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QUATERNION INTERVAL NUMBERS AND A MATRIX APPROACH TO THEM

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Abstract. This paper is an extension of the work [Quaternions: further contributions to a matrix oriented approach, Linear Alg. and its Appl., 326 (2001), 205,-213.] , in which the representation matrix of a quaternion with real number coefficients is explicitly described, and then investigated properties of the fundamental real matrix associated with a quaternion. In this new paper, we extend the powerful ideas in that study to the quaternions with real interval coefficients. Many number of concepts and techniques that we learned in a standard setting for quaternions with real number coefficients, real intervals and matrices can be carried over to quaternion interval numbers.

1. Introduction

It is well known that a complex number is a number consisting of a real and imaginary part. It can be written in the form $a + bi$, where i is the imajinary number with the defining property $i^2 = -1$. The set of all complex numbers is usually denoted by $\mathbb{C}$. From here, it can be easily said that the set of complex numbers is an extension of the set of real numbers. That is, $\mathbb{R} \subset \mathbb{C}$.

Also in literature, the set of quaternions introduced as

$$\mathbf{H}[\mathbb{R}] = \{a = a_0 + a_1 i + a_2 j + a_3 k : \ a_i \in \mathbb{R}, i = 0, 1, 2, 3\}$$

with

$$i^2 = j^2 = k^2 = -1, ij = k = -ji, \ jk = i = -kj, \ ki = j = -ik, \tag{1}$$

by Irish mathematician Sir William Rowan Hamilton in 1843, is a generalized the set of complex numbers. See for example [2] or [1] for a review.

Addition of two quaternions $a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$ is defined by

$$a + b = (a_0 + b_0) + (a_1 + b_1)\, i + (a_2 + b_2)\, j + (a_3 + b_3)\, k,$$

whereas multiplication is defined by

$$\begin{aligned} a.b = {} & (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) \\ & + (a_1 b_0 + a_0 b_1 - a_3 b_2 + a_2 b_3)\, i \\ & + (a_2 b_0 + a_3 b_1 + a_0 b_2 - a_1 b_3)\, j \\ & + (a_3 b_0 - a_2 b_1 + a_1 b_2 + a_0 b_3)\, k. \end{aligned}$$

It is well known that $\mathbf{H}$ with the operations addition and multiplication forms a skew-field (also called 'division algebra'), so that in general we have to expect $ab \neq ba$ for $a, b \in \mathbf{H}$ [3]. In [4], the representation matrix of a quaternion with real number coefficient is explicitly described, and then investigated properties of the fundamental real matrix

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associated with a quaternion. In [5], it is presented an alternative way to define quaternions is to consider to subset of the ring $M_2(\mathbb{C})$ of 2×2 matrices with complex number entries:

$$\mathbf{H}^{'} = \left\{ \begin{pmatrix} q_1 & q_2 \\ -\overline{q_2} & \overline{q_1} \end{pmatrix} : q_1, q_2 \in \mathbb{C} \right\}.$$

In that paper, it was expressed that the transformation

$$T : q = q_1 + q_2 j \in \mathbf{H} \to q^{'} = \begin{pmatrix} q_1 & q_2 \\ -\overline{q_2} & \overline{q_1} \end{pmatrix} \in \mathbf{H}^{'}$$

is bijective and preserves the operations. Furthermore, $|q|^2 = \det q^{'}$ and the eigenvalues of $q^{'}$ are $\operatorname{Re} q \pm |\operatorname{Im} q|\, i$.

In this present paper, we will define the set of quaternion intervals that it can be seen as an extension of the set of quaternions with real coefficients. Now, we introduce the definition of real intervals and their some basic properties which we will need in next sections.

Let $\widetilde{a} = [a_1, a_2] = \{x : \ a_1 \leq x \leq a_2, \ x \in \mathbb{R}\}$, where a_1 and a_2 are the left and right limits of the interval a on the real line $\mathbb{R}$, respectively. We shall use the terms "interval" and "interval number" interchangeably. If $a_1 = a_2 = a$, then $\widetilde{a} = [a, a]$ is a real number (or a degenerate interval). We use $I(\mathbb{R})$ to denote the set of all interval numbers on the real line $\mathbb{R}$.

Interval arithmetic was first suggested by Dwyer [6],[7] in 1951. Devolopment of interval arithemtic as a formal system and evidence of its value as a computational device was provided by Moore [8],[9]. After this motivation and inspiration, several authors such as Alefeld and Herzberger [10], Hansen [11], [12], [13], Neumaier [14] etc have studied interval arithmetic.

Let $x = [x_1, x_2]$,$y = [y_1, y_2] \in I(\mathbb{R})$. Then, some operations on these interval numbers are defined as follows:

- Addition: $x + y = [x_1 + y_1, x_2 + y_2]$,
- Subctraction: $x - y = [x_1 - y_2, x_2 - y_1]$,
- Multiplication: $x * y = [\min S, \max S]$, where $S = \{x_1 y_1, \ x_2 y_1, \ x_1 y_2, \ x_2 y_2\}$.

In this study, we first define the quaternion intervals set and the quaternion interval numbers. We second present the representation vector and matrix for quaternion interval numbers, and then investigate some algebraic properties of these representations, which the representation matrix is called the fundamental matrix. Hence, in here, by investigating the main features of the fundamental matrix we employ a matrix oriented approach to the topic. Finally, we compute the determinant, norm, inverse, trace, eigenvalues and eigenvectors of the representation matrix established for a general quaternion interval number.

2. Quaternion Interval Numbers and Their Vector and Matrix Representations

We come now to the main definition and thrusts of this paper which is to develop the quaternion numbers with real coefficients.

Definition 1 (**Quaternion Interval Numbers**). *The set of the quaternion interval numbers is the set*

$$\mathbf{H}\left[I\left(\mathbb{R}\right)\right] = \left\{ A = A_0 + A_1 i + A_2 j + A_3 k \ : A_i = \left[\underline{A_i}, \overline{A_i}\right]; \underline{A_i}, \overline{A_i} \in \mathbb{R}, \ i = 0, 1, 2, 3 \right\},$$

where i, j and k elements satisfy (1) *condition.*

Each element of the set $\mathbf{H}[I(\mathbb{R})]$ is called as a quaternion interval, and from the definition of the real intervals, the quaternion interval $A = A_0 + A_1 i + A_2 j + A_3 k \in \mathbf{H}[I(\mathbb{R})]$ is written by

$$A = \{a = a_0 + a_1 i + a_2 j + a_3 k : \; a_i \in A_i, \; A_i \in I(\mathbb{R}), i = 0, 1, 2, 3\}.$$

The vector representation $\mathbf{r}_A$ of a quaternion interval $A \in \mathbf{H}[I(\mathbb{R})]$ is given by

$$\mathbf{r}_A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \{\mathbf{r}_a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} : a_i \in A_i, \; A_i \in I(\mathbb{R}), i = 0, 1, 2, 3\}.$$

it is clear that the quaternion interval $A \in \mathbf{H}[I(\mathbb{R})]$ is obtained from the vector representation $\mathbf{r}_A$ and vice versa.

It is nearby to identify a quaternion interval $A \in \mathbf{H}[I(\mathbb{R})]$ with a real interval vector $\mathbf{r}_A \in \mathbb{R}^4$. We will denote such an identification by the $\cong$ symbol, i.e.,

$$A_0 + A_1 i + A_2 j + A_3 k = A \cong \mathbf{r}_A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}.$$

For a given quaternion interval $A = A_0 + A_1 i + A_2 j + A_3 k$ the quaternion interval $\overline{A} = A_0 - A_1 i - A_2 j - A_3$ is called the adjoint of A, and the vector representation, $\mathbf{r}_{\overline{A}}$, of quaternion interval $\overline{A}$, is given as

$$\mathbf{r}_{\overline{A}} = \begin{pmatrix} A_0 \\ -A_1 \\ -A_2 \\ -A_3 \end{pmatrix} = \left\{ \mathbf{r}_{\overline{a}} = \begin{pmatrix} a_0 \\ -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} : a_i \in A_i, \; A_i \in I(\mathbb{R}), i = 0, 1, 2, 3 \right\}$$

and it is shown that $\overline{A} \cong \mathbf{r}_{\overline{A}}$.

Then, the addition or subraction of quaternion intervals $A, B \in \mathbf{H}[I(\mathbb{R})]$ can be given as the vector representations($A \cong \mathbf{r}_A, B \cong \mathbf{r}_B$) of the quaternion intervals

$$A \pm B \cong \mathbf{r}_A \pm \mathbf{r}_B.$$

Now, the matrix representation of quaternion interval $A \in \mathbf{H}[I(\mathbb{R})]$ will presented.

For the matrix representation $\mathbf{R}_A$ of a quaternion interval $A \in \mathbf{H}[I(\mathbb{R})]$, firstly it can be written as

$$\begin{aligned} A &= \{a = a_0 + a_1 i + a_2 j + a_3 k : a_i \in A_i, A_i \in I(\mathbb{R}), i = 0, 1, 2, 3\} \\ &= \{a = a_0 + a_1 i + (a_2 + a_3 i) j : a_i \in A_i, A_i \in I(\mathbb{R}), i = 0, 1, 2, 3\}. \end{aligned}$$

Then, by recalling the matrix representations of the complex numbers and quaternion numbers, we have that

$$\begin{aligned}\mathbf{R}_A &= \left\{ \mathbf{R}_a = \begin{pmatrix} a_0 + a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & a_0 + a_1 i \end{pmatrix} : a_i \in A_i,\ A_i \in I(\mathbb{R}), i = 0,1,2,3 \right\} \\ &= \left\{ \mathbf{R}_a = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{pmatrix} : a_i \in A_i,\ A_i \in I(\mathbb{R}), i = 0,1,2,3 \right\} \\ &= \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ -A_1 & A_0 & -A_3 & A_2 \\ -A_2 & A_3 & A_0 & -A_1 \\ -A_3 & -A_2 & A_1 & A_0 \end{pmatrix}.\end{aligned}$$

and denoted by $\mathbf{R}_A \cong A$.

Now, the following proposition give some algebraic properties of the representation vector and matrix us.

(1) For the addition or subraction of the quaternion intervals, we write

$$A \pm B \cong \mathbf{R}_A \pm \mathbf{R}_B = \mathbf{R}_{A \pm B}.$$

(2) For scalar multiplication of $\alpha, \beta \in \mathbb{R}$ and the quaternion intervals $A, B \in \mathbf{H}[I(\mathbb{R})]$, we obtain

$$\alpha A \pm \beta B \cong \alpha \mathbf{R}_A \pm \beta \mathbf{R}_B = \mathbf{R}_{\alpha A + \beta B}.$$

(3) For the multiplication of the quaternion intervals $A, B \in \mathbf{H}[I(\mathbb{R})]$, we have

$$AB \cong \mathbf{R}_A \mathbf{R}_B = \mathbf{R}_{AB},$$

where $\mathbf{R}_A \cong A$ and $\mathbf{R}_B \cong B$.

(4) Let $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Then, the vector representation, $\mathbf{r}_{\overline{A}}$, of the adjoint of the quaternion interval $A \in \mathbf{H}[I(\mathbb{R})]$ can be written by

$$\mathbf{r}_{\overline{A}} = C \mathbf{r}_A = \begin{pmatrix} A_0 \\ -A_1 \\ -A_2 \\ -A_3 \end{pmatrix}.$$

(5) Let $\mathbf{L}_A = C \mathbf{R}_A C$. The following equalities are valid :
   (a) $\mathbf{L}_A \mathbf{r}_B \cong AB$,
   (b) $\mathbf{R}_A^T \mathbf{r}_B \cong BA$,
   (c) $\mathbf{L}_A \mathbf{R}_B^T = \mathbf{R}_B^T \mathbf{L}_A$,
   (d) $\mathbf{R}_B \mathbf{R}_A = \mathbf{R}_{\left(\mathbf{R}_{AB}^T\right)}$,
   (e) $\mathbf{L}_{\overline{A}} = \mathbf{L}_A^T$,
   (f) $\mathbf{R}_{\overline{A}} = \mathbf{R}_A^T$,
   (g) $\mathbf{R}_A^T \mathbf{R}_A = \mathbf{R}_A \mathbf{R}_A^T$,
   (h) $A\overline{A} \cong \mathbf{R}_{\overline{A}}^T \mathbf{r}_A = \mathbf{L}_A \mathbf{r}_{\overline{A}}$,
   (i) $\overline{BA} \cong C\left(R_A^T \mathbf{r}_B\right) = \left(C R_A^T\right) \mathbf{r}_B = L_A^T \left(C \mathbf{r}_B\right) = L_{\overline{A}} \mathbf{r}_{\overline{B}} \cong \overline{A}\ \overline{B}$.

2.1. **The determinant of the representation matrix $\mathbf{R}_A$.** In this section, we investigate the determinant of the representation matrix $\mathbf{R}_A$ for the quaternion interval number $A \in \mathbf{H}[I(\mathbb{R})]$.

Determinant of the representation matrix $\mathbf{R}_A$ is given as

$$\det \mathbf{R}_A = \left\{ f(a_0,a_1,a_2,a_3) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{vmatrix} : a_i \in A_i,\ A_i \in I(\mathbb{R}), \right.$$
$$\left. i = 0,1,2,3 \right\}$$
$$= \left\{ f(a_0,a_1,a_2,a_3) = \left(a_0^2 + a_1^2 + a_2^2 + a_3^2\right)^2 : a_i \in A_i,\ A_i \in I(\mathbb{R}), \right.$$
$$\left. i = 0,1,2,3 \right\}.$$

To obtain an real interval version of the set $\det \mathbf{R}_A$, a natural way is to find extreme points of the real-valued function $f(a_0,a_1,a_2,a_3) = \left(a_0^2 + a_1^2 + a_2^2 + a_3^2\right)^2$ of four variables. Hence, the calculating problem of determinant of the matrix $\mathbf{R}_A$ is reduced to the finding problem of local minumum and maximum values of f. For each $a_i, i = 0,1,2,3$, the partial derivative of f is

$$f'_{a_i}(a_0,a_1,a_2,a_3) = 4a_i\left(a_0^2 + a_1^2 + a_2^2 + a_3^2\right).$$

It is clearly that as an real interval or an union of real intervals, determining of the set $\det \mathbf{R}_A$ is surprisingly tedious. We must separately consider the cases $\underline{A_i} \geq 0$, $\overline{A_i} \leq 0$ and $\underline{A_i} < 0 < \overline{A_i}$, $i = 0,1,2,3$, (independently). This is 81 cases in all.

In some different cases of the real intervals A_i, $i = 0,1,2,3$, the real interval versions of the set $\det \mathbf{R}_A$ can be given in the following:

- $\underline{A_0} \geq 0, \underline{A_1} \geq 0, \underline{A_2} \geq 0, \underline{A_3} \geq 0$,
$$|\mathbf{R}_A| = \left[\left(\underline{A_0}^2 + \underline{A_1}^2 + \underline{A_2}^2 + \underline{A_3}^2\right)^2, \left(\overline{A_0}^2 + \overline{A_1}^2 + \overline{A_2}^2 + \overline{A_3}^2\right)^2 \right],$$
- $\underline{A_0} \geq 0, \underline{A_1} \geq 0, \underline{A_2} \geq 0, \underline{A_3} \leq 0$,
$$|\mathbf{R}_A| = \left[\left(\underline{A_0}^2 + \underline{A_1}^2 + \underline{A_2}^2 + \overline{A_3}^2\right)^2, \left(\overline{A_0}^2 + \overline{A_1}^2 + \overline{A_2}^2 + \underline{A_3}^2\right)^2 \right],$$
- $\underline{A_0} \geq 0, \underline{A_1} \geq 0, \underline{A_2} \geq 0, \underline{A_3} < 0$ and $\overline{A_3} > 0$,
$$|\mathbf{R}_A| = \left[\left(\underline{A_0}^2 + \underline{A_1}^2 + \underline{A_2}^2 + 0^2\right)^2, \max\left\{ \left(\overline{A_0}^2 + \overline{A_1}^2 + \overline{A_2}^2 + \underline{A_3}^2\right)^2, \right.\right.$$
$$\left.\left. \left(\overline{A_0}^2 + \overline{A_1}^2 + \overline{A_2}^2 + \overline{A_3}\right)^2 \right\} \right].$$

Also, similarly, another cases can be written.

2.2. **The Euclidean norm, inverse and trace of the representation matrix $\mathbf{R}_A$.** In this section, we investigate the Euclidean norm of the representation matrix $\mathbf{R}_A$ for the quaternion interval number $A \in \mathbf{H}[I(\mathbb{R})]$.

The Euclidean norm of the matrix $\mathbf{R}_A$ is given as

$$\|\mathbf{R}_A\| = \left\{ f(a_0,a_1,a_2,a_3) = 4\sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2} : a_i \in A_i,\ A_i \in I(\mathbb{R}), i = 0,1,2,3 \right\}$$

To obtain an real interval version of the set $\|\mathbf{R}_A\|$, we can find the extreme points of the real-valued function $f(a_0,a_1,a_2,a_3)=4\sqrt{a_0^2+a_1^2+a_2^2+a_3^2}$ of four variables. Hence, the problem of calculating determinant of the matrix $\mathbf{R}_A$ is reduced to the finding problem of local minumum and maximum values of f. For each $a_i, i=0,1,2,3$, the partial derivative of f is

$$f'_{a_i}(a_0,a_1,a_2,a_3)=\frac{4a_i}{\sqrt{a_0^2+a_1^2+a_2^2+a_3^2}}$$

It is clearly that to determine as an interval or the union of intervals of the set $\|\mathbf{R}_A\|$ is surprisingly tedious. We must separately consider the cases $\underline{A_i}\geq 0$, $\overline{A_i}\leq 0$ and $\underline{A_i}<0<\overline{A_i}$, $i=0,1,2,3$, (independently). This is 81 cases in all.

For the set of $\|\mathbf{R}_A\|$, we can write the following some statements:

- $\underline{A_0}\geq 0, \underline{A_1}\geq 0, \underline{A_2}\geq 0, \underline{A_3}\geq 0$,

$$\|\mathbf{R}_A\|=\left[\sqrt{\underline{A_0}^2+\underline{A_1}^2+\underline{A_2}^2+\underline{A_3}^2},\sqrt{\overline{A_0}^2+\overline{A_1}^2+\overline{A_2}^2+\overline{A_3}^2}\right],$$

- $\underline{A_0}\geq 0, \underline{A_1}\geq 0, \underline{A_2}\geq 0, \underline{A_3}\leq 0$,

$$\|\mathbf{R}_A\|=\left[\sqrt{\underline{A_0}^2+\underline{A_1}^2+\underline{A_2}^2+\overline{A_3}^2},\sqrt{\overline{A_0}^2+\overline{A_1}^2+\overline{A_2}^2+\underline{A_3}^2}\right],$$

- $\underline{A_0}\geq 0, \underline{A_1}\geq 0, \underline{A_2}\geq 0, \underline{A_3}<0$ and $\overline{A_3}>0$,

$$\begin{aligned}\|\mathbf{R}_A\| &= \left[\sqrt{\underline{A_0}^2+\underline{A_1}^2+\underline{A_2}^2+0^2},\sqrt{\overline{A_0}^2+\overline{A_1}^2+\overline{A_2}^2+\underline{A_3}^2}\right]\\ &\cup\left[\sqrt{\underline{A_0}^2+\underline{A_1}^2+\underline{A_2}^2+\underline{0}^2},\sqrt{\overline{A_0}^2+\overline{A_1}^2+\overline{A_2}^2+\overline{A_3}^2}\right]\\ &=\left[\sqrt{\underline{A_0}^2+\underline{A_1}^2+\underline{A_2}^2+0^2},\max\{\sqrt{\overline{A_0}^2+\overline{A_1}^2+\overline{A_2}^2+\underline{A_3}^2},\right.\\ &\left.\sqrt{\overline{A_0}^2+\overline{A_1}^2+\overline{A_2}^2+\overline{A_3}^2}\}\right].\end{aligned}$$

Also, similarly, another cases can be written.

Now, we will give the inverse and trace of the representation matrix $\mathbf{R}_A$, which is similarly obtained from the inverse and trace definitions of matrices, respectively.

The inverse of the matrix $\mathbf{R}_A$ is given as

$$\mathbf{R}_A^{-1}=\left\{\frac{1}{\|a\|^2}R_a^T : a_i\in A_i,\ i=0,1,2,3\right\}.$$

The trace of the matrix $\mathbf{R}_A$ is given as follows

$$\begin{aligned}trace(\mathbf{R}_A) &= trace\begin{pmatrix} A_0 & A_1 & A_2 & A_3\\ -A_1 & A_0 & -A_3 & A_2\\ -A_2 & A_3 & A_0 & -A_1\\ -A_3 & -A_2 & A_1 & A_0\end{pmatrix}\\ &= \left[4\underline{A_0},4\overline{A_0}\right].\end{aligned}$$

2.3. Eigenvalues and eigenvectors of the representation matrix $\mathbf{R}_A$.

Theorem 1. *Let $a_* = a_1 i + a_2 j + a_3 k$. If $\|a_*\| \neq 0$, the eigenvalues $\mu_s, s = 1,2,3,4$, of the representation matrix $\mathbf{R}_A$ for the quaternion interval $A \in \mathbf{H}[I(\mathbb{R})]$, which occur with algebraic multiplicity 2, are given by*

$$\mu_{1,2} = \left\{\lambda = a_0 + i\|a_*\| : i^2 = -1, a_k \in A_k, k = 0,1,2,3\right\}$$

and

$$\mu_{3,4} = \left\{\lambda = a_0 - i\|a_*\| : i^2 = -1, a_k \in A_k, k = 0,1,2,3\right\}.$$

If $\|a_*\| = 0$, the eigenvalues of $\mathbf{R}_A$, which occur algebraic multiplicity 4, are $\mu_s = a_0, s = 1,2,3,4$.

Proof. By using the interval algebra, the eigenvalues $\mu_s, s = 1,2,3,4$, of the representation matrix $\mathbf{R}_A$ for the quaternion interval $A \in \mathbf{H}[I(\mathbb{R})]$ can be written as following

$$\mu_s = \{\lambda_k \in \mathbb{C} : \mathbf{R}_a u_k = \lambda_k u_k,\ 0 \neq u \in \mathbb{C}^4, k = 1,2,3,4\ \}.$$

The problem is reduced the calculating problem of the eigenvalues, λ_k, of matrix $\mathbf{R}_a$. Now, note that

$$\mathbf{R}_a = a_0 I_4 + \mathbf{R}_{a_*},$$

where $a_* = a_1 i + a_2 j + a_3 k$. Let $\theta_k,\ k = 1,2,3,4$, be the eigenvalues of the matrix $\mathbf{R}_{a_*}$ (with associated eigenvectors x_k). Then, for each k, we have that

$$\begin{aligned}(a_0 I_4 + \mathbf{R}_{a_*})\, x_k &= a_0 I_4 x_k + \mathbf{R}_{a_*} x_k \\ &= (a_0 + \theta_k)\, x_k.\end{aligned}$$

Therefore, $\lambda_k = a_0 + \theta_k,\ k = 1,2,3,4$, are the eigenvalues of $\mathbf{R}_a$. It is well known that if θ is an eigenvalue of $\mathbf{R}_{a_*}$, then $\theta^2$is an eigenvalue of $\mathbf{R}_{a_*}^2$. From

$$\mathbf{R}_{a_*}^2 = -\|a_*\|^2 I_4,$$

we conclude that $\theta^2 = -\|a_*\|^2$. Hence, the eigenvalues of $\mathbf{R}_{a_*}$ can only be $\theta = i\ \|a_*\|$ or $\theta = -i\ \|a_*\|$. But the complex eigenvalues of the real matrix $\mathbf{R}_{a_*}$ occur in conjugate pairs, so that $\mathbf{R}_{a_*}$ has two eigenvalues $i\|a_*\|$ and two eigenvalues $-i\|a_*\|$. Then, the eigenvalues of the matrix $\mathbf{R}_a$ are obtained as

$$\lambda_{1,2} = a_0 + i\|a_*\| \text{ and } \lambda_{3,4} = a_1 - i\|a_*\|.$$

Therefore, the eigenvalues of the matrix $\mathbf{R}_A$ are given by

$$\mu_{1,2} = \left\{\lambda = a_0 + i\|a_*\| : i^2 = -1, a_k \in A_k, k = 0,1,2,3\right\}$$

and

$$\mu_{3,4} = \left\{\lambda = a_0 - i\|a_*\| : i^2 = -1, a_k \in A_k, k = 0,1,2,3\right\}.$$

The proof is completed. □

Remark 1. *It is clearly that as an real interval or the union of real intervals of the sets $\mu_{1,2,3,4}$ we meet with 27 different cases. We now establish the set $\mu_{1,2}$ in three different cases.*

Case 1. $A_0 \in I(\mathbb{R})$, $\underline{A_1} \geq 0$, $\underline{A_2} \geq 0$, $\underline{A_3} \geq 0$. In this case we have that

$$\mu_{1,2} = \left[\underline{A_0}, \overline{A_0}\right] + i\left[\sqrt{\underline{A_1}^2 + \underline{A_2}^2 + \underline{A_3}^2}, \sqrt{\overline{A_1}^2 + \overline{A_2}^2 + \overline{A_3}^2}\right].$$

Case 2. $A_0 \in I(\mathbb{R})$, $\underline{A_1} \geq 0$, $\underline{A_2} \geq 0$, $\underline{A_3} \leq 0$. In this case we have

$$\mu_{1,2} = \left[\underline{A_0}, \overline{A_0}\right] + i\left[\sqrt{\underline{A_1}^2 + \underline{A_2}^2 + \overline{A_3}^2}, \sqrt{\overline{A_1}^2 + \overline{A_2}^2 + \underline{A_3}^2}\right].$$

Case 3. $A_0 \in I(\mathbb{R})$, $\underline{A_1} \geq 0$, $\underline{A_2} \geq 0$, $\underline{A_3} < 0$ and $\overline{A_3} > 0$. In this case we obtain,

$$\mu_{1,2} = [\underline{A_0}, \overline{A_0}] + i\left\{\left[\sqrt{\underline{A_1}^2 + \underline{A_2}^2 + 0^2}, \sqrt{\overline{A_1}^2 + \overline{A_2}^2 + \underline{A_3}^2}\right] \cup \left[\sqrt{\underline{A_1}^2 + \underline{A_2}^2 + \underline{0}^2}, \sqrt{\overline{A_1}^2 + \overline{A_2}^2 + \overline{A_3}^2}\right]\right\}$$

$$= [\underline{A_0}, \overline{A_0}] + i\left[\sqrt{\underline{A_1}^2 + \underline{A_2}^2 + 0^2}, \max\{\sqrt{\overline{A_1}^2 + \overline{A_2}^2 + \underline{A_3}^2}, \sqrt{\overline{A_1}^2 + \overline{A_2}^2 + \overline{A_3}^2}\}\right].$$

Since the determinant of a matrix is the product of its eigenvalues, we have

$$\det \mathbf{R}_a = \prod_{i=1}^{4} \lambda_i = (a_0 + i\,\|a_*\|)^2\,(a_0 - i\,\|a_*\|)^2 = \|a\|^4.$$

Then

$$\det \mathbf{R}_A = \left\{\det \mathbf{R}_a = \|a\|^4 : a_i \in A_i,\ A_i \in I(\mathbb{R})\, i = 0,1,2,3\right\}$$
$$= \left\{\det \mathbf{R}_a = \left(a_0^2 + a_1^2 + a_2^2 + a_3^2\right)^2 : a_i \in A_i,\ A_i \in I(\mathbb{R})\, i = 0,1,2,3\right\}.$$

3. Discussion

The quaternion interval numbers, their representation vectors and matrices and their main features have not hitherto been studied in the literature. We initiated the study of the quaternion interval numbers in a different sense.

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