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ON NEWTON'S QUADRATURE FORMULA FOR MAPPINGS OF BOUNDED VARIATION

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Abstract. An estimation of remainder for Newton's quadrature formula for mappings of bounded variation are given. Also, some applications to special means are given.

1. Introduction

In [1] Dragomir, proved the following trapezoid inequalities for mappings with bounded variation:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping with bounded variation on [a,b].Then we have the inequality*

$$\left| \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) \right| \le \frac{1}{2}(b-a)\,V_a^b(f) \tag{1}$$

where $V_a^b(f)$ is the total variation of f on the interval [a,b]. The constant $\frac{1}{2}$ is the best possible one.

In [2], the same author proved the following Simpson type inequalities for mappings with bounded variation:

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ a mapping with bounded variation on [a,b]. Then we have the inequality*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right| \le \frac{1}{3}(b-a)\,V_a^b(f) \tag{2}$$

where $V_a^b(f)$ denotes the total variation of f on the interval [a,b].

In this paper, we consider the Newton inequality [2]:

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] \right| \leqslant \\ &\leqslant \frac{1}{6480}\left\|f^{(4)}\right\|_\infty (b-a)^5 \end{aligned} \tag{3}$$

where the mapping $f : [a, b] \to \mathbb{R}$ is assumed to be four time differentiable on the interval (a, b) and having the fourth derivative bounded on (a, b), that is $\left\|f^{(4)}\right\|_\infty = \underset{x\in(a,b)}{sup} \left|f^{(4)}(x)\right| < \infty$.

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Assume that $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a equidistance partitioning of $[a, b]$ given by, $x_k = a + \frac{b-a}{n}k$, $k = 0, \ldots, n$ and f is as above, then we have the Newton quadrature formula:

$$\int_a^b f(x)\, dx = A_n(f) + R_n(f) \tag{4}$$

where $A_n(f)$ is the quadrature rule

$$A_n\ (f) = \frac{b-a}{8n}\left\{ f(a) + 2\sum_{k=1}^{n-1} f(a+kh) + 3\sum_{k=1}^{n}\left[f\left(a + \frac{3k-2}{3}h\right) + f\left(a + \frac{3k-1}{3}h\right)\right] + f(b)\right\} \tag{5}$$

and the remainder $R_n(f)$ satisfies the estimation

$$|R_n(f)| \leqslant \frac{1}{6480}\frac{(b-a)^5}{n^4}\left\| f^{(4)}(x)\right\|_\infty \tag{6}$$

In this paper, we establish some inequalities involving functions of bounded variation and some of its applications.

2. Main Result

2.1. Newton's Inequality Mappings of Bounded Variation. The following result holds:

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be a mapping with bounded variation on $[a, b]$ and $V_a^b(f)$ its total variation on $[a, b]$. Then we have the inequality:*

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]\right| \leq \frac{5}{24}(b-a)\, V_a^b(f) \tag{7}$$

Proof. Define the mapping $s : [a, b] \to \mathbb{R}$ as

$$s(x) = \begin{cases} x - \dfrac{7a+b}{8} & , x \in \left[a, \dfrac{2a+b}{3}\right) \\ x - \dfrac{a+b}{2} & , x \in \left[\dfrac{2a+b}{3}, \dfrac{a+2b}{3}\right] \\ x - \dfrac{a+7b}{8} & , x \in \left(\dfrac{a+2b}{3}, b\right] \end{cases} \tag{8}$$

Using the integration by parts formula for Riemann-Stieltjes integral, we have:

$$\int_a^b s(x)\,df(x) =$$
$$\int_a^{\frac{2a+b}{3}} \left(x - \frac{7a+b}{8}\right) df(x) + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left(x - \frac{a+b}{2}\right) df(x) + \int_{\frac{a+2b}{3}}^{b} \left(x - \frac{a+7b}{8}\right) df(x) =$$
$$\left[\left(x - \frac{7a+b}{8}\right) f(x)\right]_a^{\frac{2a+b}{3}} + \left[\left(x - \frac{a+b}{2}\right) f(x)\right]_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} +$$
$$\left[\left(x - \frac{a+7b}{8}\right) f(x)\right]_{\frac{a+2b}{3}}^{b} - \int_a^b f(x)\,dx =$$
$$\frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \int_a^b f(x)\,dx \tag{9}$$

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < \ldots < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\upsilon(\Delta_n) \to 0$ as $n \to \infty$, where $\upsilon(\Delta_n) = \max_{i\in\{0,\ldots,n-1\}} \left(x_{i+1}^{(n)} - x_i^{(n)}\right)$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$. If $p : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then:

$$\left|\int_a^b p(x)\,dv(x)\right| = \left|\lim_{\upsilon(\Delta_n)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) \left[v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right]\right| \leqslant$$
$$\leqslant \lim_{\upsilon(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|p\left(\xi_i^{(n)}\right)\right| \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right| = \tag{10}$$
$$= \max_{x\in[a,b]} |p(x)| \sup_{\Delta_n} \sum_{i=0}^{n-1} \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right| = \max_{x\in[a,b]} |p(x)| \ \ V_a^b(v).$$

Setting $p(x) = s(x)$ and $v(x) = f(x)$, $x \in [a,b]$, in (10) we have:

$$\left|\int_a^b s(x)\,df(x)\right| \le \max_{x\in[a,b]} |s(x)| \ \ V_a^b(f) = \frac{5}{24}(b-a)\,V_a^b(f). \tag{11}$$

Since $s(x)$ is piecewise continuous on $[a,b]$ and

$$s\left(\frac{2a+b}{3} - 0\right) = \frac{5}{24}(b-a)$$
$$s\left(\frac{2a+b}{3} + 0\right) = -\frac{1}{6}(b-a)$$
$$s\left(\frac{a+2b}{3} - 0\right) = \frac{1}{6}(b-a)$$
$$s\left(\frac{a+2b}{3} + 0\right) = -\frac{5}{24}(b-a)$$

we deduce that

$$\max_{x\in[a,b]} |s(x)| = \frac{5}{24}(b-a). \tag{12}$$

From (9), (11) and (12) we will have immediately the inequalities (7). □

Corollary 1. *Suppose that* $f:[a,b]\to\mathbb{R}$ *is a differentiable mapping whose derivative is integrable on* (a,b), *i.e.,*

$$\|f'\|_1=\int_a^b|f'(x)|dx<\infty$$

Then we have the inequality

$$\left|\int_a^b f\left(x\right)dx-\frac{b-a}{8}\left[f\left(a\right)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right|\leq\frac{5}{24}\|f'\|_1(b-a) \tag{13}$$

Corollary 2. *Let* I_n *be an equidistant partitioning of* $[a,b]$ *and* f *be as in Theorem 1 Then we have the formula* (3) *and the remainder satisfies the estimation:*

$$\left|R_n\left(f\right)\right|\leq\frac{5}{24n}\left(b-a\right)V_a^b\left(f\right). \tag{14}$$

Remark 1. *If we want to approximate the integral* $\int_a^b f(x)dx$ *by Newton formula* $A_n\left(f\right)$ *with an accuracy that* $\varepsilon>0$ *we need at least* $n_\varepsilon\in\mathbb{N}$ *points for the division* I_n, *where*

$$n_\varepsilon=\left[\frac{5}{24\varepsilon}\left(b-a\right)V_a^b\left(f\right)\right]+1 \tag{15}$$

and $[r]$ *denotes the integer part of* $r\in\mathbb{R}$.

Comments. If the mapping $f:[a,b]\to\mathbb{R}$ is neither four time differentiable nor the fourth derivative is bounded on (a,b), then we can not apply the classical estimation in Newton's formula using the fourth derivative. But if we assume that f is with bounded variation, then we can use instead the formula (14).

We give hear a class of mappings which are with bounded variation but having the fourth derivative unbounded on the given interval.

Let $f_p:[a,b]\to\mathbb{R}$, $f_p(x)=(x-a)^p$, where $p\in(3,4)$. Then obviously

$$f'_p\left(x\right)=p(x-a)^{p-1},x\in(a,b)$$

and

$$f_p^{(4)}\left(x\right)=\frac{p\left(p-1\right)\left(p-2\right)\left(p-3\right)}{\left(x-a\right)^{4-p}},\ x\in\left(a,b\right).$$

It is clear that f_p is with bounded variation and the total variation is

$$V_a^b\left(f_p\right)=(b-a)^p<\infty$$

but $\lim\limits_{x\to a+}f_p^{(4)}\left(x\right)=+\infty$.

3. Applications for Special Means

Let us recall the following means:

Arithmetic mean

$A=A(a,b)=\frac{a+b}{2}$, $a,b\geqslant 0$;

Geometric mean

$G=G(a,b)=\sqrt{ab}$, $a,b\geqslant 0$;

Harmonic mean

$H=H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}}$, $a,b>0$;

Logarithmic mean

$L=L(a,b)=\frac{b-a}{\ln b-\ln a}$, $a,b>0$, $a\neq b$;

Identric mean
$I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$, $a,b>0,\ a\neq b$;
p-Logarithmic mean
$L_p = L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}$, $p\in\mathbb{R}\setminus\{-1,0\}, a,b>0,\ a\neq b$.

In is well known that L_p is monotonic nondecreasing over $p\in\mathbb{R}$ with $L_{-1}=L$ and $L_0=I$. In particular, we have the following inequalities

$$H\leq G\leq L\leq I\leq A \tag{16}$$

1. Let $f:[a,b]\to\mathbb{R}$, $(0<a<b)$, $f(x)=x^p$, $p\in\mathbb{R}\setminus\{-1,0\}$. Then:

$$\int_a^b f(x)\,dx = (b-a)L_p^p(a,b),\ \frac{f(a)+f(b)}{2} = A(a^p,b^p)$$

$$\frac{f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)}{2} = A\left(\left(\frac{2a+b}{3}\right)^p,\left(\frac{a+2b}{3}\right)^p\right)$$

$$\|f'\|_1 = |p|\,(b-a)\,L_{p-1}^{p-1}, p\in\mathbb{R}\setminus\{-1,0,1\}$$

Using the inequality (13) we get:

$$\left|L_p^p(a,b)-\frac{1}{4}A(a^p,b^p)-\frac{3}{4}A\left(\left(\frac{2a+b}{3}\right)^p,\left(\frac{a+2b}{3}\right)^p\right)\right|\leq\frac{5\,|p|}{24}(b-a)\,L_{p-1}^{p-1} \tag{17}$$

2. Let $f:[a,b]\to\mathbb{R}$, $(0<a<b)$, $f(x)=\frac{1}{x}$. Then:

$$\int_a^b f(x)\,dx = (b-a)L^{-1}(a,b),\ \frac{f(a)+f(b)}{2} = H^{-1}(a,b)$$

$$\frac{f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)}{2} = H^{-1}\left(\frac{2a+b}{3},\frac{a+2b}{3}\right)$$

$$\|f'\|_1 = \frac{(b-a)}{G^2(a,b)}.$$

Using the inequality (13) we get:

$$\left|L^{-1}(a,b)-\frac{1}{4}H^{-1}(a,b)-\frac{3}{4}H^{-1}\left(\frac{2a+b}{3},\frac{a+2b}{3}\right)\right|\leq\frac{5}{24}\frac{(b-a)}{G^2(a,b)} \tag{18}$$

3. Let $f:[a,b]\to\mathbb{R}$, $(0<a<b)$, $f(x)=\ln x$. Then:

$$\int_a^b f(x)\,dx = (b-a)\ln I(a,b),\ \frac{f(a)+f(b)}{2} = \ln G(a,b)$$

$$\frac{f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)}{2} = \ln G\left(\frac{2a+b}{3},\frac{a+2b}{3}\right)$$

$$\|f'\|_1 = \frac{(b-a)}{L(a,b)}.$$

Using the inequality (13) we get:

$$\left|\ln\left[\frac{I(a,b)}{G^{\frac{1}{4}}(a,b)\,G^{\frac{3}{4}}\left(\frac{2a+b}{3},\frac{a+2b}{3}\right)}\right]\right|\leq\frac{5}{24}\frac{(b-a)}{L(a,b)}. \tag{19}$$

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