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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING DIFFERENTIAL OPERATORS

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Abstract. In this paper we introduce a new subclass of normalized analytic functions in the open unit disc which is defined by a certain differential operator. A coefficient inequality, distortion Theorems and extreme points of differential operator for this class are given. We also discuss the boundedness properties associated with partial sums of functions in the class $TS_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n)$.

1. Introduction

Let A_n be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k \quad (n\in N). \tag{1}$$

Specially, we set $A:=A_1=\{f \text{ analytic in } U \text{ with } f(0)=0 \text{ and } f'(0)=1\}$.

The theory of differential operators plays an important roles in analytic function theory, geometric function theory and univalent function theory. The first study appeared in the year 1900 and since then many mathematicians have worked intensively in this direction. For recent work see ([7], [9], [2], [1], [5]).

For $f\in A$, we now define the differential operator $D_{\lambda,\delta}^{\sigma,s}$ as follows:

$$D_{\lambda,\delta}^{\sigma,s}f(z)=z+\sum_{k=2}^{\infty} k^s\left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma} a_k z^k, \tag{2}$$

where $\lambda\geq 0$, $C(\delta,k)=\begin{pmatrix} k+\delta-1 \\ \delta \end{pmatrix}$ and $\delta,\sigma,s\in N_0$.

Note that $D_{\lambda,\delta}^{\sigma,s}f(z)$ can be written, in terms of convolution as

$$D_{\lambda,\delta}^{\sigma,s}f(z)=\underbrace{\psi(z)*...*\psi(z)}_{\sigma-times}*\sum_{k=1}^{\infty}k^s z^k * f(z),$$

where

$$\psi(z)=\left[\frac{\lambda z}{(1-z)^2}-\frac{\lambda z}{1-z}+\frac{z}{1-z}\right]*\frac{z}{(1-z)^{\delta+1}} \quad (z\in U).$$

Note that when $\sigma=0$ we get the Sălăgean differential operator [9], when $\lambda=s=0,\sigma=1$ we obtain the Ruscheweyh operator [7], when $s=0,\sigma=1$ we obtain the differential operator given by Al-Shaqsi and Darus [1] and when $\delta=s=0$ we obtain the differential operator by Al-Oboudi [2].

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Denote by $f(z) \in T_n$ the subclass of A_n consisting of those functions f of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0\ , n \in N). \tag{3}$$

If f given by (3) then we can see that

$$D_{\lambda,\delta}^{\sigma,s} f(z) = z - \sum_{k=n+1}^{\infty} k^s \left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma} a_k z^k \tag{4}$$

Definition 1. *Let $0 < \beta \leq 1, \lambda \geq 0, \sigma, \delta, s \in N_0$ and $\gamma \in C \setminus \{0\}$. Then, the function $f \in A_n$ is said to be in the class $S_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n)$ if*

$$\left| \frac{1}{\gamma}\left(\frac{z(D_{\lambda,\delta}^{\sigma,s} f(z))'}{D_{\lambda,\delta}^{\sigma,s} f(z)} - 1\right) \right| < \beta, \;\; z \in U.$$

Further, define the class $TS_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n)$ by

$$TS_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n) = S_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n) \cap T_n, \;\; .$$

Note that the class $S_{\lambda,\delta}^{0,0}(0,\gamma,n)$ was introduced by Nasr and Aouf [3].

Recall that the function f is subordinate to g if there exists a function ω, analytic in U, with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$f(z) = g(w(z)), \quad z \in U. \tag{5}$$

We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in U, then the subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [6]).

2. Coefficient inequalities

In this section we give a characterization, via Taylor coefficients, of those T_n belonging to $S_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n)$.

Theorem 1. *Let the function $f \in T_n$ be defined by (3). Then f is in the class $TS_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n)$ if and only if*

$$\sum_{k=n+1}^{\infty} (k+\beta|\gamma|-1) k^s \left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma} a_k \leq \beta|\gamma| \tag{6}$$

for $0 < \beta \leq 1, \delta, s \in N_0, \lambda \geq 0$, $\sigma \in N_0$ and $\gamma \in C \setminus \{0\}$.

Proof. Let $f \in TS_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n)$. Then, we have

$$\Re\left\{\frac{z(D_{\lambda,\delta}^{\sigma,s} f(z))'}{D_{\lambda,\delta}^{\sigma,s} f(z)} - 1\right\} > -\beta|\gamma|.$$

Equivalently,

$$\Re\left\{\frac{-\sum_{k=n+1}^{\infty}(k-1)k^s \left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma} a_k z^k}{z - \sum_{k=n+1}^{\infty} k^s \left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma} a_k z^k}\right\} > -\beta|\gamma|.$$

Now, by choosing the values of z on the real axis and letting $z \longrightarrow 1^-$ through real values. Then the above inequality immediately yields the required condition (6).

Conversely, by applying the hypothesis (6) and letting $|z| = 1$, we obtain

$$\left|\frac{z(D^{\sigma,s}_{\lambda,\delta}f(z))'}{D^{\sigma,s}_{\lambda,\delta}f(z)} - 1\right| = \left|\frac{\sum\limits_{k=n+1}^{\infty}(k-1)k^s\left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma}a_k z^k}{z - \sum\limits_{k=n+1}^{\infty}k^s\left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma}a_k z^k}\right|$$

$$\leq \frac{\beta\,|\gamma|\,(1 - \sum\limits_{k=n+1}^{\infty}k^s\left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma}a_k)}{1 - \sum\limits_{k=n+1}^{\infty}k^s\left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma}a_k}$$

$$\leq \beta\,|\gamma|$$

Hence, by the maximum modulus theorem, we have $f \in TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$, which evidently completes the proof of Theorem 1.

Finally, the result is sharp with the extremal functions f_k be in the class $TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$ given by:

$$f_k(z) = z - \frac{\beta\,|\gamma|}{(k-1+\beta\,|\gamma|)k^s\left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma}}z^k, \tag{7}$$

for $k \geq n+1$. □

Corollary 1. *Let the function f defined by (3) be in the class $TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$. Then, we have*

$$a_k \leq \frac{\beta\,|\gamma|}{(k+\beta\,|\gamma|-1)k^s\left(C(\delta,k)[1+\lambda(k-1)]\right)^{\sigma}}, \tag{8}$$

for $k \geq n+1$. This equality is attained for the functions f given by (7).

3. Growth and Distortion Theorems

Growth and distortion properties for functions belonging to the class $TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$ will be given in the following results:

Theorem 2. *Let the function f given by (3) be in the class $TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$. Then for $|z| = r$ we have*

$$r - \frac{\beta\,|\gamma|}{(n+\beta\,|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^{\sigma}}r^{n+1}$$

$$\leq |f(z)| \leq r + \frac{\beta\,|\gamma|}{(n+\beta\,|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^{\sigma}}r^{n+1},\ n = 1,2,3...$$

with equality for

$$f(z) = z - \frac{\beta\,|\gamma|}{(n+\beta\,|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^{\sigma}}z^{n+1}.$$

Proof. In view of Theorem 1, we have

$$(n+\beta|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma \sum_{k=n+1}^{\infty} a_k$$

$$\leq \sum_{k=n+1}^{\infty} (k+\beta|\gamma|-1)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma a_k \leq \beta|\gamma|.$$

Hence

$$|f(z)| \leq r + \sum_{k=n+1}^{\infty} a_k r^k \leq r + r^{n+1} \sum_{k=n+1}^{\infty} a_k$$

$$\leq r + \frac{\beta|\gamma|}{(n+\beta|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma} r^{n+1},$$

and

$$|f(z)| \geq r - \sum_{k=n+1}^{\infty} a_k r^k \geq r - r^{n+1} \sum_{k=n+1}^{\infty} a_k$$

$$\geq r - \frac{\beta|\gamma|}{(n+\beta|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma} r^{n+1}.$$

This completes the proof. □

Theorem 3. *Let the function f given by (3) be in the class* $TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$. *Then for* $|z|=r$ *we have*

$$r - \frac{(n+1)\beta|\gamma|}{(n+\beta|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma} r^{n+1}$$

$$\leq |f'(z)| \leq r + \frac{(n+1)\beta|\gamma|}{(n+\beta|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma} r^{n+1}, \ n=1,2,3...$$

with equality for

$$f(z) = z - \frac{\beta|\gamma|}{(n+\beta|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma} z^{n+1}.$$

Proof. We have

$$|f'(z)| \leq r + \sum_{k=n+1}^{\infty} k a_k r^k \leq r + (n+1) r^{n+1} \sum_{k=n+1}^{\infty} a_k$$

$$\leq r + \frac{(n+1)\beta|\gamma|}{(n+\beta|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma} r^{n+1},$$

and

$$|f(z)| \geq r - \sum_{k=n+1}^{\infty} k a_k r^k \geq r - (n+1) r^{n+1} \sum_{k=n+1}^{\infty} a_k$$

$$\geq r - \frac{(n+1)\beta|\gamma|}{(n+\beta|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma} r^{n+1},$$

which completes the proof. □

4. Extreme points

The extreme points of the class $TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$ will be now determined

Theorem 4. *Let* $f_n(z) = z$ *and*

$$f_k(z) = z - \frac{\beta\,|\gamma|}{(k-1+\beta\,|\gamma|)(1+n)^s(C(n+1,\delta)(1+\lambda n))^\sigma} z^k,\ \ k \geq n+1.$$

Assume that f *is analytic in* U. *Then* $f \in TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=n}^{\infty} \mu_k f_k(z)$$

where $\mu_k \geq 0$ *and* $\sum\limits_{k=n}^{\infty} \mu_k = 1$.

Proof. Suppose that $f(z) = \sum\limits_{k=n}^{\infty} \mu_k f_k(z)$ with $\mu_k \geq 0$ and $\sum\limits_{k=n}^{\infty} \mu_k = 1$. Then

$$f(z) = \sum_{k=n}^{\infty} \mu_k f_k(z) = \mu_n f_n(z) + \sum_{k=n+1}^{\infty} \mu_k f_k(z)$$

$$= \mu_n z + \sum_{k=n+1}^{\infty} \mu_k (z - \frac{\beta\,|\gamma|}{(k+\beta\,|\gamma|-1)k^s(C(k,\delta)(1+\lambda(k-1)))^\sigma} z^k)$$

$$= z - \sum_{k=n+1}^{\infty} \mu_k \frac{\beta\,|\gamma|}{(k+\beta\,|\gamma|-1)k^s(C(k,\delta)(1+\lambda(k-1))^\sigma} z^k.$$

Then

$$\sum_{k=n+1}^{\infty} \mu_k (\frac{\beta\,|\gamma|}{(k+\beta\,|\gamma|-1)k^s(C(k,\delta)(1+\lambda(k-1))^\sigma})$$

$$(\frac{(k+\beta\,|\gamma|-1)k^s(C(k,\delta)(1+\lambda(k-1))^\sigma}{\beta\,|\gamma|}) = 1$$

$$\sum_{k=n+1}^{\infty} \mu_k = \sum_{k=n}^{\infty} \mu_k - \mu_n = 1 - \mu_n \leq 1.$$

Thus $TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$ by Theorem 1.

Conversely, suppose that $f \in TS^{\sigma,s}_{\lambda,\delta}(\beta,\gamma,n)$. By using (8) we may set and

$$\mu_k = \frac{(k+\beta\,|\gamma|-1)k^s\,(C(\delta,k)[1+\lambda(k-1)])^\sigma}{\beta\,|\gamma|} a_k$$

for $k \geq n+1$ and $\mu_n = 1 - \sum\limits_{k=n+1}^{\infty} \mu_k$. Then

$$f(z) = z - \sum_{k=n}^{\infty} a_k z^k = z - \sum_{k=n+1}^{\infty} \mu_k \frac{\beta\,|\gamma|}{(k+\beta\,|\gamma|-1)k^s\,(C(\delta,k)[1+\lambda(k-1)])^\sigma} z^k$$

$$= \mu_n f_n(z) + \sum_{k=n+1}^{\infty} \mu_k f_k(z) = \sum_{k=n}^{\infty} \mu_k f_k(z),$$

with $\mu_k \geq 0$ and $\sum_{k=n}^{\infty} \mu_k = 1$, which completes the proof. □

Corollary 2. *The extreme points of* $TS_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n)$ *are the functions* $f_n(z) = z$ *and*

$$f_k(z) = z - \frac{\beta\,|\gamma|}{(k-1+\beta\,|\gamma|)k^s(C(k,\delta)(1+\lambda(k-1)))^{\sigma}} z^k, \tag{9}$$

for $k \geq n+1$.

5. Partial Sums

We investigate in this section the partial sums of functions in the class $TS_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n)$. We shall obtain sharp lower bounds for the real part of its ratios. We shall follow similar works done by Silverman [8] and Khairnar and Moreon [4] about the partial sums of analytic functions. In what follows, we will use the well known result that for an analytic function ω in U

$$\Re\{\frac{1+\omega(z)}{1-\omega(z)}\} > 0,\ z \in U,$$

if and only if the inequality $|\omega(z)| \leq 1$ is satisfied.

Theorem 5. *Let* $f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \in TS_{\lambda,\delta}^{\sigma,s}(\beta,\gamma,n)$, *and define its partial sums by.* $f_n(z) = z$ *and* $f_m(z) = z - \sum_{k=n+1}^{m} a_k z^k$, $(m \geq n)$. *Then*

$$\Re\{\frac{f(z)}{f_m(z)}\} \geq\ 1 - \frac{1}{c_{m+1}} \qquad (z \in U,\ m \in N) \tag{10}$$

and

$$\Re\{\frac{f_m(z)}{f(z)}\} \geq\ \frac{c_{m+1}}{1+c_{m+1}} \qquad (z \in U,\ m \in N), \tag{11}$$

where

$$c_k = \frac{(k+\beta\,|\gamma|)k^s\,(C(\delta,k)[1+\lambda(k-1)])^{\sigma}}{\beta\,|\gamma|}. \tag{12}$$

The estimates in (10) and (11) are sharp.

Proof. To prove (10), it suffices to show that

$$c_{m+1}\{\frac{f(z)}{f_m(z)} - (1 - \frac{1}{c_{m+1}})\} \prec \frac{1+z}{1-z}, \quad z \in U.$$

By the subordination property (5), we can write

$$\frac{1 - \sum_{k=n+1}^{m} a_k z^{k-1} - c_{m+1} \sum_{k=m+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=n+1}^{m} a_k z^{k-1}} = \frac{1+\omega(z)}{1-\omega(z)},$$

for certain ω analytic in U with $|\omega(z)| \leq 1$. Notice that $\omega(0) = 0$ and

$$|\omega(z)| \leq \frac{c_{m+1} \sum\limits_{k=m+1}^{\infty} a_k}{2 - 2 \sum\limits_{k=n+1}^{m} a_k - c_{m+1} \sum\limits_{k=m+1}^{\infty} a_k}.$$

Now $|\omega(z)| \leq 1$ if and only if

$$\sum_{k=n+1}^{m} a_k + c_{m+1} \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty} c_k a_k \leq 1.$$

The above inequality holds because c_k is a non-decreasing sequence. This completes the proof of (10). Finally, it is observed that equality in (10) is attained for the function given by (8) when $z = re^{2\pi i/m}$ as $r \to 1^-$. Similarly, we take

$$(1 + c_{m+1})\{\frac{f_m(z)}{f(z)} - \frac{c_{m+1}}{1 + c_{m+1}})\} = \frac{1 - \sum\limits_{k=n+1}^{m} a_k z^{k-1} + c_{m+1} \sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{1 - \sum\limits_{k=n+1}^{m} a_k z^{k-1}} = \frac{1 + \omega(z)}{1 - \omega(z)},$$

where

$$|\omega(z)| \leq \frac{(1 + c_{m+1}) \sum\limits_{k=m+1}^{\infty} a_k}{2 - 2 \sum\limits_{k=n+1}^{m} a_k + (1 - c_{m+1}) \sum\limits_{k=m+1}^{\infty} a_k}.$$

Now $|\omega(z)| \leq 1$ if and only if

$$\sum_{k=n+1}^{m} a_k + c_{m+1} \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty} c_k a_k \leq 1.$$

This immediately leads to the assertion (11) of Theorem 5. This completes the proof of Theorem 5. □

Using a similar method, we can prove the following theorem.

Theorem 6. *If $f \in TS_{\lambda,\delta}^{\sigma,s}(\beta, \gamma, n)$ and define the partial sums, by $f_n(z) = z$ and $f_m(z) = z - \sum\limits_{k=n+1}^{m} a_k z^k$. Then*

$$\Re\{\frac{f'(z)}{f'_m(z)}\} \geq 1 - \frac{m+1}{c_{m+1}} \qquad (z \in U,\, m \in N)$$

and

$$\Re\{\frac{(f'_m(z)}{(f'(z)}\} \geq \frac{c_{m+1}}{m + 1 + c_{n+1}}.$$

where c_k is given by (12). The result is sharp for every m, with extremal functions given by (7).

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