TJMM
**3** (2011), No. 1, 15-21

# SOME DIFFERENTIAL EQUATIONS OF FIRST ORDER WITH MIXED MODIFIED ARGUMENTS AND WITH A PARAMETER

MARIA DOBRIŢOIU

Abstract. In this paper we study the solution of a generalization of a type of differential equation with mixed modified argument and with a parameter. Using an observation from V.A. Ilea [6] we present two result regarding the existence and uniqueness and the data dependence of the solution of this differential equation. An example is also given here.

## 1. Introduction

The differential equations with mixed modified arguments have been studied by K.L. Cooke, J.K. Kaplan [3] and D. Guo, V. Laksmikantham [5], like the equations which were generated as models from medicine and biology. Also, this type of differential equations have been studied by R. Precup [8], I.A. Rus and V.A. Dârzu-Ilea [12], J. Sotomayor [14] and the others.

In the paper [6] V.A. Ilea has studied the solution $x \in C[a-h, b+h] \cap C^1[a,b]$ of the following problem:

$$x'(t) = f(t, x(t), x(t-h)), x(t+h)) + \lambda, \quad t \in (a,b) \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [a-h, a] \tag{2}$$

$$x(t) = \zeta(t), \quad t \in [b, b+h], \tag{3}$$

where $a, b \in \mathbb{R}$, $a < b$, $h > 0$, $f \in C([a,b] \times \mathbb{R}^3)$, $\varphi \in C[a-h, a]$, $\zeta \in C[b, b+h]$, $\lambda \in \mathbb{R}$.

This problem is a mathematical model for a disease and the results obtained are two theorems of existence and uniqueness and of data dependence of the solution.

In this paper we will study the existence and uniqueness and the data dependence of the solution $x \in C[a-h, b+h] \cap C^1[a,b]$ of a problem which generalize the problem studied in [6], namely

$$x'(t) = f(t, x(t), x(g_1(t)), x(g_2(t))) + \lambda, \quad t \in (a,b) \tag{4}$$

$$x(t) = \varphi(t), \quad t \in [a-h, a] \tag{5}$$

$$x(t) = \zeta(t), \quad t \in [b, b+h] \tag{6}$$

where $a, b \in \mathbb{R}$, $a < b$, $h > 0$, $f \in C([a,b] \times \mathbb{R}^3)$, $g_1 \in C([a,b], [a-h, b])$, $g_2 \in C([a,b], [a, b+h])$, $\varphi \in C[a-h, a]$, $\zeta \in C[b, b+h]$, $\lambda \in \mathbb{R}$.

We will use several notations and two basic results (see [1], [2], [4], [6], [7], [9], [10], [11], [13] and [15]) and we present some of them, below.

Let $X$ be a nonempty set, $d$ a metric on $X$, $(X,d)$ a metric space and $A : X \to X$ an operator. We denote:

$A^0 := 1_X,\ A^1 := A, \ldots, A^{n+1} := A \circ A^n,\ n \in \mathbb{N}$ - the iterate operators of $A$

$F_A := \{x \in X |\ A(x) = x\}$ - the fixed points set of the operator $A$.

---

2010 *Mathematics Subject Classification.* 34G20, 47H10, 45G10,

*Key words and phrases.* Differential equation, modified argument, existence and uniqueness, data dependence.

Also, we will use the Banach space $X = C[a-h, b+h]$ endowed with the Chebyshev norm defined by the relation:

$$\|x\|_C := \max_{t\in[a-h,b+h]} |x(t)| \ , \quad for\ all\ x \in C[a-h, b+h]\ . \tag{7}$$

In order to study the existence and uniqueness of the solution of the problem (4)+(5)+(6), in the section 2, we need the *Contraction Principle* (see [1], [2], [9], [11] and [13]) and a result obtained by V.A. Ilea in [6], which we present below.

**Theorem 1.** *(Contraction Principle) Let $(X, d)$ be a complete metric space and $A : X \to X$ an $\alpha$-contraction, ($\alpha < 1$). In these conditions we have:*

*(i)* $F_A = \{x^*\}$;
*(ii)* $A^n(x_0) \to x^*$, *as* $n \to \infty$;
*(iii)* $d(x^*, A^n(x_0)) \le \dfrac{\alpha^n}{1-\alpha} d(x_0, A(x_0))$.

**Theorem 2.** *(V.A. Ilea [6]) Suppose that the following conditions are satisfied:*

*(i) there exists $L_f > 0$ such that*

$$|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \le L_f\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right),$$

*for all $t \in [a, b]$, $u_i, v_i \in \mathbb{R}$, $i = \overline{1,3}$;*
*(ii)* $6L_f(b-a) < 1$.

*In these conditions the problem (1)+(2)+(3) has a unique solution. Moreover, if $(x^*, \lambda^*)$ is the unique solution of this problem, then the following relations are fulfilled:*

$$x^* = \lim_{n\to\infty} A^n(t), \quad \forall x \in C[a-h, b+h] \tag{8}$$

*and*

$$\lambda^* = \frac{\zeta(b) - \varphi(a)}{b-a} - \frac{1}{b-a}\int_a^b f(s, x^*(s), x^*(s-h)), x^*(s+h))ds. \tag{9}$$

In the section 3, for the study of data dependence of the solution of the problem (4)+(5)+(6), we need the *General data dependence theorem* (see [9], [11], [13]).

**Theorem 3.** *(General data dependence theorem). Let $(X, d)$ be a complete metric space and $A, B : X \to X$ two operators. We suppose that:*

*(i) $A$ is an $\alpha$-contraction;*
*(ii) $x^*_B \in F_B$;*
*(iii) there exists $\eta > 0$ such that*

$$d(A(x), B(x)) < \eta$$

*for all $x \in X$.*

*In these conditions we have*

$$d(x^*_A, x^*_B) \le \frac{\eta}{1-\alpha},$$

*where $x^*_A$ is the unique fixed point of $A$.*

Finally, an example is given.

## 2. Existence and uniqueness

Using the *Contraction Principle* in the study of existence and uniqueness of the solution of the problem (4)+(5)+(6) in the Banach space $C[a-h,b+h]\cap C^1[a,b]$, the following result was obtained:

**Theorem 4.** *Suppose that the following conditions are satisfied:*

*(i)* $f\in C([a,b]\times\mathbb{R}^3)$, $g_1\in C([a,b],[a-h,b])$, $g_2\in C([a,b],[a,b+h])$;

*(ii)* $\varphi\in C[a-h,a]$, $\zeta\in C[b,b+h]$;

*(iii) there exists* $L_f>0$ *such that*

$$|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)|\le L_f\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right)$$

*for all* $t\in[a,b]$, $u_i,v_i\in\mathbb{R}$, $i=\overline{1,3}$;

*(iv)* $3L_f(b-a)<1$.

*In these conditions the problem (4)+(5)+(6) has a unique solution. Moreover, if* $(x^*,\lambda^*)$ *is the unique solution of this problem, then the following relations hold:*

$$x^*=\lim_{n\to\infty}A^n(x),\quad \forall x\in C[a-h,b+h] \tag{10}$$

*and*

$$\lambda^*=\frac{\zeta(b)-\varphi(a)}{b-a}-\frac{1}{b-a}\int_a^b f(s,x^*(s),x^*(g_1(s)),x^*(g_2(s)))ds. \tag{11}$$

*Proof.* The problem (4)+(5)+(6) is equivalent with the following integral equation with modified argument:

$$x(t)=\begin{cases}\varphi(t),\quad t\in[a-h,a]\\ \varphi(a)+\int_a^t f(s,x(s),x(g_1(s)),x(g_2(s)))ds+\lambda(t-a),\quad t\in[a,b]\\ \zeta(t),\quad t\in[b,b+h].\end{cases} \tag{12}$$

The function $x$ is continuous in $t=b$, and therefore we deduce that

$$\lambda=\frac{\zeta(b)-\varphi(a)}{b-a}-\frac{1}{b-a}\int_a^b f(s,x(s),x(g_1(s)),x(g_2(s)))ds. \tag{13}$$

Now, we consider the Banach space $C[a-h,b+h]$ endowed with the Chebyshev norm defined by the relation (7) and the operator $A:C[a-h,b+h]\longrightarrow C[a-h,b+h]$ defined by the relation:

$$A(x)(t)=\begin{cases}\varphi(t),\quad t\in[a-h,a]\\ \varphi(a)+\frac{t-a}{b-a}\left[\zeta(b)-\varphi(a)\right]-\\ -\frac{t-a}{b-a}\int_a^b f(s,x(s),x(g_1(s)),x(g_2(s)))ds+\\ +\int_a^t f(s,x(s),x(g_1(s)),x(g_2(s)))ds,\quad t\in[a,b]\\ \zeta(t),\quad t\in[b,b+h].\end{cases} \tag{14}$$

From the conditions $(i)$ and $(ii)$ it results that the operator $A$ is well defined.

The set of the solutions of the problem (4)+(5)+(6) coincides with the set of fixed points of the operator $A$, i.e. with $F_A$. Therefore, our study is reduced to a fixed point problem.

We have:

$$|A(x)(t) - A(y)(t)| =$$

$$= \left| -\frac{t-a}{b-a}\int_a^b [f(s,x(s),x(g_1(s)),x(g_2(s))) - f(s,y(s),y(g_1(s)),y(g_2(s)))]\, ds + \right.$$

$$\left. + \int_a^t [f(s,x(s),x(g_1(s)),x(g_2(s))) - f(s,y(s),y(g_1(s)),y(g_2(s)))]\, ds \right| \leq$$

$$\leq \left| \int_a^b [f(s,x(s),x(g_1(s)),x(g_2(s))) - f(s,y(s),y(g_1(s)),y(g_2(s)))]\, ds \right| \leq$$

$$\leq \int_a^b |f(s,x(s),x(g_1(s)),x(g_2(s))) - f(s,y(s),y(g_1(s)),y(g_2(s)))|\, ds,$$

for all $t \in [a-h, b+h]$.

Using the Lipschitz condition $(iii)$ we have

$$|A(x)(t) - A(y)(t)| \leq$$

$$\leq L_f \int_a^b [|x(s) - y(s)| + |x(g_1(s)) - y(g_1(s))| + |x(g_2(s)) - y(g_2(s))|]\, ds.$$

Next, using the Chebyshev norm on $C[a-h, b+h]$, defined by the relation (7), we obtain

$$\|A(x) - A(y)\|_{C[a-h,b+h]} \leq 3L_f(b-a)\, \|x-y\|_{C[a-h,b+h]}\, .$$

Therefore the operator $A$ satisfies a Lipschitz condition with the constant $\alpha = 3L_f(b-a) > 0$ and by the condition $(iv)$ it results that the operator $A$ is an $\alpha-$contraction with the coefficient $\alpha = 3L_f(b-a)$.

Now, by the conditions $(i)-(iv)$ and applying the *Contraction Principle*, it results that the integral equation with modified argument (12) has a unique solution and therefore the problem (4)+(5)+(6) has a unique solution and the relations (10) and (11) are fulfilled. The proof is complete. □

## 3. Data dependence

Applying the *General data dependence theorem* we obtain the following result of data dependence of the solution of the problem (4)+(5)+(6).

**Theorem 5.** *Suppose that the following conditions are satisfied:*

*(i) there exists* $f_1, f_2 \in C([a,b] \times \mathbb{R}^3)$, $\varphi_1, \varphi_2 \in C[a-h, a]$, $\zeta_1, \zeta_2 \in C[b, b+h]$;

*(ii) there exists* $g_1 \in C([a,b],[a-h,b])$, $g_2 \in C([a,b],[a,b+h])$;

*(iii) there exists* $\eta_1, \eta_2, \eta_3 > 0$, *such that*

$$|f_1(t,u,v,w) - f_2(t,u,v,w)| \leq \eta_1, \quad \text{for all } t \in [a,b],\ u,v,w \in \mathbb{R},$$

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta_2, \quad \text{for all } t \in [a-h, a],$$

*and*

$$|\zeta_1(t) - \zeta_2(t)| \leq \eta_3, \quad \text{for all } t \in [b, b+h].$$

*In these conditions, if* $(x_1^*, \lambda_1^*)$ *is the unique solution of the problem (4)+(5)+(6) with* $f_1$, $\varphi_1$, $\zeta_1$, *and* $(x_2^*, \lambda_2^*)$ *is a solution of this problem with* $f_2$, $\varphi_2$, $\zeta_2$, *then the following relations are fulfilled:*

$$|x_1^* - x_2^*| \le \frac{\eta_1(b-a) + 2\eta_2 + \eta_3}{1 - 3L_{f_1}(b-a)} \tag{15}$$

*and*

$$|\lambda_1^* - \lambda_2^*| \le \eta_1 + \frac{\eta_2 + \eta_3}{b-a}. \tag{16}$$

*Proof.* Let $(x_1^*, \lambda_1^*)$ be the unique solution of the problem (4)+(5)+(6) with $f_1$, $\varphi_1$, $\zeta_1$. We consider the operator $A : C[a-h, b+h] \longrightarrow C[a-h, b+h]$ defined by the relation:

$$A(x)(t) = \begin{cases} \varphi_1(t), \quad t \in [a-h, a] \\ \varphi_1(a) + \frac{t-a}{b-a}\left[\zeta_1(b) - \varphi_1(a)\right] - \\ -\frac{t-a}{b-a} \int\limits_a^b f_1(s, x(s), x(g_1(s)), x(g_2(s)))ds + \\ + \int\limits_a^t f_1(s, x(s), x(g_1(s)), x(g_2(s)))ds, \quad t \in [a, b] \\ \zeta_1(t), \quad t \in [b, b+h]. \end{cases} \tag{17}$$

Let $(x_2^*, \lambda_2^*)$ be a solution of the problem (4)+(5)+(6) with $f_2$, $\varphi_2$, $\zeta_2$. We consider the operator $B : C[a-h, b+h] \longrightarrow C[a-h, b+h]$ defined by the relation:

$$B(x)(t) = \begin{cases} \varphi_2(t), \quad t \in [a-h, a] \\ \varphi_2(a) + \frac{t-a}{b-a}\left[\zeta_2(b) - \varphi_2(a)\right] - \\ -\frac{t-a}{b-a} \int\limits_a^b f_2(s, x(s), x(g_1(s)), x(g_2(s)))ds + \\ + \int\limits_a^t f_2(s, x(s), x(g_1(s)), x(g_2(s)))ds, \quad t \in [a, b] \\ \zeta_2(t), \quad t \in [b, b+h]. \end{cases} \tag{18}$$

From the conditions $(i)$ and $(ii)$ it results that the operators $A$ and $B$ are well defined.

From the following estimation:

$$|A(x)(t) - B(x)(t)| \le |\varphi_1(a) - \varphi_2(a)| +$$

$$+ \left|\frac{t-a}{b-a}\left[\varphi_1(a) - \varphi_2(a)\right]\right| + \left|\frac{t-a}{b-a}\left[\zeta_1(b) - \zeta_2(b)\right]\right| +$$

$$+ \left|\int\limits_a^b \left[f_1(s, x(s), x(g_1(s)), x(g_2(s))) - f_2(s, x(s), x(g_1(s)), x(g_2(s)))\right] ds\right| \le$$

$$\le 2\left|\varphi_1(a) - \varphi_2(a)\right| + \left|\zeta_1(b) - \zeta_2(b)\right| +$$

$$+ \int\limits_a^b \left|f_1(s, x(s), x(g_1(s)), x(g_2(s))) - f_2(s, x(s), x(g_1(s)), x(g_2(s)))\right| ds$$

and using the condition $(iii)$ and the Chebyshev norm on $C[a-h, b+h]$, defined by the relation (7), we obtain:

$$\|A(x) - B(x)\|_{C[a-h,b+h]} \le 2\eta_2 + \eta_3 + \eta_1(b-a).$$

Now, applying the *General data dependence theorem* it results the relations (15) and (16) and the proof is complete. □

## 4. Example

In what follows, we consider the problem

$$x'(t) = \frac{1}{15}\left[x(t) + (t-1)^2 x\left(\frac{t}{2}\right) + x\left(\frac{t^2+2}{3}\right)\right] + \lambda, \quad t \in [0,2] \tag{19}$$

$$x(t) = 2t+1, \quad t \in [-1,0] \tag{20}$$

$$x(t) = 2t-1, \quad t \in [2,3] \tag{21}$$

where $f \in C\left([0,2] \times \mathbb{R}^3\right)$,

$f(t,u_1,u_2,u_3) = \frac{1}{15}\left[u_1 + (t-1)^2 u_2 - u_3\right], \; t \in [0,2], \; u_1,u_2,u_3 \in \mathbb{R}$,
$g_1 \in C([0,2],[-1,2]), \; g_1(t) = \frac{t}{2}, \; t \in [0,2]$,
$g_2 \in C([0,2],[0,3]), \; g_2(t) = \frac{t^2+2}{3}, \; t \in [0,2]$,
$\varphi \in C[-1,0], \; \varphi(t) = 2t+1, \; t \in [-1,0]$
$\zeta \in C[2,3], \zeta(t) = 2t-1, \; t \in [2,3]$
and $x \in C[-1,3] \cap C^1[0,2]$.

The problem (19)+(20)+(21) is equivalent with the following integral equation with modified argument:

$$x(t) = \begin{cases} 2t+1, \quad t \in [-1,0] \\ 1 + \int\limits_0^t \frac{1}{15}\left[x(s) + (s-1)^2 x\left(\frac{s}{2}\right) + x\left(\frac{s^2+2}{3}\right)\right] ds + \lambda t, \quad t \in [0,2] \\ 2t-1, \quad t \in [2,3]. \end{cases} \tag{22}$$

The function $x$ is continuous in $t=2$ and therefore we deduce that

$$\lambda = 1 - \frac{1}{30}\int\limits_0^2 \left[x(s) + (s-1)^2 x\left(\frac{s}{2}\right) + x\left(\frac{s^2+2}{3}\right)\right] ds. \tag{23}$$

The operator $A : C[-1,3] \longrightarrow C[-1,3]$, attached to the equation (22), defined by the relation:

$$A(x)(t) = \begin{cases} 2t+1, \quad t \in [-1,0] \\ 1 + t - \frac{t}{30}\int\limits_0^2 \left[x(s) + (s-1)^2 x\left(\frac{s}{2}\right) + x\left(\frac{s^2+2}{3}\right)\right] ds + \\ + \int\limits_0^t \frac{1}{15}\left[x(s) + (s-1)^2 x\left(\frac{s}{2}\right) + x\left(\frac{s^2+2}{3}\right)\right] ds, \; t \in [0,2] \\ 2t-1, \quad t \in [2,3], \end{cases} \tag{24}$$

is an $\alpha$-contraction with the coefficient $\alpha = \frac{1}{5} < 1$.

The conditions of the Theorem 4 being satisfied, it results that the problem (19)+(20)+(21) has a unique solution $x^* \in C[-1,3] \cap C^1[0,2]$.

## References

[1] Berinde, V., *Generalized contractions and applications*, Cub Press 22, Baia Mare, 1997.

[2] Coman, G., Rus, I., Pavel, G., Rus, I. A., *Introducere în teoria ecuaţiilor operatoriale*, Editura Dacia, Cluj–Napoca, 1976.

[3] Cooke, K.L., Kaplan, J.K., A periodicity theresold theorem for epidemics and population growth, *Math. Biosci.*, 31(1976), no. 1-2, 87-104.

[4] Dobriţoiu, M., *Ecuaţii integrale cu argument modificat*, Presa Universitară Clujeană, Cluj-Napoca, 2009.

[5] Guo, D., Lakshmikantham, V., Positive solutions of nonlinear integral equations arising in infectious diseases, *J.Math.Anal.Appl.*, 134 (1988), 1–8.

[6] Ilea, V.A., *Ecuaţii diferenţiale cu modificarea mixtă a argumentului*, Presa Universitară Clujeană, 2006.

[7] Mureşan, V., *Ecuaţii diferenţiale cu modificarea afină a argumentului*, Transilvania Press, Cluj–Napoca, 1997.

[8] Precup, R., Some existence results for differential equations with both retarded and advanced arguments, *Mathematica*, Tome 44(67), Nr. 1, 2002, 31–38.

[9] Rus, I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979.

[10] Rus, I.A., Iancu, C., *Modelare matematică*, Casa de editură Transilvania Press, Cluj–Napoca, 2000.

[11] Rus, I. A., *Generalized contractions and applications*, Cluj University Press, Cluj–Napoca, 2001.

[12] Rus, I.A., Dârzu-Ilea, V.A., First order functional-differential equations with both advanced and retarded arguments, *Fixed Point Theory*, 5(2004), no.1, 103-115.

[13] Rus, I.A., Petruşel, A., Petruşel, G., *Fixed Point Theory*, Cluj University Press, 2008.

[14] Sotomayor, J., Smooth dependence of solution of differential equation on initial data: a simple proof, *Bol. Soc. Brasil*, 4, 1(1973), 55–59.

[15] Yosida, K., *Équations différentielles et integrales*, DUNOD, Paris, 1971.

University of Petroşani
Department of Mathematics & Computer Science
Universităţii 20, 332006, Petroşani, România
*E-mail address*: mariadobritoiu@yahoo.com