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ON THE AL-SALAM AND VERMA ORTHOGONAL q-POLYNOMIALS

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Abstract. The aim of this paper is to highlight a new integral representation of the Al-Salam and Verma regular form through a true function by investigating the quadratic operator σ, duality and the H_q-classical character of the Wall form where H_q is the q-difference operator. Some other characterizations are given in the general case and in the symmetric one corresponding to the Al-Salam and Verma polynomials.

1. Introduction Preliminaries and First Results

1.1. **Introduction.** Let $\{B_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg B_n = n$, $n \geq 0$ MPS. When the MPS $\{B_n\}_{n\geq 0}$ is orthogonal MOPS, it satisfy the second order recurrence relation [6]

$$\begin{cases} B_0(x) = 1\ ,\ B_1(x) = x - \beta_0\ , \\ B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n \geq 0. \end{cases} \tag{1}$$

In [8] Geronimus considered polynomial sets PS of the form

$$B_n(x) = \sum_{k=0}^{n} a_{n-k} b_k w_k(x), \quad n \geq 0 \tag{2}$$

where $w_0(x) = 1,\ w_k(x) = \prod_{l=1}^{k}(x - x_l),\ k \geq 1,\ a_0 = b_0 = 1$ and $b_n \neq 0,\ n \geq 0$. He raised the following problem: For what sequences of real or complex numbers $(x_n)_{n\geq 1}$, $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ is the PS in (2) orthogonal? He did not solve this problem. However he gave some necessary and sufficient conditions on the coefficients, (a_n), (b_n), and the sequence (x_n). This problem has remained unsolved in its full generality. Some special choices of the sequence (x_k) lead to complete solutions. Chihara [4,5] showed that when $x_k = 0$, $k \geq 1$ then (2) forms Brenke type set and in this case he obtained all such OPS. Al-Salam and Verma [2] showed that the Al-Salam and Verma MOPS $\{B_n\}_{n\geq 0}$ satisfying the second order recurrence relation (1) with

$$\begin{cases} \beta_n = (-1)^{n+1}\alpha \\ \gamma_{2n+1} = Eq^{n+1}(1 - \gamma q^n) \qquad ,\quad n \geq 0 \\ \gamma_{2n+2} = E\gamma q^{n+1}(1 - q^{n+1}) \end{cases} \tag{3}$$

where $E > 0$, $\ 0 < \gamma < 1$, $\ 0 < q < 1$ and $\alpha \in \mathbb{R}$, appears as the only solution of the Geronimus problem when $x_{2k+1} = x_1$, $k \geq 0$, $x_{2k} = x_2$, $k \geq 1$ and $a_1 = 0$. For other partial resolution of the Geronimus problem see [1,3].

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In [2], it is proved that the Al-Salam and Verma polynomials are orthogonal with respect to the discrete distribution $\phi(x)$ which has the spectral points at

$$x_k = \pm(\alpha^2 + Eq^{k+1})^{\frac{1}{2}},\ k \geq 0 \tag{4}$$

and jumps

$$d\phi(\pm x_k) = (\gamma;q)_\infty \frac{\gamma^k}{(q;q)_k},\ k \geq 0 \tag{5}$$

since this OPS turn out to be related to the Wall polynomials $\{W_n(.;b,q)\}_{n\geq 0}$ given by

$$W_n(x;b,q) = (-1)^n (b;q)_n q^{\frac{1}{2}n(n+1)} {}_2\phi_1 \begin{bmatrix} q^{-n}, & 0; & x \\ & b & \end{bmatrix},\ n \geq 0$$

and orthogonal with respect to the discrete distribution $\psi(x)$ which has the spectral points at q^{k+1}, $k \geq 0$ and jumps $d\psi(q^{k+1}) = (b;q)_\infty \dfrac{b^k}{(q;q)_k}$, $k \geq 0$ [6] with the notations [10]

$$(a;q)_0 := 1\ ;\ (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1})\ ,\ n \geq 1\ ;\ (a;q)_\infty := \prod_{k=0}^{+\infty}(1-aq^k)\ ;\ |q| < 1.$$

The Wall polynomials satisfies the second order recurrence relation (1) with

$$\begin{cases} \beta_n = \{b + q - b(1+q)q^n\}q^n, & n \geq 0, \\ \gamma_{n+1} = b(1-q^{n+1})(1-bq^n)q^{2n+2}, & n \geq 0, \end{cases} \tag{6}$$

In [10], the second author and P. Maroni have exhaustively described the H_q-classical orthogonal q-polynomials where H_q is the q-difference operator [9]. Particularly, they meet again the Wall polynomials orthogonal with respect to the Wall form $\mathcal{W}(b,q)$ ($b \neq 0$, $b \neq q^{-n}$, $n \geq 0$) and a new integral representation and some discrete measure are given. The Wall form satisfies the q-distributional equation

$$H_q\big(x\mathcal{W}(b,q)\big) - b^{-1}(q-1)^{-1}(q^{-1}x + b - 1)\mathcal{W}(b,q) = 0, \tag{7}$$

From which they derive:
the moments

$$(\mathcal{W}(b,q))_n = q^n (b;q)_n,\ n \geq 0 \tag{8}$$

and the following representations

$$\mathcal{W}(b,q) = (b;q)_\infty \sum_{k=0}^{+\infty} \frac{b^k}{(q;q)_k} \delta_{q^{1+k}},\ 0 < q < 1,\ 0 < b < 1, \tag{9}$$

$$\mathcal{W}(b,q) = \frac{1}{(bq^{-1};q^{-1})_\infty} \sum_{k=0}^{+\infty} \frac{q^{-\frac{1}{2}k(k+1)}}{(q^{-1};q^{-1})_k} (-b)^k \delta_{q^{1+k}},\quad q > 1,\ b \neq q^{\pm l},\ l \geq 0, \tag{10}$$

$$\langle \mathcal{W}(b,q), f \rangle = K \int_0^1 x^{\frac{\ln b}{\ln q} - 1} (x;q)_\infty f(x)dx,\ f \in \mathcal{P},\ 0 < q < 1,\ 0 < b < 1, \tag{11}$$

where $K^{-1} = \displaystyle\int_0^1 x^{\frac{\ln b}{\ln q} - 1}(x;q)_\infty dx$.

From the characterizations of H_q-classical orthogonal polynomials in [10,11], one may write the structure relation and the second order linear q-difference equation satisfied by the Wall polynomials. For all $n \geq 0$

$$x(H_q W_{n+1})(x;b,q) = [n+1]_q \Big\{ W_{n+1}(x;b,q) + (1-bq^n)q^{n+1} W_n(x;b,q) \Big\}, \tag{12}$$

$$x\Big(H_q \circ H_{q^{-1}} W_{n+1}\Big)(x;b,q) + b^{-1}(q-1)^{-1}(q^{-1}x+b-1)\Big(H_{q^{-1}}W_{n+1}\Big)(x;b,q)+$$
$$+(bq(1-q))^{-1}[n+1]_q W_{n+1}(x;b,q) = 0, \tag{13}$$

where

$$[n]_q = \frac{q^n-1}{q-1},\ n \geq 0,\ q \neq 1. \tag{14}$$

Our goal is to bring out a new integral representation of the Al-Salam and Verma form and also a discrete measure for $q > 1$ in a simpler manner by investigating the new results for the Wall form and the quadratic decomposition. Particularly, we recover again the discrete measure (4)-(5), the moments and the connections with the Wall polynomials [2]. The symmetric case of the Al-Salam and Verma polynomials ($\alpha = 0$ in (3)) is studied into details; this case turn out to be related to the OPS of Brenke type $\{Y_n(.;b,q)\}_{n\geq 0}$ having the recurrence formula [6]

$$\begin{cases} Y_0(x;b,q) = 1\ ,\ \ Y_1(x;b,q) = x, \\ Y_{n+2}(x;b,q) = xY_{n+1}(x;b,q) - \widetilde{\gamma}_{n+1}Y_n(x;b,q),\ n \geq 0, \\ \widetilde{\gamma}_{2n+2} = b(1-q^{n+1})q^{n+1}\ ;\ \widetilde{\gamma}_{2n+1} = (1-bq^n)q^{n+1},\ n \geq 0, \\ b \neq 0,\ b \neq q^{-n},\ n \geq 0 \end{cases} \tag{15}$$

and respecting the H_q-semiclassical character [7]. As a consequence, the structure relation satisfied by any symmetric Al-Salam and Verma polynomial is established.

1.2. **Preliminaries and First Results.** Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of the form $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. Particularly, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. Let us introduce some useful operations in $\mathcal{P}'$. For any form u, any polynomial g and any $(a,b,c) \in \mathbb{C} - \{0\} \times \mathbb{C}^2$, we let $H_q u$, σu gu, $h_a u$,$\tau_b u$ and δ_c, be the forms defined by duality [10,12]

$$\langle H_q u, f\rangle = -\langle u, H_q f\rangle \quad , \quad \langle \sigma u, f\rangle = \langle u, \sigma f\rangle \quad , \quad \langle gu, f\rangle = \langle u, gf\rangle \quad ,$$
$$\langle h_a u, f\rangle = \langle u, h_a f\rangle \quad , \quad \langle \tau_b u, f\rangle = \langle u, \tau_{-b} f\rangle \quad , \quad \langle \delta_c, f\rangle = f(c)$$

where $(H_q f)(x) = \frac{f(qx)-f(x)}{(q-1)x}$, $q \in \widetilde{\mathbb{C}} := \big\{z \in \mathbb{C},\ \ z \neq 0,\ z^n \neq 1,\ n \geq 1\big\}$, $(\sigma f)(x) = f(x^2)$, $(h_a f)(x) = f(ax)$ and $(\tau_{-b} f)(x) = f(x+b)$ for any $f \in \mathcal{P}$. We have the well known formulas [7,12]

$$f(x)\sigma u = \sigma(f(x^2)u)\ , f \in \mathcal{P}\ ,\ u \in \mathcal{P}', \tag{16}$$
$$\sigma(H_q u) = (q+1)H_{q^2}(\sigma(xu))\ ,\ u \in \mathcal{P}'. \tag{17}$$

The form u is called regular if we can associate with it a sequence of monic polynomials $\{B_n\}_{n\geq 0}$, $\deg B_n = n$, $n \geq 0$ MPS such that

$$\langle u, B_m B_n\rangle = r_n \delta_{n,m}\ ,\ n,m \geq 0\ ;\ r_n \neq 0\ ,\ n \geq 0.$$

The sequence $\{B_n\}_{n\geq 0}$ is then said orthogonal with respect to u; it is a monic orthogonal polynomials sequence MOPS. In this case, $\{B_n\}_{n\geq 0}$ fulfils (1) where $\beta_n = \dfrac{\langle u, xB_n^2\rangle}{r_n}$ and $\gamma_{n+1} = \dfrac{r_{n+1}}{r_n} \neq 0$ for all $n \geq 0$.

The shifted MOPS $\{\widehat{B}_n := A^{-n}\big(h_A \circ \tau_{-b} B_n\big)\}_{n\geq 0}$ is then orthogonal with respect to $\widehat{u} = h_{A^{-1}} \circ \tau_{-b} u$ and satisfies (1) with

$$\widehat{\beta}_n = \frac{\beta_n - b}{A} \quad , \quad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{A^2} \quad , \quad n \geq 0. \tag{18}$$

Moreover, the form u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any form will be normalized. Also, the form u is said to be positive definite if and only if $\beta_n \in \mathbb{R}$, $n \geq 0$ and $\gamma_{n+1} > 0$, $n \geq 0$. Lastly, when u is regular, u is symmetric if and only if $(u)_{2n+1} = 0$, $n \geq 0$ or equivalently $\beta_n = 0$, $n \geq 0$.

Denoting $\mathcal{SV}(\alpha) := \mathcal{SV}(\alpha, \gamma, E, q)$ the regular form associated with the Al-Salam and Verma polynomials. From (3), the form $\mathcal{SV}(\alpha)$ is regular if and only if

$$E\gamma \neq 0\,,\ \gamma \neq q^{-n},\ n \geq 0.$$

Also, the form $\mathcal{SV}(\alpha)$ is positive definite if and only if

$$E > 0,\ 0 < \gamma < 1,\ 0 < q < 1\ ;\ E < 0,\ \gamma > 1,\ q > 1\ ;\ E > 0,\ \gamma < 0,\ q > 1.$$

Let $\{B_n\}_{n\geq 0}$ be an MPS. Is is possible to associate with it two MPSs $\{P_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and two polynomials sequences $\{a_n(x)\}_{n\geq 0}$, $\{b_n(x)\}_{n\geq 0}$ such that

$$B_{2n}(x) = P_n(x^2) + xa_{n-1}(x^2) \quad , \quad B_{2n+1}(x) = xR_n(x^2) + b_n(x^2),\ n \geq 0, \tag{19}$$

where $\deg a_n \leq n$, $\deg b_n \leq n$ and $a_{-1}(x) = 0$. In [12], P. Maroni gave necessary and sufficient conditions for the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ to be orthogonal. The result is written as follows in [13] by the third author and P. Maroni.

Proposition 1. *Let $\{B_n\}_{n\geq 0}$ orthogonal with respect to the form w and satisfy the recurrence relation* (1)*, where*

$$\beta_n = (-1)^n \beta_0,\ n \geq 0. \tag{20}$$

Then there exist two MOPSs $\{P_n\}_{n\geq 0}$, with respect to u, and $\{R_n\}_{n\geq 0}$, with respect to v, fulfilling the following relations:

$$\begin{cases} P_0(x) = 1\ ,\ P_1(x) = x - \gamma_1 - \beta_0^2\ , \\ P_{n+2}(x) = (x - \gamma_{2n+2} - \gamma_{2n+3} - \beta_0^2)P_{n+1}(x) - \gamma_{2n+1}\gamma_{2n+2}P_n(x),\ n \geq 0, \end{cases} \tag{21}$$

$$\begin{cases} R_0(x) = 1\ ,\ R_1(x) = x - \gamma_1 - \gamma_2 - \beta_0^2\ , \\ R_{n+2}(x) = (x - \gamma_{2n+3} - \gamma_{2n+4} - \beta_0^2)R_{n+1}(x) - \gamma_{2n+2}\gamma_{2n+3}R_n(x),\ n \geq 0, \end{cases} \tag{22}$$

$$P_{n+1}(x) = R_{n+1}(x) + \gamma_{2n+2}R_n(x),\ n \geq 0, \tag{23}$$

$$(x - \beta_0^2)R_n(x) = P_{n+1}(x) + \gamma_{2n+1}P_n(x),\ n \geq 0, \tag{24}$$

since, in (19)*, $a_n(x) = 0$ and $b_n(x) = -\beta_0 R_n(x),\ n \geq 0$.*

Moreover, the forms u, v, and w satisfy

$$u = \sigma w, \tag{25}$$

$$\sigma(xw) = \beta_0(\sigma w), \tag{26}$$

$$v = \frac{1}{\gamma_1}\big(x - \beta_0^2\big)(\sigma w). \tag{27}$$

Now, this result will be applied to the Al-Salam and Verma polynomials. Consequently, it is possible to associate with the Al-Salam and Verma $\{B_n\}_{n\geq 0}$ two MOPSs $\{P_n\}_{n\geq 0}$ with respect to u and $\{R_n\}_{n\geq 0}$ with respect to v, fulfilling the following relations

$$B_{2n}(x) = P_n(x^2) \quad , \quad B_{2n+1}(x) = (x + \alpha)R_n(x^2) \quad , \quad n \geq 0, \tag{28}$$

$$\begin{cases} P_0(x) = 1\ \ ,\ P_1(x) = x - \alpha^2 - Eq(1-q)\ , \\ P_{n+2}(x) = \left(x - \alpha^2 - Eq^{n+1}\left(\gamma + q - \gamma(1+q)q^{n+1}\right)\right)P_{n+1}(x) - \\ \qquad -\gamma E^2(1-q^{n+1})(1-\gamma q^n)q^{2n+2}P_n(x)\ ,\ n \geq 0, \end{cases} \tag{29}$$

$$\begin{cases} R_0(x) = 1 \ , \ R_1(x) = x - \alpha^2 - Eq(1-\gamma q) \ , \\ R_{n+2}(x) = \left(x - \alpha^2 - Eq^{n+2}\left(1+\gamma-\gamma(1+q)q^{n+1}\right)\right)R_{n+1}(x) - \\ \qquad -\gamma E^2(1-q^{n+1})(1-\gamma q^{n+1})q^{2n+3}R_n(x) \ , \ n \geq 0. \end{cases} \tag{30}$$

Moreover, the forms u, v and $\mathcal{SV}(\alpha)$ satisfy

$$u = \sigma \mathcal{SV}(\alpha), \tag{31}$$

$$-\alpha u = \sigma\left(x\mathcal{SV}(\alpha)\right), \tag{32}$$

$$v = (Eq(1-\gamma))^{-1}(x-\alpha^2)\sigma\mathcal{SV}(\alpha). \tag{33}$$

Remark 1. *From assumption of regularity, it is necessary that* $P_{n+1}(\alpha^2) \neq 0 \ , \ n \geq 0$ *according to (28).*

Proposition 2. *We have the following connections*

$$h_{E^{-1}} \circ \tau_{-\alpha^2} u = \mathcal{W}(\gamma, q) \quad , \quad \gamma \neq 0, \ \gamma \neq q^{-n}, \ n \geq 0, \tag{34}$$

$$h_{E^{-1}} \circ \tau_{-\alpha^2} v = \mathcal{W}(\gamma q, q) \quad , \quad \gamma \neq 0, \ \gamma \neq q^{-n-1}, \ n \geq 0, \tag{35}$$

where $\mathcal{W}(b,q)$ *(*$b=\gamma$ *;* $b = \gamma q$ *respectively) is the Wall form.*

Proof. From (29)-(30) denoting the recurrence coefficients of $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ by

$$\begin{cases} \beta_0^P = \alpha^2 + Eq(1-q) \ , \\ \beta_{n+1}^P = \alpha^2 + Eq^{n+1}\left(\gamma + q - \gamma(1+q)q^{n+1}\right) \ , \ n \geq 0, \\ \gamma_{n+1}^P = \gamma E^2(1-q^{n+1})(1-\gamma q^n)q^{2n+2} \ , \ n \geq 0, \end{cases} \tag{36}$$

$$\begin{cases} \beta_0^R = \alpha^2 + Eq(1-\gamma q) \ , \\ \beta_{n+1}^R = \alpha^2 + Eq^{n+2}\left(1+\gamma - \gamma(1+q)q^{n+1}\right) \ , \ n \geq 0, \\ \gamma_{n+1}^R = \gamma E^2(1-q^{n+1})(1-\gamma q^{n+1})q^{2n+3} \ , \ n \geq 0. \end{cases} \tag{37}$$

According to (18) and with the choice $A = E, B = \alpha^2$ for (36)-(37), we get respectively

$$\begin{cases} \widehat{\beta}_n^P = q^n\left(\gamma + q - \gamma(1+q)q^n\right) \ , \ n \geq 0, \\ \widehat{\gamma}_{n+1}^P = \gamma(1-q^{n+1})(1-\gamma q^n)q^{2n+2} \ , \ n \geq 0, \end{cases}$$

$$\begin{cases} \widehat{\beta}_n^R = \left(\gamma q + q - \gamma q(1+q)q^n\right)q^n \ , \ n \geq 0, \\ \widehat{\gamma}_{n+1}^R = \gamma q(1-q^{n+1})(1-\gamma q q^n)q^{2n+2} \ , \ n \geq 0. \end{cases}$$

Comparing with (6), we recognize the recurrence coefficients of Wall for $b = \gamma$ and for $b = \gamma q$ respectively. Thus the desired results (34)-(35). □

Corollary 1. *The following system holds*

$$\begin{cases} B_{2n}(x) = W_n\left(\dfrac{x^2-\alpha^2}{E}; \gamma, q\right), \ n \geq 0, \\ B_{2n+1}(x) = (x+\alpha)W_n\left(\dfrac{x^2-\alpha^2}{E}; \gamma q, q\right), \ n \geq 0. \end{cases} \tag{38}$$

Proof. The formulas in the system (38) are consequence from the connections in (34)-(35), the results in (19) and the notation of the Wall polynomials. □

Proposition 3. *The moments of $\mathcal{SV}(\alpha)$ are*

$$(\mathcal{SV}(\alpha))_n = (-1)^n \sum_{k=0}^{[\frac{n}{2}]} \begin{pmatrix} [\frac{n}{2}] \\ k \end{pmatrix} (\gamma;q)_k\, \alpha^{n-2k}(Eq)^k, \quad n \geq 0. \tag{39}$$

Proof. From definitions, we have

$$\begin{aligned}(\mathcal{SV}(\alpha))_{2n+1} = \langle \sigma(x\mathcal{SV}(\alpha)), x^n\rangle &= -\alpha(u)_n && \text{(by (32))}\\ &= -\alpha\langle \tau_{\alpha^2} \circ h_E \mathcal{W}(\gamma,q), x^n\rangle && \text{(by (34))}\\ &= -\alpha\langle \mathcal{W}(\gamma,q), (Ex+\alpha^2)^n\rangle \\ &= -\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} E^k \alpha^{2n+1-2k} (\mathcal{W}(\gamma,q))_k\end{aligned}$$

and by virtue of (31) we get $(\mathcal{SV}(\alpha))_{2n} = \langle \sigma(\mathcal{SV})(\alpha), x^n\rangle = (u)_n$.
Particularly, $(\mathcal{SV}(\alpha))_{2n+1} = -\alpha(\mathcal{SV}(\alpha))_{2n}$, $n \geq 0$. Taking into account (8) we deduce (39). □

2. Main results

2.1. **Integral representation and discrete measure of $\mathcal{SV}(\alpha)$.** In what follows we are going to use the logarithmic function denoted by Log : $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$ defined by

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z,\ z \in \mathbb{C}\setminus\{0\},\ -\pi < \operatorname{Arg} z \leq \pi,$$

Log is the principal branch of log and includes ln : $\mathbb{R}^+ \setminus \{0\} \longrightarrow \mathbb{R}$ as a special case. Consequently, the principal branch of the square root is

$$\sqrt{z} = \sqrt{|z|}\ \mathrm{e}^{i\,\frac{\operatorname{Arg} z}{2}},\ z \in \mathbb{C}\setminus\{0\},\ -\pi < \operatorname{Arg} z \leq \pi.$$

Now, consider a polynomial f and let us split up this polynomial according to its even and odd parts

$$f(z) = f^e(z^2) + z f^o(z^2), \quad z \in \mathbb{C}. \tag{40}$$

Consequently, for any branch of the square root, we get

$$\begin{cases} f^e(z) = \dfrac{f(\sqrt{z}) + f(-\sqrt{z})}{2} \\ f^o(z) = \dfrac{f(\sqrt{z}) - f(-\sqrt{z})}{2\sqrt{z}} \end{cases}, z \in \mathbb{C} - \{0\}. \tag{41}$$

According to all the above results, we are able to give an integral representation and a discrete measure for $\mathcal{SV}$ for any positive definite case.

Proposition 4. *For $\alpha \in \mathbb{R}$, $E > 0$, $0 < \gamma < 1$, $0 < q < 1$ and $f \in \mathcal{P}$ the form $\mathcal{SV}(\alpha)$ has the following integral representation*

$$\begin{aligned}\langle \mathcal{SV}(\alpha), f\rangle = \frac{K}{E^{\frac{\ln\gamma}{\ln q}}} \int_{|\alpha|}^{\sqrt{\alpha^2+E}} \Big(x^2-\alpha\Big)^{\frac{\ln\gamma}{\ln q}-1} \big(x-\alpha\big)\Big(\frac{x^2-\alpha^2}{E};q\Big)_\infty f(x)dx - \\ -\frac{K}{E^{\frac{\ln\gamma}{\ln q}}} \int_{-\sqrt{\alpha^2+E}}^{-|\alpha|} \Big(x^2-\alpha\Big)^{\frac{\ln\gamma}{\ln q}-1} \big(x-\alpha\big)\Big(\frac{x^2-\alpha^2}{E};q\Big)_\infty f(x)dx,\end{aligned} \tag{42}$$

where K is the normalization constant in (11).

Proof. Let $f \in \mathcal{P}$ satisfying (40) and $\alpha \in \mathbb{R}$, $E > 0$, $0 < \gamma < 1$, $0 < q < 1$. We may write

$$\langle \mathcal{SV}(\alpha), f\rangle = \langle \mathcal{SV}(\alpha), f^e(x^2)\rangle + \langle x\mathcal{SV}(\alpha), f^o(x^2)\rangle.$$

According to the definitions, (31)-(32) and (34), the last expression becomes

$$\langle \mathcal{SV}(\alpha), f\rangle = \langle \tau_{\alpha^2} \circ h_E \mathcal{W}(\gamma,q), f^e\rangle - \alpha\langle \tau_{\alpha^2} \circ h_E \mathcal{W}(\gamma,q), f^o\rangle.$$

Thus,

$$\langle \mathcal{SV}(\alpha), f\rangle = \Big\langle \mathcal{W}(\gamma, q), f^e(Ex+\alpha^2) - \alpha f^o(Ex+\alpha^2)\Big\rangle. \tag{43}$$

Applying (11) then (43) yields to

$$\langle \mathcal{SV}(\alpha), f\rangle = K\int_0^1 x^{\frac{\ln\gamma}{\ln q}-1}(x;q)_\infty \Big(f^e(Ex+\alpha^2) - \alpha f^o(Ex+\alpha^2)\Big)\,dx.$$

By the change of variable $t = Ex+\alpha^2$, we get

$$\langle \mathcal{SV}(\alpha), f\rangle = \frac{K}{E^{\frac{\ln\gamma}{\ln q}}}\int_{\alpha^2}^{\alpha^2+E}\Big(t-\alpha^2\Big)^{\frac{\ln\gamma}{\ln q}-1}\Big(\frac{t-\alpha^2}{E};q\Big)_\infty \big(f^e(t)-\alpha f^o(t)\big)\,dt. \tag{44}$$

But from (41) we have

$$f^e(t) - \alpha f^o(t) = \frac{1}{2\sqrt{t}}\Big\{(\sqrt{t}-\alpha)f(\sqrt{t}) + (\sqrt{t}+\alpha)f(-\sqrt{t})\Big\}, \quad t>0. \tag{45}$$

Replacing (45) in (44), the formula in (44) becomes
$2\,E^{\frac{\ln\gamma}{\ln q}}\,K^{-1}\langle \mathcal{SV}(\alpha), f\rangle$

$$= \int_{\alpha^2}^{\alpha^2+E}\Big(t-\alpha^2\Big)^{\frac{\ln\gamma}{\ln q}-1}\Big(\frac{t-\alpha^2}{E};q\Big)_\infty \Big\{\big(1-\frac{\alpha}{\sqrt{t}}\big)f(\sqrt{t}) + \big(1+\frac{\alpha}{\sqrt{t}}\big)f(-\sqrt{t})\Big\}\,dt$$

$$= \int_{\alpha^2}^{\alpha^2+E}\Big(t-\alpha^2\Big)^{\frac{\ln\gamma}{\ln q}-1}\Big(\frac{t-\alpha^2}{E};q\Big)_\infty (1-\frac{\alpha}{\sqrt{t}})f(\sqrt{t})\,dt +$$

$$+\int_{\alpha^2}^{\alpha^2+E}\Big(t-\alpha^2\Big)^{\frac{\ln\gamma}{\ln q}-1}\Big(\frac{t-\alpha^2}{E};q\Big)_\infty (1+\frac{\alpha}{\sqrt{t}})f(-\sqrt{t})\,dt$$

since the two integrals in the last sum exist. Using the change of variable $x = \sqrt{t}$ in the first integral and $x = -\sqrt{t}$ in the second one, it gives to the desired representation (42). □

Proposition 5. *The form $\mathcal{SV}(\alpha)$ has the following discrete representations: for $E>0$, $0<\gamma<1$, $0<q<1$*

$$\mathcal{SV}(\alpha) = (\gamma;q)_\infty \sum_{k=0}^{+\infty}\frac{\gamma^k}{(q;q)_k}\Delta_k, \tag{46}$$

for $E>0$, $\gamma<0$, $q>1$ or $E<0$, $\gamma>1$, $q>1$

$$\mathcal{SV}(\alpha) = \frac{1}{(\gamma q^{-1};q^{-1})_\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k q^{-\frac{1}{2}k(k+1)}\gamma^k}{(q^{-1};q^{-1})_k}\Delta_k, \tag{47}$$

where for all $k\geq 0$

$$\Delta_k = \frac{\Big(1-\frac{\alpha}{\sqrt{Eq^{1+k}+\alpha^2}}\Big)\delta_{\sqrt{Eq^{1+k}+\alpha^2}} + \Big(1+\frac{\alpha}{\sqrt{Eq^{1+k}+\alpha^2}}\Big)\delta_{-\sqrt{Eq^{1+k}+\alpha^2}}}{2}. \tag{48}$$

Proof. Let $f\in\mathcal{P}$. Taking into account (40), according to (43) and by virtue of (41) we have

$$\langle \mathcal{SV}(\alpha), f\rangle = \big\langle \mathcal{W}(\gamma,q), f^e(Ex+\alpha^2) - \alpha f^o(Ex+\alpha^2)\big\rangle$$

and

$$f^e(z) - \alpha f^o(z) = \frac{\big(1-\frac{\alpha}{\sqrt{z}}\big)f(\sqrt{z}) + \big(1+\frac{\alpha}{\sqrt{z}}\big)f(-\sqrt{z})}{2}, \quad z\in\mathbb{C}-\{0\}.$$

Finally, by virtue of (9)-(10) we obtain the discrete measure in (46)-(47) with (48). □

2.2. **Some characterizations concerning the symmetric case of the Al-Salam and Verma polynomials.** This section is devoted to the study of the symmetric case of the Al-Salam and Verma polynomials $\{B_n\}_{n\geq 0}$ $(\alpha = 0)$ orthogonal with respect to the regular form $\mathcal{SV}(0)$. The system (3) becomes

$$\begin{cases} \beta_n = 0 \\ \gamma_{2n+1} = Eq^{n+1}(1-\gamma q^n) \quad , \quad n \geq 0. \\ \gamma_{2n+2} = E\gamma q^{n+1}(1-q^{n+1}) \end{cases} \tag{49}$$

The choice $A^2 = E$, $B = 0$ in (18) yields to

$$\begin{cases} \widehat{\beta}_n = 0 \\ \widehat{\gamma}_{2n+1} = q^{n+1}(1-\gamma q^n) \quad , \quad n \geq 0. \\ \widehat{\gamma}_{2n+2} = \gamma q^{n+1}(1-q^{n+1}) \end{cases} \tag{50}$$

By comparison with (15) we obtain

$$\widehat{B}_n = Y_n(.;\gamma,q), \quad n \geq 0. \tag{51}$$

Consequently, according to [7], its corresponding regular form $\widehat{\mathcal{SV}}(0)$, $\gamma \neq 0$, $\gamma \neq q^{-n}$, $n \geq 0$ is $H_{\sqrt{q}}$-semiclassical of class one for $\gamma \neq 0$, $\gamma \neq \sqrt{q}$, $\gamma \neq q^{-n}$, $n \geq 0$ satisfying the q-distributional equation

$$H_{\sqrt{q}}\big(x\widehat{\mathcal{SV}}(0)\big) - \gamma^{-1}(\sqrt{q}-1)^{-1}\big\{q^{-1}x^2+\gamma-1\big\}\widehat{\mathcal{SV}}(0) = 0.$$

For more details about the H_q-semiclassical character see [11]. Particularly, the moments, integral representations and discrete measure of $\widehat{\mathcal{SV}}(0)$ are given in [11] and are recovered in the above study for $\alpha = 0$.

On the other hand, by virtue of (51) and Proposition 1 we meet the well known result [2]

$$\widehat{B}_{2n}(x) = W_n\Big(x^2;\gamma,q\Big) \;;\; \widehat{B}_{2n+1}(x) = xW_n\Big(x^2;\gamma q,q\Big), \quad n \geq 0. \tag{52}$$

Finally, from its H_q-semiclassical character another time, it is straightforward to bring out the structure relation satisfied by any shifted symmetric Al-Salam and Verma polynomial $\widehat{B}_{n+1}$ according to (52), (12) and (17). We obtain

$$x\big(H_{\sqrt{q}}\widehat{B}_{2n+2}\big)(x) = (1+\sqrt{q})[n+1]_q x\widehat{B}_{2n+1}(x),\ n \geq 0, \tag{53}$$

$$\begin{aligned} x\big(H_{\sqrt{q}}\widehat{B}_{2n+3}\big)(x) = &\left\{1+\frac{\sqrt{q}}{1-\sqrt{q}}\big(1-\gamma^{-1}+q^n(1-q)\big)\right\}\widehat{B}_{2n+3}(x)+ \\ &+\frac{\sqrt{q}}{1-\sqrt{q}}\big(\gamma^{-1}-q^n\big)\widehat{B}_{2n+2}(x),\ n \geq 0. \end{aligned} \tag{54}$$

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