w = \phi(z)$, where $\phi(z) = ze^{1-z}$, *i.e.*,

$$w(0, B, G) = m(\phi(B \cap S)). \tag{6}$$

for any Borel set $B \subset \mathbb{C}$. Here $dm = d\theta/2\pi$. From (6), note that $w(0, \cdot, G)$ is a unit Borel measure which is supported on $S$, *i.e.*, $w(0, S, G) = 1$ and supp $w(0, \cdot, G) = S$. For any polynomial $P_n(z)$ , the normalized counting measure of its zeros is defined by

$$\nu_n(P_n) = \frac{1}{n} \sum_{P_n(z_i)=0} \delta_{z_i}, \tag{7}$$

where $\delta_z$ is the unit point mass at $z$ where all zeros are considered with multiplicity.

## 2. Auxiliary Results

This section contains various results which have been utilized to prove the main theorems.

**Proposition 1.** *Let $(R_o, R)$ a pair of numbers with $0 < R_o < R \le 1$. If a function $f$ is analytic in $G_R$, then there exists a sequence of polynomials $\{P_n(z)\}$ such that*

$$E_n^w(f, \overline{G}_{R_o}) \le \left\| f(z) - e^{-nz} P_n(z) \right\|_{\overline{G}_{R_o}} \le K'' \frac{1}{R-\varepsilon-R_0} M(R-\varepsilon, f) \left(R_o/R-\varepsilon\right)^{n+1}, \tag{8}$$

*where $M(R-\varepsilon, f) = \max_{z \epsilon G_{R-\varepsilon}} |f(z)|$ and $K''$ is a large number.*

*Proof.* The following weighted equilibrium problem provides important tools in the derivation of the proof. For the weighted energy integral

$$I_E(\mu) = \int\int ln \frac{1}{|z-t| w(z) w(t)}\, d\mu(z)\, d\mu(t), \quad \mu \in M(E), \tag{9}$$

find

$$\tau_E^* = \inf_{\mu \in M(E)} I_E(\mu). \tag{10}$$

and identify the extremal measure $\mu_E \in M(E)$ for which the infimum in (10) is attained. Here $M(E)$ denotes the class of all positive Borel measure $\mu$ on $\mathbb{C}$ which are supported on $E$ and have total mass unity, *i.e.*, $\mu(\mathbb{C}) = 1$.
The logarithmic potential of a Borel measure $\mu$, with compact support which is defined as

$$V^\mu(z) = \int \ln \frac{1}{|z-t|} d\mu(t).$$

It follows from Theorem I.3.3 of [10] that the solution of the weighted energy problem in (10) for the weight function $w(z) = e^{Re\, z}$, $z \in \mathbb{C}$ of (9) on $\overline{G}_{R_o}$, $0 < R_o \le 1$, is given by

$$\mu_{\overline{G}_{R_o}} = w(0, \cdot, G_{R_o}),$$

and

$$V^{\mu_{\overline{G}_{R_o}}}(z) + Q(z) = \begin{cases} 1 - ln R_o, & z \in \overline{G}_{R_o} \\ 1 - ln|\phi(z)|, & z \in \mathbb{C} \setminus G_{R_o}, \end{cases} \tag{11}$$

where $Q(z) = Re\, z$ and $\phi(z) = z e^{1-z}$.

Now, suppose that $f(z)$ is analytic in $G_R$. For each $n \ge 0$, let $z_1^{(n+1)}, z_2^{(n+1)}, \ldots, z_{n+1}^{(n+1)}$ be $n+1$ points in $G_R$. In view of the Hermite interpolation formula, the polynomial $P_n(z)$ which interpolates $e^{nz} f(z)$ at these points is given by, see [12],

$$e^{nz} f(z) - P_n(z) = \frac{w_{n+1}(z)}{2\pi i} \int_{S_{R-\varepsilon}} \frac{f(t) e^{nt}}{(t-z) w_{n+1}(t)}\, dt, \tag{12}$$

where $w_{n+1}(z) = \prod_{k=1}^{n+1} (z - z_k^{(n+1)})$ and $z \in G_{R-\varepsilon}$; $\varepsilon > 0$ is small enough so that the set $\{z_1^{(n+1)}, z_2^{(n+1)}, \ldots, z_{n+1}^{(n+1)}\}$ is contained in $G_{R-\varepsilon}$. Division by $e^{nz}$ in (2.5) yields

$$f(z) - e^{-nz} P_n(z) = \frac{e^{-nz} w_{n+1}(z)}{2\pi i} \int_{S_{R-\varepsilon}} \frac{f(t)\, dt}{(t-z) e^{-nt} w_{n+1}(t)}, \quad z \in G_{R-\varepsilon}.$$

Let $\nu_n(w_n)$ be the normalized counting measure of zeros of $w_n(z)$ defined as (see (7)),

$$\nu_n(w_n) = \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}(n), \quad n \ge 1.$$

Obviously,

$$|w_n(z)| = \exp\{-nV^{\nu_n(w_n)}(z)\}, \quad n \geq 1. \tag{13}$$

For each $R_o$ with $0 < R_o < R$, choosing an interpolation in (12) which satisfies

$$\left\{z_k^{(n+1)}\right\}_{k=1}^{n+1} \subset S_{R_o} \tag{14}$$

and

$$\nu_n(w_n) \to w(0, \cdot, G_{R_o}) \text{ as } n \to \infty. \tag{15}$$

Note that at (15) the convergence is weak-convergence. As an example of an interpolation where (14) and (15) are valid, one can take the pre-images of equally spaced points on $|w| = R_o$ under the conformal map $w = \phi(z) = ze^{1-z}$, *i.e.*, for $\eta = \phi^{-1}$, we define

$$z_k^{(n)} = \eta\left(R_o e^{i2\pi k/n}\right), \ 1 \leq k \leq n, \ n = 1, 2, \ldots.$$

In view of (13)-(15), we have

$$\lim_{n\to\infty} |w_n(z)|^{1/n} = \lim_{n\to\infty} \exp\{-V^{\nu_n(w_n)}(z)\} = \exp\{-V^{w(0,\cdot,G_{R_o})}(z)\}, \tag{16}$$

which holds locally uniformly in $\mathbb{C} \setminus \overline{G}_{R_o}$. Taking any $\varepsilon > 0$ small enough so that $R_o + \varepsilon < R - \varepsilon$ in (12) we obtain

$$\left\|f(z) - e^{-nz}P_n(z)\right\|_{\overline{G}_{R_o}} \leq \frac{\|e^{-nz}w_{n+1}(z)\|_{S_{R_o}} \|f\|_{S_{R-\varepsilon}}}{2\pi \text{ dist}\,(S_{R-\varepsilon}, S_{R_o}) \min_{t\in S_{R-\varepsilon}} |e^{-nt}w_{n+1}(t)|}.$$

We see that the immediate outcome of (11) is

$$V^{w(0,\cdot,G_{R_o})}(z) = -\ln|z|, \tag{17}$$

where $z \in \mathbb{C} \setminus G_{R_O}$. From (11), it is obvious that $V^{w(0,\cdot,G_{R_o})}(z)$ is continuous on $S_{R_o} = \text{supp } w(0, \cdot, G_{R_o})$ and, therefore, is continuous in $\mathbb{C}$ by Theorem II.3.5 of [10], see also Theorem 1.7 in [4]. Assume that $z \in S_{R_o}$, so that from (11), $-\ln|z| + \text{Re } z = 1 - \ln R_o$. Then with (17), it gives

$$V^{w(0,\cdot,G_{R_o})}(z) + Q(z) = -\ln|z| + \text{Re } z = 1 - \ln R_o, \quad Q(z) = \text{Re } z.$$

It can be easily seen from the above definition that $V^{w(0,\cdot,G_{R_o})} + Q(z)$ is harmonic in $G_{R_o}$ and is identically constant on the boundary $S_{R_o}$. From (11) we have

$$V^{w(0,\cdot,G_{R_o})} + Q(z) = 1 - \ln R_o, \quad z \in \overline{G}_{R_o}. \tag{18}$$

Using (16) and (18), we obtain

$$\| f(z) - e^{-nz}P_n(z) \|_{\overline{G}_{R_o}} \leq K' \frac{1}{R - \varepsilon - R_0} M(R-\varepsilon, f) \frac{e^{R_e z}(e^{ln(R_0)-1})^{n+1}}{e^{R_e t}(e^{ln(R_0-\varepsilon)-1})^{n+1}},$$

which follows from :

$$e^{-nz}w_{n+1}(z) = e^z(e^{-z}w_{n+1}(z)^{1/(n+1)})^{n+1},$$

so

$$\left|e^z(e^{-z}w_{n+1}(z)^{1/(n+1)})^{n+1}\right|$$

is behaves like

$$|e^z|(e^{-Q-V})^{n+1} = e^{R_e z}(e^{ln(R_0)-1})^{n+1},$$

by (11) and (16).

So we get

$$\|f(z) - e^{-nz}P_n(z)\|_{\overline{G}_{R_0}} \le K'' \frac{1}{R-\varepsilon-R_0} M(R-\varepsilon, f) \left(\frac{R_0}{R-\varepsilon}\right)^{n+1}.$$

□

**Remark 1.** *In proving Proposition 1, we have used the technique developed by Pritsker and Varga.*

**Proposition 2.** *A function $f$ has a singularity on $S_R$ if and only if*

$$\limsup_{n\to\infty} \left[E_n^w(f, \overline{G}_{R_o})\right]^{1/n} = \frac{R_o}{R}.$$

*Proof.* If $f(z)$ is analytic in $G_R$, then by Proposition 1,

$$\limsup_{n\to\infty} \left(E_n^w(f, \overline{G}_{R_o})\right)^{1/n} \le \frac{R_o}{R-\varepsilon},$$

for all $R-\varepsilon$ sufficiently near to $R$ and so

$$\limsup_{n\to\infty} \left(E_n^w(f, \overline{G}_{R_o})\right)^{1/n} \le \frac{R_o}{R}, \tag{19}$$

However, the strict inequality in (19) is equivalent to the analyticity of $f(z)$ in $G_\rho$ for some $\rho$ with $R < \rho < 1$, which is a contradiction. Thus $f(z)$ has a singularity on $S_R$ if and only if equality holds in (19). □

**Proposition 3.** *For any polynomial $P_n(z)$ of degree $\le n$, we have*

$$\left|e^{-nz}P_n(z)\right| \le \|e^{-nz}P_n(z)\|_{S_{R_o}} \left(\frac{|\phi(z)|}{R_o}\right)^n,$$

*where $z \in \mathbb{C} \setminus G_{R_O}$, $n \ge 0$ and $0 < R_o \le 1$.*

*Proof.* Since $ln(|\phi(z)|/R_o)$ is the green function, the statement of the proposition follows on the similar lines as that of the Bernstein-Walsh lemma (see[11]). □

**Proposition 4.** *Let $f \in H_R$ and let $R_o$ be a fixed number $(0 < R_o < R)$. Then the function $g(z) = \sum_{n=0}^{\infty} E_n^w(f, \overline{G}_{R_o})z^n$ is analytic in a disk centered at origin whose radius is $R/R_0$ and for every $r, R_o \le r < R$,we have*

$$M(r, f) \le a_o + 2g(r/R_o), \tag{20}$$

*where $a_o$ is not a constant it depends on $r = |z|$.*

*Proof.* Since $f \in H_R$ so $f(z)$ is analytic in $G_R$. Consider the function

$$g(z) = \sum_{n=0}^{\infty} E_n^w\left(f, \overline{G}_{R_o}\right) z^n.$$

As $\lim_{n\to\infty} \left[E_n^w\left(f, \overline{G}_{R_o}\right)\right]^{1/n} = R_o/R$, by Proposition 2. It follows that g(z) is analytic in a disk centered at the origin whose radius is $R/R_o$.

By uniform convergence on $\overline{G}_{R_o}$, the function $f(z)$ can be represented in telescopic series:

$$f(z) = e^{-nz}P_n(z) + \sum_{k=n}^{\infty} \left(e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\right),\ z \in \overline{G}_{R_o}.$$

where $P_{n's}$ are best approximation polynomials for which

$$\|f - e^{-kz}P_k(z)\|_{\overline{G}_{R_0}} = E_k^w.$$

Thus,

$$|f(z)| \leq |e^{-nz}P_n(z)| + \sum_{k=n}^{\infty} \left|(e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z))\right|. \tag{21}$$

and

$$\begin{aligned}\left|e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\right| &\leq \left\|e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\right\|_{\overline{G}_{R_o}} \\ &\leq \left\|f - e^{-(k+1)z}P_{k+1}(z)\right\|_{\overline{G}_{R_o}} + \left\|f - e^{-kz}P_k(z)\right\|_{\overline{G}_{R_o}} \\ &= E^w_{k+1}\left(f, \overline{G}_{R_o}\right) + E^w_k\left(f, \overline{G}_{R_O}\right) \\ &\leq 2E^w_k\left(f, \overline{G}_{R_o}\right).\end{aligned}$$

In view of Proposition 3, we get

$$\left|e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\right| \leq 2E^w_k\left(f, \overline{G}_{R_o}\right)\left(\frac{|\phi(z)|}{R_o}\right)^k, k \geq n, z\epsilon\mathbb{C} \setminus G_{R_0}.$$

Hence the inequality (21) yields

$$|f(z)| \leq a_o + 2\sum_{k=n}^{\infty} E^w_k\left(f, \overline{G}_{R_o}\right)\left(\frac{|\phi(z)|}{R_o}\right)^k.$$

If $z \in S_r$, i.e.,$|\phi(z)| = r$,then

$$|f(z)| \leq a_o + 2\sum_{k=n}^{\infty} E^w_k\left(f, \overline{G}_{R_o}\right)\left(\frac{r}{R_o}\right)^k. \tag{22}$$

The last series in (22) converges inside $G_R$ since we can majorate it by $g$. Now (22) implies (20). Hence the proof is completed. □

In order to prove the main results we need the concepts of order and type of a function of a single complex variable which is analytic in the disc $|z| < R$. Let $f(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in $|z| < R$, $0 < R < 1$. The order $\rho_o$ and type $T_o$ of $f(z)$ are defined in a analogous manner to (3) and (4). The coefficient characterization of $\rho_o$ and $T_o$ for $f(z)$ are as follows;

$$\rho_o = \limsup_{n\to\infty} \frac{\ln^+ \ln^+(|b_n|R^n)}{\ln n - \ln^+ \ln^+ |b_n|R^n}, \tag{23}$$

$$T_o = \frac{\rho_o^{\rho_o}}{(\rho_o + 1)^{\rho_o+1}} \limsup_{n\to\infty} \frac{(\ln^+(|b_n|R^n))^{\rho_o+1}}{n^{\rho_o}}, \; 0 < \rho_o < \infty. \tag{24}$$

The expression for $\rho_o$ is due to Beuermann [6], while for $T_o$ was obtained by Kapoor [3].

## 3. Main Results

**Theorem 1.** *Let $f \in H_R$ be order $\rho$ and let $0 < R_o < R < 1$ . Then*

$$\rho = \limsup_{n\to\infty} \frac{\ln^+ \ln^+\left(E^w_n(f, \overline{G}_{R_o})(R/R_o)^n\right)}{\ln n - \ln^+ \ln^+\left(E^w_n(f, \overline{G}_{R_o}(R/R_o)^n\right)}. \tag{25}$$

*Proof.* Let the limit superior on the right hand side of (25) be denoted by $\alpha$. Obviously, $0 \leq \alpha \leq \infty$. First let $0 < \alpha < \infty$ and $\alpha'$ be an arbitrary number such that $0 < \alpha' < \alpha$. Then by the definition of $\alpha$ there exists a sequence $\{n_k\}$ of positive integers tending to $\infty$ such that

$$\ln\left(E^w_{n_k}(f, \overline{G}_{R_o})(R/R_o)^{n_k}\right) > n_k^{\alpha'/(1+\alpha')}, \; k = 1, 2 \ldots. \tag{26}$$

Using (8) with (26) we get

$$\ln M(r,f) \geq n_k^{\alpha'/(1+\alpha')} - n_k \ln(R/r) - ln(R_o/r) - \ln K, \tag{27}$$

for all large $k$ and all $r$ sufficiently near to $R$. Let $\{r_k\}$ be a sequence defined by

$$n_k = (3\ln(R/r_k))^{-(1+\alpha')}. \tag{28}$$

Then $r_k \to R$ as $k \to \infty$. Now using (27) and (28) for all sufficiently large $k$, we have

$$\ln M(r_k,f) \geq \frac{1}{3^{\alpha'}} (\ln(R/r_k))^{-\alpha'} - \ln(R/r_k)\,[3\ln(R/r_k)]^{-(1+\alpha')} - ln(R_o/r_k) - \ln K$$

or

$$\ln^+ \ln^+ M(r_k,f) \geq -\ln 3^{\alpha'} - \alpha' \ln\ln(R/r_k) + O(1).$$

Since $\ln(R/(R-r_k) \simeq -\ln\ln(R/r_k)$ as $k \to \infty$, above relation gives

$$\limsup_{k\to\infty} \frac{\ln^+ \ln^+ M(r_k,f)}{\ln(R/R-r_k)} \geq \alpha'.$$

Since $\alpha' < \alpha$ is arbitrary, we have $\rho \geq \alpha$. This is valid for $\alpha = 0$. For $\alpha = \infty$ this ensures $\rho = \infty$.

For reverse inequality apply (23) to $g(z/R_0)$ to see that the order of $g(z/R_0)$ is $\alpha$. Now $\rho \leq \alpha$ follows from (20) and (3). This completes the proof. □

**Theorem 2.** *Let $f \in H_R$, $0 < R_0 < R < 1$. Then $f$ is of order $\rho(0 < \rho < \infty)$ and type $T(0 \leq T \leq \infty)$ if and only if*

$$\frac{(\rho+1)^{\rho+1}}{(\rho)^\rho} T = \limsup_{n\to\infty} \frac{\left(\ln^+\left(E_n^w\left(f,\overline{G}_{R_o}\right)(R/R_o)^n\right)\right)^{\rho+1}}{n^\rho}. \tag{29}$$

*Proof.* Let $f \in H_R$ be of order $\rho$ and type $T(T < \infty)$. But by (4) there exists $r' = r'(\varepsilon)$ for any $\varepsilon > 0$ such that

$$\ln M(r,f) \leq (T+\varepsilon)(R/(R-r))^\rho \tag{30}$$

for $r' < r < R$. Equations (8) and (30) combine to yield

$$\ln^+\left(E_n^w\left(f,\overline{G}_{R_o}\right)(R/R_o)^n\right) \leq (T+\varepsilon)(R/(R-r))^\rho + nln(R/r) + ln(R_0/r) + \ln^+ K \tag{31}$$

for all $r$ sufficiently near to $R$ and sufficiently large $n$. Consider a sequence $\{r_n^*\}$ as

$$\frac{R}{R-r_n^*} = \left(\frac{n(\rho+1)^{(\rho+1)/\rho}}{\rho(T+\varepsilon)}\right)^{1/(\rho+1)}. \tag{32}$$

Obviously $r_n^* \to R$ as $n \to \infty$. In view of (32), (31) gives

$$\ln^+\left(E_n^w\left(f,\overline{G}_{R_o}\right)(R/R_o)^n\right) \leq \frac{(T+\varepsilon)^{1/(\rho+1)}(n)^{\rho/(\rho+1)}(\rho+1)}{\rho^{\rho/(\rho+1)}}(1+o(1))$$

For all sufficiently large values of $n$.
Proceeding to limits in above inequality, we get

$$T \geq \frac{(\rho)^\rho}{(\rho+1)^{\rho+1}} \limsup_{n\to\infty} \frac{\left(\ln^+\left(E_n^w\left(f,\overline{G}_{R_o}\right)(R/R_o)^n\right)\right)^{\rho+1}}{n^\rho}. \tag{33}$$

Inequality (33) is obviously true when $T = \infty$.
For reverse inequality apply (24) to $g(z/R_0)$ to see that the type of $g(z/R_0)$ is $T$. Now reverse inequality in (33) follows from (20) and (4). Thus necessary part of the proof is done.

For the sufficiency part suppose that right hand side of (29) denoted by $v$ and $0 < v < \infty$. Then,(29) implies (25) and hence by Theorem 1,$f(z)$ is of order $\rho$. Now if $v = 0$, then $f$ is of order at most $\rho$ and the reverse inequality in (33) gives that, if $f$ is of order $\rho$, then its type is zero. Therefore, if $v = 0$, then $f$ is of growth $(\rho, o)$. Similarly, if $v = \infty$, then $f$ if of order at least $\rho$ and (33) shows that, if $f$ is of order $\rho$, then $T = \infty$. Hence, $v = \infty$, then $f$ is of growth $(\rho, \infty)$. Hence the proof is completed. □

## References


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