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A COMPANION OF OSTROWSKI'S INEQUALITY WITH APPLICATIONS

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ABSTRACT. An inequality for a companion of Ostrowski's integral inequality is proved. Applications to a composite quadrature rule and to probability density functions are considered.

1. INTRODUCTION

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows [5]:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

For recent results concerning Ostrowski's inequality see [1, 2]. Also, the reader may be refer to the monograph [5] where various inequalities of Ostrowski type are discussed.

In [4], Guessab and Schmeisser have proved among others, the following companion of Ostrowski's inequality:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be satisfies the Lipschitz condition, i.e., $|f(t) - f(s)| \leq M|t-s|$. Then for each $x \in [a, \frac{a+b}{2}]$, we have the inequality*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{8} + 2\left(\frac{x - \frac{3a+b}{4}}{b-a}\right)^2\right](b-a)M \tag{2}$$

The inequality $1/8$ is the best possible in (2) in the sense that it cannot be replaced by a smaller constant.

We may also note that the best inequality in (2) is obtained for $x = \frac{3a+b}{4}$, giving the trapezoid type inequality

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{(b-a)}{8}M \tag{3}$$

The constant $1/8$ is sharp in (3) in the sense mentioned above.

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Companions of Ostrowski's integral inequality for absolutely continuous functions was considered by Dragomir in [6], as follows :

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have the inequalities*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \begin{cases} \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty, & f' \in L_\infty[a,b] \\ \frac{2^{1/q}}{(q+1)^{q+1}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}-\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\left\|f'\right\|_{[a,b],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \text{ and } f' \in L_p[a,b] \\ \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left\|f'\right\|_{[a,b],1} \end{cases} \tag{4}$$

for all $x \in [a, \frac{a+b}{2}]$.

In [7], Dragomir established some inequalities for this companion for mappings of bounded variation. Also, Liu [8], introduced some companions of an Ostrowski type integral inequality for functions whose derivatives are absolutely continuous. Recently, Barnett et al. [3], have proved some companions for the Ostrowski inequality and the generalized trapezoid inequality.

The aim of this paper is to study the companion of Ostrowski inequality (3) for differentiable bounded mappings.

2. Main Results

Our main result may be stated as follows:

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, and let $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$ and $\gamma \leq f'(x) \leq \Gamma$, $\forall x \in [a, b]$, then the following inequality holds,*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq (b-a)\left[\frac{1}{16}+\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]\cdot(\Gamma+\gamma), \tag{5}$$

for all $x \in [a, \frac{a+b}{2}]$.

Proof. Let us define the mapping

$$p(x,t) = \begin{cases} t-a, & t \in [a,x] \\ t-\frac{a+b}{2}, & t \in (x, a+b-x] \\ t-b, & t \in (a+b-x, b] \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$.

Integrating by parts

$$\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt = \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{6}$$

We also have

$$\int_a^b p(x,t)\,dt = 0. \tag{7}$$

Let $C = \frac{\Gamma+\gamma}{2}$. From (6) and (7), it follows that

$$\frac{1}{b-a}\int_a^b p(x,t)\left[f'(t)-C\right]dt = \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt.$$

On the other hand, we have

$$\left|\frac{1}{b-a}\int_a^b p(x,t)\left[f'(t)-C\right]dt\right| \le \frac{1}{b-a}\cdot\max_{t\in[a,b]}\left|f'(t)-C\right|\int_a^b |p(x,t)|\,dt. \tag{8}$$

Since

$$\max_{t\in[a,b]}\left|f'(t)-C\right| \le \frac{\Gamma+\gamma}{2} \tag{9}$$

and

$$\begin{aligned}\frac{1}{b-a}\cdot\int_a^b |p(x,t)|\,dt &= \frac{4(x-a)^2+(a+b-2x)^2}{4(b-a)}\\ &= (b-a)\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]\end{aligned} \tag{10}$$

therefore, from (8)–(10), it follows that

$$\begin{aligned}&\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\qquad\le (b-a)\left[\frac{1}{16}+\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]\cdot(\Gamma+\gamma),\end{aligned}$$

for all $x \in [a, \frac{a+b}{2}]$. □

Corollary 1. *In Theorem 4, choose $x = \frac{3a+b}{4}$, therefore (5) becomes*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)}{16}\cdot(\Gamma+\gamma). \tag{11}$$

Corollary 2. *In Theorem 4, choose $x = a$, then we have the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)}{8}\cdot(\Gamma+\gamma). \tag{12}$$

An inequality of Ostrowski's type may be stated as follows:

Corollary 3. *Let f as in Theorem 4. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le (b-a)\left[\frac{1}{16}+\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]\cdot(\Gamma+\gamma), \tag{13}$$

for all $x\in[a,\frac{a+b}{2}]$.

Remark 1. *In Corollary 3, choose $x=a$, we have*

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{(b-a)}{8}\cdot(\Gamma+\gamma), \tag{14}$$

and for $x=\frac{a+b}{2}$, we have the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{(b-a)}{8}\cdot(\Gamma+\gamma). \tag{15}$$

3. A Composite Quadrature Formula

Let $I_n: a=x_0<x_1<\cdots<x_n=b$ be a division of the interval $[a,b]$ and $h_i=x_{i+1}-x_i$, $(i=0,1,2,\cdots,n-1)$.

Consider the general quadrature formula

$$Q_n(I_n,f):=\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i. \tag{16}$$

The following result holds.

Theorem 5. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, where $a,b\in I$ with $a<b$. If $f'\in L^1[a,b]$ and $\gamma\le f'(x)\le\Gamma$, $\forall x\in[a,b]$. Then, we have*

$$\int_a^b f(t)\,dt=Q_n(I_n,f)+R_n(I_n,f). \tag{17}$$

where, $Q_n(I_n,f)$ is defined by formula (16), and the remainder satisfies the estimates

$$|R_n(I_n,f)|\le\frac{(\Gamma-\gamma)}{16}\cdot\sum_{i=0}^{n-1}h_i. \tag{18}$$

Proof. Applying inequality (11) on the intervals $[x_i,x_{i+1}]$, we may state that

$$R_i(I_i,f)=\int_{x_i}^{x_{i+1}}f(t)\,dt-\frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i.$$

Summing the above inequality over i from 0 to $n-1$, we get

$$\begin{aligned}R_n(I_n,f)&=\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}f(t)\,dt-\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i\\&=\int_a^b f(t)\,dt-\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i,\end{aligned}$$

which follows form (11), that

$$|R_n(I_n,f)| = \left|\int_a^b f(t)\,dt - \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i\right|$$
$$\leq \frac{(\Gamma-\gamma)}{16}\cdot\sum_{i=0}^{n-1}h_i.$$

which completes the proof. □

4. Applications to probability density functions

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x) = Pr(X\leq x) = \int_a^b f(t)dt$.

Theorem 6. *With the assumptions of Theorem 4, we have the inequality*

$$\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\frac{b-E(X)}{b-a}\right| \leq (b-a)\left[\frac{1}{16}+\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]\cdot(\Gamma+\gamma),$$

for all $x\in[a,\frac{a+b}{2}]$, *where* $E(X)$ *is the expectation of* X.

Proof. In the proof of Theorem 4, let $f = F$, and taking into account that

$$E(X)=\int_a^b t\,dF(t) = b-\int_a^b F(t)\,dt.$$

We left the details to the interested reader. □

Corollary 4. *In Theorem 6, choose* $x=\frac{3a+b}{4}$, *we get*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right|\leq\frac{(b-a)}{16}\cdot(\Gamma+\gamma).$$

Corollary 5. *In Theorem 6, if* F *is symmetric about the x-axis, i.e.,* $F(a+b-x) = F(x)$, *we have*

$$\left|F(x)-\frac{b-E(X)}{b-a}\right|\leq(b-a)\left[\frac{1}{16}+\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]\cdot(\Gamma+\gamma),$$

for all $x\in[a,\frac{a+b}{2}]$.

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