TJMM
2 (2010), No. 2, 149-152

A NOTE ON THE NUMERICAL RANGE OF A COMPLEX TRIDIAGONAL MATRIX

FATMA FEYZA TOPAL AND AHMET İPEK

Abstract. This paper is an extension of the work [Numerical range of a continuant matrix, Appl.Math.Lett., 14 (2001) 213-216.], in which the numerical range of a continuant matrix is explicitly described as an elliptic disc. In this new paper, we improve the results of that work.

1. Introduction

The numerical range (also known as the fields of values) of A is the set

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, |x| = 1\}.$$

Since the numerical range can often give information that the spectrum alone cannot give, we meet the necessity to compute it in many cases. Many of the proofs of the interesting results obtained for the numerical range are done by reducing the problem to considering the numerical range of a 2×2 complex matrix.

Lemma 1. *Let* $A = \begin{bmatrix} a & c \\ -\overline{c} & b \end{bmatrix} \in M_2$, *with* $a, b \in \mathbb{R}$. *Then* $W(A)$ *is an elliptic disc centered at* $(a+b)/2$, *with horizontal axis of length* $|a-b|$, *and vertical axis of length* $2|c|$.

Theorem 1. ([1], *Theorem* 1)*Let* $A \in \mathbb{C}^{n\times n}$ *and suppose* S *a nonzero subspace of* $\mathbb{C}^n$. *Then,*

$$W(A) = \underset{x,y}{\cup} W(A_{xy}),$$

where x *and* y *vary over all unit vectors in* S *and* $S^{\perp}$, *respectively, and where*

$$A_{xy} = \begin{pmatrix} x^*Ax & x^*Ay \\ y^*Ax & y^*Ay \end{pmatrix}.$$

Also, obtaining rather good information about the eigenvalues of tridiagonal matrices arisen some difference, differential equations and delay differential equations, the field of values $W(\cdot)$ is used.

Eiermann ([2], Corollary 4) proved that the numerical range of the matrix $A = [a_{ij}]$ such that

$$A_{ij} = \begin{cases} a_{i,i+1} = a, & i = 1,2,3,...,n-1 \\ a_{i+1,i} = b, & i = 1,2,3,...,n-1 \\ a_{ij} = 0, & |i-j| > 1 \end{cases}$$

is an elliptic disc.

As a consequence of Eirmann's results, Chien and Huang ([3], Lemma 2) obtained that the numerical range of the matrix B of the form

2010 *Mathematics Subject Classification.* 47A13; 15-00.
Key words and phrases. Numerical range; Tridiagonal matrix.

$$B=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \ddots & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -1 & \ddots \end{bmatrix}_{n\times n}$$

is the set

$$W(B)=\left\{z\in\mathbb{C}: z=2i\sin\theta\cos\left(\frac{\pi}{n+1}\right), 0\leq\theta<\pi\right\}.$$

In the same paper of Chien and Huang, it was proved that the numerical range of the matrix of the form

$$C=\begin{bmatrix} a & 1 & 0 & 0 & \cdots & 0 \\ -1 & b & 1 & 0 & \cdots & 0 \\ 0 & -1 & a & \ddots & \cdots & 0 \\ \vdots & \cdots & \ddots & b & \ddots & \vdots \\ 0 & \cdots & \cdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -1 & \ddots \end{bmatrix}_{n\times n} \quad \text{with } a,b\in\mathbb{R} \tag{1}$$

is an elliptic disc centered at $(a+b)/2$, with horizontal axis of length $|a-b|$, and vertical axis of length $4\cos\left(\frac{\pi}{n+1}\right)$.

The main purpose of this paper is to develop the results of [3].

2. Main Results

We come now to the main thrusts of this paper which is to develop the numerical range for a tridiagonal matrix containing the matrix in (1).

Theorem 2. *Let D be the matrix of the form*

$$D=\begin{bmatrix} a & c & 0 & 0 & \cdots & 0 \\ -\overline{c} & b & c & 0 & \cdots & 0 \\ 0 & -\overline{c} & a & \ddots & \cdots & 0 \\ \vdots & \cdots & \ddots & b & \ddots & \vdots \\ 0 & \cdots & \cdots & \ddots & \ddots & c \\ 0 & \cdots & \cdots & 0 & -\overline{c} & \ddots \end{bmatrix}$$

with $a,b\in\mathbb{R}, c\in\mathbb{C}$. Then, $W(D)$ is an elliptic disc centered at $(a+b)/2$, with horizontal axis of length $|a-b|$, and vertical axis of length $4|c|\cos\frac{\pi}{n+1}$.

Proof. Let e_i, $i=1,2,...,n$, be the n-dimensional i-th unit coordinate vector, i.e., the i-th entry of e_i is 1 and its remaining entries are 0. Also, let S be the subspace of $\mathbb{C}^n$ spanned by $\{e_1, e_{3,...}\}$. Then $S^\perp$ is the subspace spanned by $\{e_2, e_{4,...}\}$ .Now, let x and y vectors be vary over unit vectors in S and $S^\perp$, respectively.

Then, we have that

$$D_{x,y} = \begin{pmatrix} x^*Dx & x^*Dy \\ y^*Dx & y^*Dy \end{pmatrix}$$
$$= \begin{pmatrix} a & \begin{array}{c} c\left(\overline{x_1}y_2 - \overline{x_3}y_2 + \right. \\ \left. +\overline{x_3}y_4 - \overline{x_5}y_4 + ...\right) \end{array} \\ \begin{array}{c} -\overline{c}\left(-\overline{x_1}y_2 + \overline{x_3}y_2 - \right. \\ \left. -\overline{x_3}y_4 + \overline{x_5}y_4 - ...\right) \end{array} & b \end{pmatrix}.$$

Since

$$max\left|\overline{x_1}y_2 - \overline{x_3}y_2 + \overline{x_3}y_4 - \overline{x_5}y_4 + ...\right| = 2\left|c\right|\cos\frac{\pi}{n+1}$$

and by Lemma 1 and Theorem 1, we obtain the result. □

Corollary 1. *If the diagonal entries of D in Theorem 2 are not in the pattern $a, b, a, b, ...$, the $W(D)$ is not always an elliptic disc.*

2.1. **Numerical Considerations.**

Example 1. *The image in Figure 1 of the numerical range of the matrix*

$$E = \begin{bmatrix} 1 & 1-i & 0 \\ -1-i & 2 & 1-i \\ 0 & -1-i & 1 \end{bmatrix} \tag{2}$$

providing the criteries of Theorem 2 indicates that $W(E)$ is an elliptic disc.

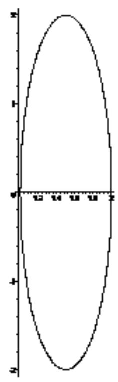

FIGURE 1

Example 2. *The image in Figure 2 of the numerical range of the matrix*

$$F = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 5 \end{bmatrix} \tag{3}$$

providing the criteries of Corollary 1 indicates that $W(F)$ is not an elliptic disc.

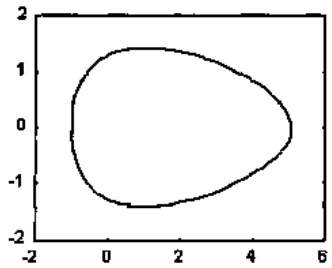

Figure 2

References

[1] M.-T. Chien, On the numerical range of tridiagonal operators, Linear Algebra and its Applications, 246 (1996) 203-214.
[2] M. Eiermann, Fields of values and iterative methods, Linear Algebra and Its Applications 180 (1993) 167-197.
[3] M.-T. Chien and J.-M. Huang, Numerical Range of a Continuant Matrix, Applied Mathematics Letters, 14 (2001) 213-216.

Mustafa Kemal University
Faculty of Art and Science
Department of Mathematics, Campus, Hatay, Turkey
E-mail address: ftopal@mku.edu.tr