TJMM
2 (2010), No. 2, 141-148

ON A SHARP INEQUALITY FOR THE MEDIANS OF A TRIANGLE

JIAN LIU

ABSTRACT. In this paper, we prove that the known inequality which involving the upper bounds of median sums for the triangle is sharp. We also prove a stronger conjecture for this inequality, which is equivalent to a inequality posed by H.Y.Yin in [1]. Finally, a similar conjecture checked by the computer is put forward.

1. INTRODUCTION

Let ABC be a triangle with medians m_a, m_b, m_c and semi-perimeter s, then we have the simple inequality (see [2]):

$$m_a + m_b + m_c < 2s, \tag{1}$$

where constant 2 is optimal.

In 2000, Chu Xiao-Guang and Yang Xue-Zhi [3] established a stronger inequality:

Theorem 1. *In any triangle ABC with medians m_a, m_b, m_c, semi-perimeter s, inradius r, and circumradius R, the following inequality holds:*

$$(m_a + m_b + m_c)^2 \leq 4s^2 - 16Rr + 5r^2. \tag{2}$$

The equality if and only if triangle ABC is equilateral.

The inequality (2) gives an excellent upper bound of expression $(m_a + m_b + m_c)^2$. But the authors of the the paper [3] have not considered a natural question: Find the maximum value for λ such that inequality

$$(m_a + m_b + m_c)^2 \leq 4s^2 - \lambda Rr + (2\lambda - 27)r^2 \tag{3}$$

holds for all triangle ABC.

One of the aim of this paper is to prove the following related conclusion:

Theorem 2. *Let λ be positive real numbers such that inequality (3) holds for all triangle ABC, then $\lambda_{max} = 16$.*

The inequality (3) just becomes (2) when $k = 16$. This means that inequality (2) is the strongest one in all inequalities such as (3). Out of the blue, Sun Wen-Cai recently posed a stronger conjecture (in a personal communication), he guessed the inequality below holds:

Theorem 3. *In any triangle ABC with sides a, b, c, medians m_a, m_b, m_c, inradius r, and circumradius R, the following inequality holds:*

$$\frac{(m_a + m_b + m_c)^2}{a^2 + b^2 + c^2} \leq 2 + \frac{r^2}{R^2}. \tag{4}$$

The equality if and only if triangle ABC is equilateral.

2010 *Mathematics Subject Classification.* 51M16.
Key words and phrases. triangle, median, Euler inequality, Gerretsen inequality, L'Hôpital Rule.

By applying the known identity $a^2+b^2+c^2=2s^2-8Rr-2r^2$ and Euler inequality (see [2], [4]):

$$R \geq 2r, \tag{5}$$

we can easily show that inequality (4) is stronger than (2).

After Sun posed inequality (4), I soon found that it is equivalent to the following conjecture inequality which was posed by Yin Hua Yan [1] in 2000:

$$\frac{m_b m_c + m_c m_a + m_a m_b}{a^2+b^2+c^2} \leq \left(\frac{5}{8}+\frac{r^2}{2R^2}\right). \tag{6}$$

In the fact, by the well known identity

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2). \tag{7}$$

we can easily know that inequality (4) is equivalent to (6).

In Section 3, we will prove inequality (4) by applying the method of $R-r-s$, see [5]-[6].

2. The Proof of Theorem 2

Proof. If $R \neq 2r$, then the inequality (2) is equivalent to

$$\lambda \leq \frac{4s^2-27r^2-(m_a+m_b+m_c)^2}{r(R-2r)},$$

By the known identities $sr^2=(s-a)(s-b)(s-c), abc=4Rrs$, we further see that the above inequality is equivalent to

$$\lambda \leq \frac{4s^3-27(s-a)(s-b)(s-c)-s(m_a+m_b+m_c)^2}{abc-8(s-a)(s-b)(s-c)}. \tag{8}$$

We now suppose ABC is an isosceles triangle with sides $x,1,1(x\neq 1)$, putting $b=c=1, a=x$, then using median formula $m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ and the known identities:

$$s=\frac{1}{2}(a+b+c),\ R=\frac{abc}{4\sqrt{(s-a)(s-b)(s-c)}},\ r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

we get

$$s=\frac{1}{2}x+1,\ m_a=\frac{1}{2}\sqrt{4-x^2},\ m_b=m_c=\frac{1}{2}\sqrt{1+2x^2},\ R=\frac{1}{\sqrt{4-x^2}},\ r=\frac{x\sqrt{4-x^2}}{2(x+2)}.$$

Plugging $a=x, b=c=1$ and the above relations into (8), after some calculations we obtain

$$\lambda \leq \frac{12x^3-22x^2+20x-2(x+2)\sqrt{(4-x^2)(1+2x^2)}+8}{x(x-1)^2}. \tag{9}$$

Let

$$f(x)=12x^3-22x^2+20x-2(x+2)\sqrt{(4-x^2)(1+2x^2)}+8,\ g(x)=x(x-1)^2.$$

Then it follows that

$$f'(x)=\frac{12x^4+16x^3-28x^2-28x-8+4(9x^2-11x+5)\sqrt{(4-x^2)(1+2x^2)}}{\sqrt{(4-x^2)(1+2x^2)}},$$

$$g'(x)=3x^2-4x+1.$$

Note that

$$\lim_{x\to 0} f(x)=0, \lim_{x\to 0} g(x)=0.$$

According to inequality (9) and the L'Hôpital's Rule, we get

$$\begin{aligned}
&\lambda \le \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}\\
&= \lim_{x\to 0}\frac{12x^4+16x^3-28x^2-28x-8+4(9x^2-11x+5)\sqrt{(4-x^2)(1+2x^2)}}{(3x^2-4x+1)\sqrt{(4-x^2)(1+2x^2)}}\\
&= 16.
\end{aligned}$$

Thus we complete the proof of Theorem 2. □

Remark 1. *In* [3], *Chu Xiao-Guang and Yang Xue-Zhi also obtained inverse inequality of (2):*

$$(m_a+m_b+m_c)^2 \ge 4s^2 - 28Rr + 29r^2. \tag{10}$$

In the same way as in the proof of Theorem 2, we can also prove the following conclusion: Let λ be positive real numbers such that

$$(m_a+m_b+m_c)^2 \ge 4s^2 - \lambda Rr + (2\lambda - 27)r^2. \tag{11}$$

holds for all triangles ABC, then $\lambda_{min} = 28$.

3. The Proof of Theorem 3

In order to prove Theorem 3, we need some lemmas.

Lemma 1. [3] *In all triangle ABC the following inequality holds:*

$$4m_bm_c \le 2a^2 + bc - \frac{4s(s-a)(b-c)^2}{2a^2+bc}, \tag{12}$$

with equality if and only if $b = c$.

Lemma 2. *In all triangle ABC the following inequalities hold:*

$$16Rr - 5r^2 \le s^2 \le \frac{R(4R+r)^2}{2(2R-r)}, \tag{13}$$

with equality if and only if $\triangle ABC$ is equilateral.

The first inequality

$$s^2 \ge 16Rr - 5r^2 \tag{14}$$

is the well-known Gerretsen inequality (see [2],[4]). The second inequality

$$s^2 \le \frac{R(4R+r)^2}{2(2R-r)} \tag{15}$$

is Kooi inequality (see [2]), which is stronger than another Gerretsen inequality:

$$s^2 \le 4R^2 + 4Rr + 3r^2. \tag{16}$$

In addition, Kooi inequality is equivalent to Garfunkel-Bankoff inequality (see [7]-[9]):

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \ge 2 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{17}$$

where A, B, C are the angles of triangle ABC. In [10], the author gave a generalization of the equivalent form of the above inequality.

Lemma 3. *In all triangle ABC the following inequality holds:*

$$
\begin{aligned}
&8Rr(115R^2+47r^2)s^4+(64R^6-8448rR^5+6484R^4r^2-35204R^3r^3\\
&+9260R^2r^4-1912Rr^5+80r^6)s^2+4r(292R^5-485R^4r+1727R^3r^2\\
&+137R^2r^3+16Rr^4+4r^5)(4R+r)^2\geq 0, \qquad (18)
\end{aligned}
$$

with equality if and only if triangle ABC if equilateral.

Proof. We denote the right hand side of (18) by $Q_1(s^2)$ in which s^2 is being seen as a variable and put

$$
\begin{aligned}
T=&\,16Rr(115R^2+47r^2)s^2+64R^6-8448rR^5+6484R^4r^2-35204R^3r^3\\
&+9260R^2r^4-1912Rr^5+80r^6.
\end{aligned}
$$

We will prove inequality $Q_1(s^2)\geq 0$ divides into two cases.

Case 1. $T\geq 0$.

According to the property of parabolas and Gerretsen's inequality chain $16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2$, $Q_1(s^2)$ is monotone increasing on interval $[16Rr-5r^2, 4R^2+4Rr+3r^2]$ in this case. Therefore, in order to prove $Q_1(s^2)\geq 0$ we need to prove that $Q_1(16Rr-5r^2)\geq 0$, namely

$$
\begin{aligned}
&8Rr(115R^2+47r^2)(16Rr-5r^2)^2+(64R^6-8448rR^5+6484R^4r^2-35204R^3r^3\\
&+9260R^2r^4-1912Rr^5+80r^6)(16Rr-5r^2)+4r(292R^5-485R^4r+1727R^3r^2\\
&+137R^2r^3r+16Rr^4+4r^5)(4R+r)^2\geq 0,
\end{aligned}
$$

after simplifying, that is

$$8r(R-2r)(2464R^6-14720R^5r+30270R^4r^2-24559R^3r^3+7851R^2r^4-1265Rr^5+24r^6)\geq 0,$$

By Euler's inequality (5), it suffices to prove that

$$2464R^6-14720R^5r+30270R^4r^2-24559R^3r^3+7851R^2r^4-1265Rr^5+24r^6>0, \quad (19)$$

which is equivalent to

$$2464d^6+14848d^5r+30910d^4r^2+23041d^3r^3+(737d^2-441dr+3402r^2)r^4>0,$$

where $d=R-2r\geq 0$. It is easy to show that $737d^2-441dr+3402r^2>0$, hence inequality (19) is proved. The proof of inequality (18) in the first case is complete.

Case 2. $T<0$.

We can get the following identity:

$$2Q_1(s^2)=-t_2T+M_1t_1+16drM_2+256d^8, \quad (20)$$

where

$$
\begin{aligned}
&d=R-2r\\
&t_1=s^2-16Rr+5r^2,\\
&t_2=4R^2+4Rr+3r^2-s^2,\\
&M_1=64d^6-320d^5r+6804d^4r^2+50796d^3r^3+112788d^2r^4+97560dr^5+27216r^6,\\
&M_2=560d^6+1749d^5r-1442d^4r^2-8991d^3r^3-1956d^2r^4+14529dr^5+10206r^6.
\end{aligned}
$$

From Euler inequality (5), Gerretsen inequalities (14), (16) and the hypothesis $T<0$ we have $d\geq 0, t_1\geq 0, -t_2T\geq 0$, thus to prove $Q_1(s^2)\geq 0$ we need prove $M_1>0, M_2>0$. Since $64d^6-320d^5r+6804d^4r^2=4d^4(16d^2-80dr+1701r^2)>0$, hence $M_1>0$. By applying the monotonicity of the function, we easily prove that

$$560x^6+1749x^5-1442x^4-8991x^3-1956x^2+14529x+10206>0. \quad (21)$$

Then inequality $M_1 > 0$ follows immediately. Thus, we complete the proof of $Q_1(s^2) \geq 0$ in the second case.

Combing the arguments of the two cases, inequality (18) holds for all triangle ABC. The equality in (18) occurs if and only if $R = 2r$, namely $\triangle ABC$ is equilateral. The proof of Lemma 3 is complete. □

Lemma 4. *In all triangle ABC the following inequality holds:*

$$s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3 \leq 0, \tag{22}$$

with equality if and only if ABC is isosceles.

Inequality (22) is the fundamental triangle inequality, which has various equivalent forms, see [2], [4], [5], [6], [9].

Lemma 5. *In all triangle ABC holds*

$$\begin{aligned}&(-R^2 + 56Rr - 4r^2)s^6 + (8R^4 - 480R^3r - 891R^2r^2 + 80Rr^3 - 12r^4)s^4\\ &+(4R + r)(244R^4 + 1420R^3r + 533R^2r^2 + 56Rr^3 - 12r^4)s^2r\\ &-(33R^2 + 16Rr + 4r^2)(4R + r)^4r^2 \geq 0,\end{aligned} \tag{23}$$

with equality if and only if triangle ABC is equilateral.

Proof. Denote the right hand side of (26) by Q_2, we can rewrite it as follows:

$$\begin{aligned}Q_2 = {}&(s^2 - 16Rr + 5r^2)[(-R^2 + 56Rr - 4r^2)s^4 + (8R^4 - 496R^3r + 10R^2r^2\\ &-264Rr^3 + 8r^4)s^2 + r(1104R^5 - 2052R^4r + 6192R^3r^2 - 3517R^2r^3\\ &+1456Rr^4 - 52r^5)] + 4r^2[2304(R - 2r)^4 + 14924(R - 2r)^3r\\ &+36261(R - 2r)^2r^2 + 39188(R - 2r)r^3 + 15876r^4](R - 2r)^2.\end{aligned} \tag{24}$$

So, if the following holds

$$\begin{aligned}&(-R^2 + 56Rr - 4r^2)s^4 + (8R^4 - 496R^3r + 10R^2r^2 - 264Rr^3 + 8r^4)s^2\\ &+r(1104R^5 - 2052R^4r + 6192R^3r^2 - 3517R^2r^3 + 1456Rr^4 - 52r^5) \geq 0,\end{aligned} \tag{25}$$

then $Q_2 \geq 0$ follows from the first inequality of Lemma (2) and Euler's inequality (5). If inequality (25) is reverse, by Kooi inequality (15), it is sufficient to prove that

$$\begin{aligned}&\left[\frac{R(4R + r)^2}{2(2R - r)} - 16Rr + 5r^2\right]\left[(-R^2 + 56Rr - 4r^2)s^4 + (8R^4 - 496R^3r\right.\\ &+10R^2r^2 - 264Rr^3 + 8r^4)s^2 + r(1104R^5 - 2052R^4r + 6192R^3r^2 - 3517R^2r^3\\ &\left.+1456Rr^4 - 52r^5)\right] + 4r^2\left[2304(R - 2r)^4 + 14924(R - 2r)^3r + 36261(R - 2r)^2r^2\right.\\ &\left.+39188(R - 2r)r^3 + 15876r^4\right](R - 2r)^2 \geq 0.\end{aligned} \tag{26}$$

This can be changed into

$$\frac{R - 2r}{2(2R - r)}M_3 \geq 0, \tag{27}$$

where

$$\begin{aligned}M_3 = {}&-(4R - 5r)(4R - r)(R^2 - 56Rr + 4r^2)s^4 + 2(4R - 5r)(4R - r)(4R^4 - 248R^3r\\ &+5R^2r^2 - 132Rr^3 + 4r^4)s^2 + r(1104R^5 - 1956R^4r + 1256R^3r^2\\ &-856R^2r^3 - 16Rr^4 - 4r^5)(4R + r)^2.\end{aligned}$$

After analyzing, we obtain the following identity:

$$\begin{aligned}M_3 = {}&(16R^4 + 1413R^2r^2 + 20r^4)\left[-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3\right]\\ &+2R(4R - r)(4R - 5r)r^5 + Q_1(s^2),\end{aligned} \tag{28}$$

where $Q_1(s^2)$ denotes the value of the right hand side of (18) which is non-negative. Therefore, $M_3 \geq 0$ follows from the fundamental inequality (22), hence inequality (27) is proved.

Combing the above arguments, we deduce that $Q_2 \geq 0$ holds for every triangle. The equality in $Q_2 \geq 0$ is valid when triangle ABC is equilateral. The proof of Lemma 5 comes to the end. □

We are now in a position to prove Theorem 3. Next, let $\sum, \prod$ denote the cyclic sums and the cyclic products respectively.

Proof. As we have said, inequality (4) is equivalent to (6). So we have to prove that

$$4\sum m_b m_c \leq \left(\frac{5}{2} + \frac{2r^2}{R^2}\right)\sum a^2. \tag{29}$$

By Lemma 1, it suffices to prove that

$$\sum(2a^2+bc) - 4s\sum\frac{(s-a)(b-c)^2}{2a^2+bc} \leq \left(\frac{5}{2} + \frac{2r^2}{R^2}\right)\sum a^2,$$

i.e.

$$\begin{aligned}&\sum a\sum(b+c-a)(2b^2+ca)(2c^2+ab)(b-c)^2\\&+\left(\frac{R^2+4r^2}{2R^2}\sum a^2 - \sum bc\right)\prod(2a^2+bc) \geq 0.\end{aligned}$$

Putting

$$\begin{aligned}M_4 = {}& 2R^2\sum a\sum(b+c-a)(2b^2+ca)(2c^2+ab)(b-c)^2\\&+\left[(R^2+4r^2)\sum a^2 - 2R^2\sum bc\right]\prod(2a^2+bc).\end{aligned} \tag{30}$$

then we need to prove $M_4 \geq 0$. To do so, we first make some computations.

It is easy to check the identities:

$$\prod(2a^2+bc) = 9(abc)^2 + 2abc\sum a^3 + 4\sum b^3c^3, \tag{31}$$

and

$$\begin{aligned}&\sum(b+c-a)(2b^2+ca)(2c^2+ab)(b-c)^2\\&= 2\sum bc\sum a^5 - 6abc\sum a^4 + 2\sum b^2c^2\sum a^3 - abc\sum bc\sum a^2\\&-5(abc)^2\sum a - 4\sum a\sum b^3c^3 + 14abc\sum b^2c^2.\end{aligned} \tag{32}$$

Then using $\sum a = 2s$ and the following well-known identities:

$$abc = 4Rrs \tag{33}$$
$$\sum bc = s^2 + 4Rr + r^2, \tag{34}$$
$$\sum a^2 = 2s^2 - 8Rr - 2r^2, \tag{35}$$
$$\sum a^3 = 2s^3 - (12Rr + 6r^2)s, \tag{36}$$
$$\sum a^4 = 2s^4 - (16Rr + 12r^2)s^2 + 2r^2(4R + r)^2, \tag{37}$$
$$\sum a^5 = 2s^5 - (20Rr + 20r^2)s^3 + (80R^2r^2 + 60Rr^3 + 10r^4)s, \tag{38}$$
$$\sum b^2c^2 = s^4 + (-8Rr + 2r^2)s^2 + (4R + r)^2r^2, \tag{39}$$
$$\sum b^3c^3 = s^6 + (-12Rr + 3r^2)s^4 + 3r^4s^2 + r^3(4R + r)^3, \tag{40}$$

We further obtain

$$\prod(2a^2 + bc) = 4s^6 - (32Rr - 12r^2)s^4 + 12(2R - r)^2r^2s^2 + 4(4R + r)^3r^3, \tag{41}$$

and

$$\begin{aligned}&\sum(b + c - a)(2b^2 + ca)(2c^2 + ab)(b - c)^2\\&= 16rs^3\left[(R - 4r)s^2 + r(2R^2 + 25Rr - 4r^2)\right].\end{aligned} \tag{42}$$

Plugging $\sum a = 2s$,(34),(35),(41), and (42) into the expression of M_4, we can get

$$M_4 = 16r^2M_5, \tag{43}$$

where

$$\begin{aligned}M_5 = {}& 2s^8 - (17R^2 + 24Rr - 4r^2)s^6 + R(40R^3 + 96R^2r + 69Rr^2 - 32r^3)s^4\\&-(R - 2r)(4R + r)(12R^3 + 12R^2r + 19Rr^2 - 2r^3)s^2r - (R^2 + 2r^2)(4R + r)^4r^2.\end{aligned}$$

Therefore, in order to prove $M_4 \geq 0$ we need to prove that $M_5 \geq 0$. After studying, we get the following identity:

$$M_5 \equiv Q_2 + 2\left[s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3\right]^2, \tag{44}$$

where Q_2 is equal to the value of the left hand side of inequality (23). Hence $M_5 \geq 0$, inequality $M_4 \geq 0$ and inequality (29) are proved. It is easy to see that the equality in (29) if and only if triangle ABC is equilateral. This completes the proof of Theorem 3. □

4. A Conjecture

The author has proved the following refinement of inequality (1):

$$m_a + m_b + m_c < 2(b^2c^2 + c^2a^2 + a^2b^2)^{\frac{1}{4}} < 2s. \tag{45}$$

This result prompts us to propose a conjecture similar to Theorem 3, which is checked by the computer:

Conjecture 1. *In all triangle ABC, the following inequality holds:*

$$\frac{(m_a + m_b + m_c)^4}{b^2c^2 + c^2a^2 + a^2b^2} \leq 16 - \frac{13r^2}{4R^2}. \tag{46}$$

Remark 2. *We know that inequality (46) can not be derived from inequality (2) or inequality (4).*

References

[1] Hu.-Y.yin, 110 conjecture inequalities involving Ceva segments and radii of a triangle, *Research in Ineqalities*, Tibet People's Press, Lhasa, 2000, 313-322. (in Chinese)

[2] O.Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović, AND P.M. Vasić. *Geometric Inequality.* Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.

[3] Xi.-G.chu AND Xu.-Z.Yang, On some inequalities involving medians of a triangle, *Research in Ineqalities*, Tibet People's Press, Lhasa, 2000, 234-247. (in Chinese)

[4] D.Mitrinović, J.E.Pečarić AND V.Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1989.

[5] W.J.Blundon, Inequalities associated with the triangle, *Canad. Math. Bull.*, **8**(1965), 615-626.

[6] O.Bottema, Inequalities for R, r and s, *Univ. Beograd.Publ.Elektrotehn. Fak. Ser. Mat. Fiz.*, No. **338-352**(1971)27-36.

[7] J.Garfunkel, Problem 825, *Crux Math.*, **9**(1983), 79.

[8] L.Bankoff, Solution of Problem 825, *Crux Math.*, **10**(1984), 168.

[9] Sh.-H.wu AND M.Bencze, An equivalent form of the fundamental triangle inequality, *J. Inequal. Pure Appl. Math.*, **10**(1)(2009), Art.16.

[10] J.Liu, An extension of an equivalent form of the Kooi inequality, *Journal of Guizhou University (Natural Sciences).*, **23**(2)(2005), 60-62. (in Chinese)

East China Jiaotong University
Nanchang City, Jiangxi Province, 330013, P.R.China
E-mail address: dfj168@163.com