TJMM
2 (2010), No. 1, 105-119

THE GREEN'S MATRIX AND THE GREEN'S TYPE INTEGRAL FORMULA FOR AN ELASTIC STRIP

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ABSTRACT. An efficient unified method to derive Green's matrices, called the incompressible influence elements method (IIEM), had been elaborated and published earlier by V. D. Şeremet [Handbook of Green's Functions and Matrices, WIT press, Southampton, Boston, 2003]. The main point of this method is general integral representations for Green's matrices in terms of Green's functions for Poisson's equation. This paper uses above mentioned representations to derive the Green's matrix and the Green's type integral formula for a boundary value problem (BVP) for an elastic strip. All results are obtained exactly and in elementary functions. To obtain these results some Green's functions for Posson's equation for a strip are derived. An exact solution for a particular BVP for an elastic strip also is included.

1. Introduction

The Green's matrix plays the leading role in finding the solutions in integrals for boundary value problems (BVP) for partial differential equations of elasticity theory. But take note that the issue under discussion here is not a trivial mathematical problem at all. Therefore deriving every new Green's matrix, especially in an explicit form, is always highly appreciated by experts in the field. Such kind of results can be considered a valuable contribution to the theory of elasticity. Traditionally, for derivation of the Green's matrices the same spectra of mathematical methods are applied, which usually are used for derivation of the Green's functions for Poisson's equation. This situation was not accepted by the author [9]. This is why he has been dealing with the research into the development of a unified method to derive Green's matrices in elasticity for many years. The outcome is a new method which represents in the same time an efficient unified method called the incompressible influence element method (IIEM). IIEM is based on the integral representations of the Green's matrix components (solution of the fundamental Lame's equations) via respective incompressible Green's matrices components (solution of the incompressible Lame's equations, when volume dilatation is equal to zero). The main advantage of this method is that, if we know Green's function for Poisson's equation, then the respective Green's matrix can be derived by unified mathematical procedures. In the most complicated case these mathematical procedures consist in: 1/. Derivation of the incompressible Green's matrix on the base of known already Green's functions for Poisson's equation; 2/. In some cases of complicated boundary conditions it is necessary to solve simple boundary integral equations with respect to volume dilatation and to compute some boundary integrals. This method was mostly developed for Cartesian system of coordinates. So, the best practical results achieved by IIEM have been obtained to derive Green's matrices just for Cartesian canonical domains and published in the books [9, 10, 13]. So, before appearance of these books there had been few problems for the domains

2010 *Mathematics Subject Classification.* 34B27, 35J08.
Key words and phrases. integral representations, Green's functions, incompressible influence element, Green's matrices, Maxwell's reciprocity theorem, volume dilatation, elasticity.

of the Cartesian for the solution of which Green's matrices have been derived. Among them there were well-known problems such as Kelvin's problem for an unbounded elastic space [14], the problems for the elastic half-space [1, 6-8], as well as some others, e.g. those for the domains discussed in the monographs [4, 5]. Indeed the book [9] produced an impact in this field, because it contains many Green's functions for Poisson's equation, the influence functions for the elastic volume dilatation and for Green's matrices, derived effectively using IIEM. Most of them are presented in a closed form. It is regrettable, but the elaborated already IIEM works for derivation of the Green's matrices for elastic isotropic Cartesian domains only. The extension of IIEM in the cases of elastic isotropic polar domains can be found in [11, 12].

The main objective of this paper is to derive the Green's matrix and the Green's type integral formula for a BVP for a homogeneous, elastic, isotropic strip. To achieve this goal we use general integral representations of the Green's matrices in terms of fundamental solutions with the exactitude of regular functions (F+R solutions) for Poisson's equations, suggested in [9, 10]. If, for the concrete body we suppose these F+R solutions to certain homogeneous boundary conditions that are analogical to the respective homogeneous boundary conditions for Green's matrices components, then they (F+R solutions) became Green's functions for Poisson's equation. As a result of this mathematical procedure the general integral representations became substantially simpler. In addition, if the boundary conditions are of locally-mixed type (normal tractions and tangential displacements or normal displacements and tangential tractions are given on the surface), then as a rule all integrals vanish and Green's matrices are expressed closely in terms of Green's functions for Poisson's equation only.

2. Constructive formulas for the Green's matrix of an elastic strip in terms of Green's functions for Poisson's equation

2.1. **General integral representations for Green's matrices in Cartesian coordinates.** The formulation of three dimensional (3D) BVPs in the case of Green's matrices constructing for the isotropic homogeneous elastic V body with the surface Γ consist from the following set of Lame's fundamental differential equations in partial derivatives [7]:

$$\mu\nabla^2 U_i^{(k)}(x,\xi) + (\lambda+\mu)\,\Theta_{,\,i}^{(k)}(x,\xi) = -\delta_{ik}\delta(x-\xi)\,; \quad i,k=1,2,3 \tag{1}$$

and from respective homogeneous boundary conditions. In Eq. (1) λ, μ are Lame's constants of elasticity; Green's matrix components $U_i^{(k)}(x,\xi)$ are the displacements at point $x \equiv (x_1,x_2)$ along the axis ox_i. These displacements are the result of the action in the direction of the axis ox_k of the unit concentrated forces $\delta_{ik}\delta(x-\xi)$, applied at point $\xi \equiv (\xi_1,\xi_2)$ of the Cartesian system of coordinates; ∇^2 is Laplace's operator in Cartesian coordinates $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$; $\delta(x-\xi)$ being δ Dirac's function; δ_{ik} being Kronecker's symbols $\delta_{ik}=0$, $i\neq k$; $\delta_{ik}=1$; $i=k$; $i,k=1,2,3$. The volume dilatation, determined by the formula

$$\Theta^{(k)} = U_{j,j}; \quad j=1,2,3, \tag{2}$$

satisfies the following differential equation of the Poisson's type:

$$\nabla^2\Theta^{(k)}(x,\xi) = -(\lambda+2\mu)^{-1}\frac{\partial}{\partial x_k}\delta(x-\xi)\,; \quad i,k=1,2,3, \tag{3}$$

obtained from Eq. (1) using the rule (2).

Beside, displacements must satisfy to the Maxwell's theorem of reciprocity:

$$U_i^{(k)}(x,\xi) = U_k^{(i)}(\xi,x)\,. \tag{4}$$

In addition for unbounded domains the displacements $U_i^{(k)}\,(x,\xi)$ must vanish at infinity.

Using Eq. (3) and others mathematical formulas, the set of Lame's equations (1) was given to three independent Poisson's type differential equations [9, 10, 13], Then, representing the solutions of obtained Poisson's equations via fundamental solutions with the exactitude of regular functions (F+R solutions), were obtained three general integral formula for the Green's matrix components. In this paper we use the following general integral formula for the Green's matrix components, presented in [9, 10, 13]:

$$\begin{gathered} U_i^{(k)}(x,\xi) = \tilde{U}_i^{(k)}(x,\xi) + \\ \int_\Gamma \left[G_i(y,x)\frac{\partial}{\partial n_\Gamma} - \frac{\partial}{\partial n_\Gamma}G_i(y,x)\right]\left[U_i^{(k)}(y,\xi) + \beta y_i\,\Theta^{(k)}(y,\xi)\right]d\Gamma(y) - \\ \beta x_i \int_\Gamma \left[\frac{\partial\Theta^{(k)}(y,\xi)}{\partial n_\Gamma} - \Theta^{(k)}(y,\xi)\frac{\partial}{\partial n_\Gamma}\right]G_\Theta(y,x)d\Gamma(y);\;\; \beta = \frac{\lambda+\mu}{2\mu} \end{gathered} \tag{5}$$

where n_Γ is the external normal to the surface Γ;G_i and G_Θ are some fundamental solutions with the exactitude up to regular functions (F+R solution) for Poisson's equation. So, the functions G_i and $G_\Theta$represent some F+R solutions to the following equations:

$$\nabla^2 G_i = -\delta\,(x-\xi)\;;\;\; \nabla^2 G_\Theta = -\delta\,(x-\xi)\;, \tag{6}$$

which represent the Lame's equations (1) when $\mu = 1$ and the Poisson's ratio is equal to 0,5: $\nu = 0,5$, so that volume dilatation is equal to zero: $\Theta^{(k)} = 0$. The displacements $\tilde{U}_i^{(k)}(x,\xi)$ in Eq. (5) are determined by the following formula:

$$\tilde{U}_i^{(k)}(x,\xi) = A\left[\left(B\delta_{ik} - \xi_i\frac{\partial}{\partial\xi_k}\right)G_i(x,\xi) + x_i\frac{\partial}{\partial\xi_k}G_\Theta(x,\xi)\right], \tag{7}$$

where $A = \beta\,(\lambda+2\mu)^{-1}$; $B = (\lambda+3\mu)\,/\,(\lambda+\mu)$ are some elasticity constants.

Respectively, for volume dilatation $\Theta^{(k)}(x,\xi)$, determined by Eq. (3) we have the following integral representation [9, 10, 13]:

$$\Theta^{(k)}(x,\xi) = -\frac{1}{\lambda+2\mu}\frac{\partial}{\partial\xi_k}G_\Theta(x,\xi) + \int_\Gamma\left[\frac{\partial\Theta^{(k)}(y,\xi)}{\partial n_\Gamma} - \Theta^{(k)}(y,\xi)\frac{\partial}{\partial n_\Gamma}\right]G_\Theta(y,x)d\Gamma(y) \tag{8}$$

The procedure of writing the functions G_i, G_Θ and deriving the Green's matrix for an elastic strip, on the base of the integral representations in Eqs. (5), (7) and (8), is presented in the next sections.

2.2. **Derivation of the Green's matrix for an elastic strip in terms of Green's functions for Poisson's equation.** In this section we show how to apply the general integral representations in Eqs. (5), (7) and (8) to derive the Green's matrix for a BVP for an elastic strip. So, we need to construct the influence function for the dilatation -$\Theta^{(k)}\,(x,\xi)$ and for the components of the displacements Green's matrix -$U_i^{(k)}\,(x,\xi)$ for a 2D BVP for the elastic strip $V \equiv (-\infty \le x_1 \le \infty,\quad 0 \le x_2 \le a_2)$ with the boundary $\Gamma \equiv \Gamma_{20}\bigcup\Gamma_{21}$, where $\Gamma_{20} \equiv (-\infty < x_1 < \infty,\; x_2 = 0)$and $\Gamma_{21} \equiv (-\infty < x_1 < \infty,\; x_2 = a_2)$ are the boundary straight lines. The strip is in the state of plane deformation.

2.2.1. *Formulation of the BVP for the elastic strip.* So, to derive Green's matrix for the elastic strip V we need to solve 2D BVP that consists from Lame's equations:

$$\mu\nabla^2 U_i^{(k)}\,(x,\xi) + (\lambda+\mu)\,\Theta_{,\,i}^{(k)}\,(x,\xi) = -\delta_{ik}\delta\,(x-\xi)\,;\quad i,k = 1,2 \tag{9}$$

and the following locally mixed boundary conditions belonging to sliding fixation type:

$$U_2^{(k)} = \sigma_{21}^{(k)} = 0\,;\;\; y_2 = 0, a_2;\;\; -\infty < y_1 < \infty\,. \tag{10}$$

In addition the displacements $U_2^{(k)}$ and the derivatives $\partial U_1^{(k)} \Big/ \partial x_1$ must vanish at infinity:

$$U_2^{(k)}\,|_{x_1=\pm\infty} < \infty;\; \frac{\partial U_1^{(k)}}{\partial x_1}\,|_{x_1=\pm\infty} = 0 \tag{11}$$

and the Maxwell's reciprocity theorem (4) must be satisfied.

This means that for the given BVP it is necessary to construct the influence functions for the unit concentrated body forces $\int_V \delta_{ik}\delta\,(x-\xi)\,dV\,(\xi) = \delta_{ik}$, i,k=1,2, applied at the inner points $\xi \equiv (\xi_1, \xi_2)$ of the strip and directed along the coordinate axis $0x_k$, and influenced both the volume dilatation (influence functions for the volume dilatation – $\Theta^{(k)}\,(x,\xi)$), and displacements (the components of the displacements Green's matrices – $U_i^{(k)}\,(x,\xi)$) at the points of observation $x \equiv (x_1, x_2)$ along the direction of the coordinate axis $0x_i$.

Computational Procedure

To derive Green's matrix for the elastic strip V we need to solve 2D BVP in Eqs (9)–(11). To achieve this goal we use the general integral representations for Green's matrices $U_i^{(k)}\,(x,\xi)$ (5) and (7), at $i,k = 1,2$. Also we use the integral representation for volume dilatation $\Theta^{(k)}\,(x,\xi)$ (8).

So, the general integral representations for the Green's matrix components $U_i^{(k)}\,(x,\xi)$ (5), (7) and for volume dilatation $\Theta^{(k)}\,(x,\xi)$ (8), in the case of the strip V, can be written in the following forms:

$$\begin{gathered} U_1^{(k)}\,(x,\xi) = A\left[\left(B\delta_{ik} - \xi_i\frac{\partial}{\partial\xi_k}\right)G_i(x,\xi) + x_i\frac{\partial}{\partial\xi_k}G_\Theta(x,\xi)\right] - \\ \int\limits_{-\infty}^{\infty}\left(\frac{\partial G_1\,(y_1,0;x)}{\partial n_{\Gamma_{20}}} - G_1\,(y_1,0;x)\,\frac{\partial}{\partial n_{\Gamma_{20}}}\right)\times\left(U_1^{(k)}\,(y_1,0;\xi) + \beta y_1\Theta^{(k)}\,(y_1,0;\xi)\right)dy_1 - \\ \int\limits_{-\infty}^{\infty}\left(\frac{\partial G_1\,(y_1,a_2;x)}{\partial n_{\Gamma_{21}}} - G_1\,(y_1,a_2;x)\,\frac{\partial}{\partial n_{\Gamma_{21}}}\right)\left(U_1^{(k)}\,(y_1,a_2;\xi) + \beta y_1\Theta^{(k)}\,(y_1,a_2;\xi)\right)dy_1 + \\ \beta x_1\int\limits_{-\infty}^{\infty}\left(\frac{\partial G_\Theta\,(y_1,0;x)}{\partial n_{\Gamma_{20}}} - G_\Theta\,(y_1,0;x)\,\frac{\partial}{\partial n_{\Gamma_{20}}}\right)\Theta^{(k)}\,(y_1,0;\xi)dy_1 + \\ \beta x_1\int\limits_{-\infty}^{\infty}\left(\frac{\partial G_\Theta\,(y_1,a_2;x)}{\partial n_{\Gamma_{21}}} - G_\Theta\,(y_1,a_2;x)\,\frac{\partial}{\partial n_{\Gamma_{21}}}\right)\Theta^{(k)}\,(y_1,a_2;\xi)dy_1 \end{gathered} \tag{12}$$

for displacements $U_1^{(k)}\,(x,\xi)$, and

$$\begin{gathered} U_2^{(k)}\,(x,\xi) = A\left[\left(B\delta_{2k} - \xi_2\frac{\partial}{\partial\xi_k}\right)G_2(x,\xi) + x_2\frac{\partial}{\partial\xi_k}G_\Theta(x,\xi)\right] - \\ \int\limits_{-\infty}^{\infty}\left(\frac{\partial G_2\,(y_1,0;x)}{\partial n_{\Gamma_{20}}} - G_2\,(y_1,0;x)\,\frac{\partial}{\partial n_{\Gamma_{20}}}\right)\times\left(U_2^{(k)}\,(y_1,0;\xi) + \beta y_2\Theta^{(k)}\,(y_1,0;\xi)\right)dy_1 - \end{gathered}$$

$$\int\limits_{-\infty}^{\infty}\left(\frac{\partial G_2\left(y_1,a_2;x\right)}{\partial n_{\Gamma_{21}}}-G_2\left(y_1,a_2;x\right)\frac{\partial}{\partial n_{\Gamma_{21}}}\right)\left(U_2^{(k)}\left(y_1,a_2;\xi\right)+\beta y_2\Theta^{(k)}\left(y_1,a_2;\xi\right)\right)dy_1+$$
$$\beta x_1\int\limits_{-\infty}^{\infty}\left(\frac{\partial G_\Theta\left(y_1,0;x\right)}{\partial n_{\Gamma_{20}}}-G_\Theta\left(y_1,0;x\right)\frac{\partial}{\partial n_{\Gamma_{20}}}\right)\Theta^{(k)}\left(y_1,0;\xi\right)dy_1+$$
$$\beta x_1\int\limits_{-\infty}^{\infty}\left(\frac{\partial G_\Theta\left(y_1,a_2;x\right)}{\partial n_{\Gamma_{21}}}-G_\Theta\left(y_1,a_2;x\right)\frac{\partial}{\partial n_{\Gamma_{21}}}\right)\Theta^{(k)}\left(y_1,a_2;\xi\right)dy_1 \tag{13}$$

for displacements $U_2^{(k)}\left(x,\xi\right)$, and

$$\Theta^{(k)}(x,\xi)=-\frac{1}{\lambda+2\mu}\frac{\partial}{\partial\xi_k}G_\Theta(x,\xi)+$$
$$\int\limits_{-\infty}^{\infty}\left[\frac{\partial\Theta^{(k)}(y_1,0;\xi)}{\partial n_{\Gamma_{20}}}+\Theta^{(k)}(y_1,0;\xi)\frac{\partial}{\partial n_{\Gamma_{20}}}\right]G_\Theta(y_1,0;x)dy_1+ \tag{14}$$
$$\int\limits_{-\infty}^{\infty}\left[\frac{\partial\Theta^{(k)}(y_1,a_2;\xi)}{\partial n_{\Gamma_{21}}}-\Theta^{(k)}(y_1,a_2;\xi)\frac{\partial}{\partial n_{\Gamma_{21}}}\right]G_\Theta(y_1,a_2;x)dy_1$$

for volume dilatation $\Theta^{(k)}(x,\xi)$.

We have to notice that in representations (12)–(14) the functions $\Theta^{(k)}$, G_i and G_Θ are unknown yet. They have to be determined in the following two items.

In the item 1 let us prove that the boundary conditions in Eq. (10) lead to zero normal derivative of volume dilatation on the boundary semi-straight-line Γ_{20}, that means

$$U_2^{(k)}\left(y,\xi\right)=\sigma_{21}^{(k)}\left(y,\xi\right)=0\;\Rightarrow\;\frac{\partial\Theta^{(k)}\left(y,\xi\right)}{n_{\Gamma_{20}}}=-\frac{\partial\Theta^{(k)}\left(y,\xi\right)}{\partial y_2}=0;\; y\equiv(y_1,0)\in\Gamma_{20}. \tag{15}$$

To prove equality $\partial\Theta^{(k)}\left(y_1,0;\xi\right)/\partial y_2=0$ from Eq. (14), first of all we see that the boundary condition with respect to tangential stresses in Eq. (14) and the equilibrium equation

$$\sigma_{2j\,,\,j}^{(k)}=0;\; j,k=1,2 \tag{16}$$

leads to the following relations:

$$\sigma_{21,1}^{(k)}=0\;\Rightarrow\;\sigma_{22,2}^{(k)}=0. \tag{17}$$

From the equation (2) for $\Theta^{(k)}$ (where $j,k=1,2$), from equation (16) and from the Hooke's law for the derivatives $\sigma_{22,2}^{(k)}$ it follows:

$$\sigma_{22,2}^{(k)}=2\mu U_{2,22}^{(k)}+\lambda\Theta_{,2}^{(k)}=\left(\lambda+2\mu\right)\Theta_{,2}^{(k)}-2\mu U_{1,12}^{(k)}=0. \tag{18}$$

Next, from the first boundary conditions (14) and from Hooke's law for the derivative $\sigma_{21,1}^{(k)}=0$

$$\sigma_{21,1}^{(k)}=\mu\left(U_{2\,,\,11}^{(k)}+U_{1\,,\,21}^{(k)}\right)=0, \tag{19}$$

follows

$$\left.\begin{array}{l}U_2^{(k)}=0\to U_{2,11}^{(k)}=0;\\ \sigma_{21,1}^{(k)}=0\to\mu\left(U_{2,11}^{(k)}+U_{1,21}^{(k)}\right)=0;\end{array}\right\}\Rightarrow U_{1,21}^{(k)}=0. \tag{20}$$

Finally, Eqs. (17) and (19) lead to zero normal derivative of volume dilatation on the boundary half-plane Γ_{20}

$$\Theta^{(k)}_{,2} = 0 \to \left[\partial\Theta^{(k)}(y,\xi) \big/ \partial n_{\Gamma_{20}}\right] = 0 \tag{21}$$

that coincides with Eq. (14). So we have proved that on boundary Γ_{20} the normal derivative of volume dilatation is equal to zero. In the same way we can prove the boundary conditions in Eq. (10) lead to zero normal derivative of volume dilatation on the boundary semi-straight-line Γ_{21}, that means

$$U_2^{(k)}(y,\xi) = \sigma_{21}^{(k)}(y,\xi) = 0 \;\Rightarrow \frac{\partial\Theta^{(k)}(y,\xi)}{n_{\Gamma_{21}}} = -\frac{\partial\Theta^{(k)}(y,\xi)}{\partial y_2} = 0;\; y \equiv (y_1, a_2) \in \Gamma_{21}. \tag{22}$$

Finally, the obtained in the item 1 results can be written as follows:

$$\begin{gathered} U_2^{(k)}(y_1,0;\xi) = \sigma_{21}^{(k)}(y_1,0;\xi) = 0 \;\Rightarrow \\ \frac{\partial\Theta^{(k)}(y_1,0;\xi)}{n_{\Gamma_{20}}} = -\frac{\partial\Theta^{(k)}(y_1,0;\xi)}{\partial y_2} = 0;\; y \equiv (y_1,0) \in \Gamma_{20}; \\ U_2^{(k)}(y_1,a_2;\xi) = \sigma_{21}^{(k)}(y_1,a_2;\xi) = 0 \;\Rightarrow \\ \frac{\partial\Theta^{(k)}(y_1,a_2;\xi)}{n_{\Gamma_{21}}} = -\frac{\partial\Theta^{(k)}(y_1,a_2;\xi)}{\partial y_2} = 0;\; y \equiv (y_1,a_2) \in \Gamma_{21}. \end{gathered} \tag{23}$$

In the item 2 we prove that boundary conditions (10) lead to the equality

$$U_{1,2}^{(k)} = 0; y_2 = 0, a_2;\;\; -\infty < y_1 < \infty \tag{24}$$

Indeed from boundary conditions (10) follows:

$$\left.\begin{array}{l} U_2^{(k)} = 0 \to U_{2,1}^{(k)} = 0 \\ \sigma_{21}^{(k)} = \mu\left(U_{1,2}^{(k)} + U_{2,1}^{(k)}\right) = 0\,; \end{array}\right\} \to\; U_{1,2}^{(k)} = 0; y_2 = 0, a_2;\;\; -\infty < y_1 < \infty\,.$$

Note that from boundary conditions, described in Eqs. (23)–(24) follows the following equivalent boundary conditions

$$\begin{gathered} U_2^{(k)}(y_1,0;\xi) = \frac{\partial U_1^{(k)}(y_1,0;\xi)}{\partial n_{\Gamma_{20}}} = 0;\; \Rightarrow \frac{\partial\Theta^{(k)}(y_1,0;\xi)}{\partial n_{\Gamma_{20}}} = 0;\; y \equiv (y_1,0) \in \Gamma_{20}; \\ U_2^{(k)}(y_1,a_2;\xi) = \frac{\partial U_1^{(k)}(y_1,a_2;\xi)}{\partial n_{\Gamma_{21}}} = 0 \;\Rightarrow \frac{\partial\Theta^{(k)}(y_1,a_2;\xi)}{\partial n_{\Gamma_{21}}} = 0;\; y \equiv (y_1,a_2) \in \Gamma_{21} \end{gathered} \tag{25}$$

At the end we have to suppose the F+R fundamental solutions $G_i$and $G_\Theta$for Poisson's equation in representations (12)–(14) to the following homogeneous boundary conditions analogical to those in Eq. (25):

$$\begin{gathered} G_2(y_1,0;\xi) = \frac{\partial G_1(y_1,0;\xi)}{\partial n_{\Gamma_{20}}} = 0;\; \Rightarrow \frac{\partial G_\Theta(y_1,0;\xi)}{\partial n_{\Gamma_{20}}} = 0;\; y \equiv (y_1,0) \in \Gamma_{20}; \\ G_2(y_1,a_2;\xi) = \frac{\partial G_1(y_1,a_2;\xi)}{\partial n_{\Gamma_{20}}} = 0;\; \Rightarrow \frac{\partial G_\Theta(y_1,a_2;\xi)}{\partial n_{\Gamma_{20}}} = 0;\; y \equiv (y_1,a_2) \in \Gamma_{21} \end{gathered} \tag{26}$$

from which follows that the Green's functions for Poisson equation are the same, $G_1(x,\xi) \equiv G_\Theta(x,\xi)$.

In addition the conditions (10) and $G_1(x,\xi) \equiv G_\Theta(x,\xi)$ lead to the following conditions for the Green's functions:

$$G_2\,|_{x_1=\pm\infty} < \infty;\; \frac{\partial G_1}{\partial x_1}\,|_{x_1=\pm\infty} = \frac{\partial G_\Theta}{\partial x_1}\,|_{x_1=\pm\infty} = 0. \tag{27}$$

Finally, substituting Eqs. (25)–(26) into integral representations (12)–(14) we can see that all integrals in representations for $U_1^{(k)}(x,\xi)$ and $\Theta^{(k)}(x,\xi)$ vanish. So, we obtain the final constructive formulas for components $U_1^{(k)}(x,\xi)$ of Green's matrix $U_i^{(k)}(x,\xi)$ and volume dilatation $\Theta^{(k)}(x,\xi)$ in terms of Green's functions for Poisson's equations in the following form:

$$U_1^{(k)}(x,\xi) = A\left(B\delta_{1k} + (x_1-\xi_1)\frac{\partial}{\partial\xi_k}\right)G_1(x,\xi), \tag{28}$$

$$\Theta^{(k)}(x,\xi) = -\frac{1}{\lambda+2\mu}\frac{\partial}{\partial\xi_k}G_\Theta(x,\xi). \tag{29}$$

In integral representations (13) one integral do not vanish only and we obtain the following simpler integral representation for $U_2^{(k)}(x,\xi)$:

$$\begin{aligned} U_2^{(k)}(x,\xi) = A\left[\left(B\delta_{2k} - \xi_2\frac{\partial}{\partial\xi_k}\right)G_2(x,\xi) + x_2\frac{\partial}{\partial\xi_k}G_\Theta(x,\xi)\right] - \\ \int\limits_{-\infty}^{\infty}\left(\frac{\partial G_2(y_1,a_2;x)}{\partial n_\Gamma}\right)\beta a_2\Theta^{(k)}(y_1,a_2;\xi)\,dy_1. \end{aligned} \tag{30}$$

The integral (30) can be written in the following way:

$$\begin{aligned} \int\limits_{-\infty}^{\infty}\left(\frac{\partial G_2(y_1,a_2;x)}{\partial n_\Gamma}\right)\beta a_2\Theta^{(k)}(y_1,a_2;\xi)\,dy_1 = \\ Aa_2\int\limits_{-\infty}^{\infty}\left(\frac{\partial G_2(y_1,a_2;x)}{\partial y_2}\right)\frac{\partial}{\partial\xi_k}G_\Theta(y_1,a_2;\xi)\,dy_1, \end{aligned} \tag{31}$$

where the expression for volume dilatation $\Theta^{(k)}(y_1,a_2;\xi)$ from Eq. (29) was applied. Father for computing of the integral (30) we need to have expressions for Green's functions $G_2(x,\xi)$ and $G_\Theta(x,\xi) \equiv G_1(x,\xi)$ for Poisson's equation for a strip. In the next section we present an example of derivation of the Green's function $G_2(x,\xi)$.

3. Green's functions for Poisson's equation for an elastic strip

To construct the Green's function $G_2(x,\xi)$ for Poisson's equation $\nabla_x^2 G_2(x,\xi) = -\delta(x-\xi)$ for the strip $V \equiv (-\infty \le x_1 \le \infty, \quad 0 \le x_2 \le a_2)$ under the following boundary conditions

$$G_2 = 0\,;\;\; x_2 = 0, a_2;\; -\infty \le x_1 \le \infty. \tag{32}$$

Here, the function $G_2(x,\xi)$ should be restricted at the infinity, i.e. $G_2|_{x_1=\pm\infty} < \infty$.

Computational Procedure

In general, to construct Green's functions for Poisson's equation are developed many classical methods, presented in many books, including [2, 3, 11]. In this section we use the special method of separation of variables, presented in the book [11]. So, in accordance with this method to derive the Green's function G_2 we use the following general trigonometric series:

$$G_2 = a_0 + \sum_{m=1}^{\infty} a_m \sin\nu_1 x_2 + \sum_{m=1}^{\infty} b_m \cos\nu_1 x_2, \tag{33}$$

where the coefficients a_0, a_m, b_m are the functions of the variables x_1. The boundary conditions of this problem simplify the general series and reduce it to the form:

$$G_2 = \sum_{m=1}^{\infty} a_m \sin \nu_1 x_2, \quad \nu_1 = \frac{m\pi}{a_2};\ m = 1, 2, 3, ... \tag{34}$$

In so doing, the condition $G_2 \left|_{x_1=\pm\infty}\right. < \infty$ leads to the equivalent conditions $a_m \left|_{x_1=\pm\infty}\right. < \infty$. By substituting the latter expression (34) for the function G_2 into Poisson's equation $\nabla_x^2 G_2 (x, \xi) = -\delta (x - \xi)$ we obtain

$$\sum_{m=1}^{\infty} \left(a_m'' - \nu_1^2 a_m\right) \sin \nu_1 x_2 = -\delta (x_1 - \xi_1)\, \delta (x_2 - \xi_2) \tag{35}$$

the equality $\delta (x - \xi) = \delta (x_1 - \xi_1)\, \delta (x_2 - \xi_2)$ being taken into account in the method of separation of variables.

Let us multiply both parts of the obtained equation (35) by $\sin \nu_2 x_2$; $\nu_2 = \frac{s\pi}{a_2}$; $s = 1, 2, 3, ...$, then integrate it with respect to the variables x_2 and find the integrals

$$\int_0^{a_2} \sin \frac{s\pi x_2}{a_2} \sin \frac{m\pi x_2}{a_2} dx_2 = \begin{cases} 0;\ s \neq m, \\ a_2/2;\ s = m,\ \nu_1 = \nu_2 \end{cases}; \tag{36}$$

$$\int_0^{a_2} \delta (x_1 - \xi_1)\, \delta (x_2 - \xi_2) \sin \nu_2 x_2 dx_2 = \delta (x_1 - \xi_1) \sin \nu_1 \xi_2;\ \nu_1 = \nu_2. \tag{37}$$

The latter is calculated by making use of the property

$$\int_V f(x)\ \delta (x - \xi)\, dV(x) = f(\xi) \tag{38}$$

of the Dirac function.

Hence, we obtain the following equation used to determine the function a_m

$$\left(a_m'' - \nu_1^2 a_m\right) \frac{a_2}{2} = -\delta (x_1 - \xi_1) \sin \nu_1 \xi_2. \tag{39}$$

Assuming ξ_2 to be a parameter and taking the notation $a_m = (2/a_2)\, \bar{a}_m \sin \nu_1 \xi_2$ we obtain the 1D boundary-value problem for the function $\bar{a}_m$:

$$\left(\bar{a}_m'' - \nu_1^2 \bar{a}_m\right) = -\delta (x_1 - \xi_1); \quad \bar{a}_m \left|_{x_1=-\infty}\right. < \infty;\ \bar{a}_m \left|_{x_1=\infty}\right. < \infty. \tag{40}$$

To construct the Green's functions for this BVP we use a standard technique given in [4, 5, 9, 11]. The general Answer to the latter homogeneous equation (40) is written as

$$\bar{a}_m = \begin{cases} c_1 e^{-\nu_1 x_1} + c_2 e^{\nu_1 x_1};\ x_1 \leq \xi_1; \\ k_1 e^{-\nu_1 x_1} + k_2 e^{\nu_1 x_1};\ x_1 \geq \xi_1. \end{cases} \tag{41}$$

Then, from the conditions of conjugality at the point $x_1 = \xi_1$

$$\bar{a}_m (x_1 = \xi_1 - 0) = \bar{a}_m (x_1 = \xi_1 + 0)\ ; \quad \bar{a}_m' (x_1 = \xi_1 - 0) - \bar{a}_m' (x_1 = \xi_1 + 0) = 1 \tag{42}$$

we get a set of two simultaneous linear algebraic equations

$$\begin{gathered} (c_1 - k_1)\, e^{-\nu_1 \xi_1} + (c_2 - k_2)\, e^{\nu_1 \xi_1} = 0\,; \\ \nu_1 \left[(c_1 - k_1)\, e^{-\nu_1 \xi_1} - (c_2 - k_2)\, e^{\nu_1 \xi_1}\right] = -1\,. \end{gathered} \tag{43}$$

From the boundary conditions at infinite it follows

$$\bar{a}_m (x_1 = -\infty) < \infty \Rightarrow c_1 = 0; \quad \bar{a}_m (x_1 = \infty) < \infty \Rightarrow k_2 = 0. \tag{44}$$

When accounting for the obtained values $c_1 = k_2 = 0$ the solution of the considered set of equations is reduced to the form

$$k_1 = (2\nu_1)^{-1} e^{\nu_1 \xi_1}; \quad c_2 = (2\nu_1)^{-1} e^{-\nu_1 \xi_1}; \quad c_1 = k_2 = 0. \tag{45}$$

Therefore, for the influence function $\bar{a}_m$ the following expression is obtained

$$\bar{a}_m = \begin{cases} (2\nu_1)^{-1} e^{\nu_1(x_1-\xi_1)}; \ x_1 \le \xi_1 ; \\ (2\nu_1)^{-1} e^{-\nu_1(x_1-\xi_1)}; \ x_1 \ge \xi_1 . \end{cases} \tag{46}$$

Taking into consideration the expression for the function $a_m = (2/a_2)\,\bar{a}_m \sin\nu_1\xi_2$ the Green's function $G\,(x,\xi)$ is written as

$$G_2\,(x,\xi) = \begin{cases} G_{2l}\,(x,\xi) = \frac{2}{a_2}\sum_{m,n=1}^{\infty}\frac{1}{2\nu_1}e^{\nu_1(x_1-\xi_1)}\sin\nu_1 x_2 \sin\nu_1\xi_2, \ x_1 \le \xi_1; \\ G_{2r}\,(x,\xi) = \frac{2}{a_2}\sum_{m,n=1}^{\infty}\frac{1}{2\nu_1}e^{-\nu_1(x_1-\xi_1)}\sin\nu_1 x_2 \sin\nu_1\xi_2, \ x_1 \ge \xi_1. \end{cases} \tag{47}$$

Here, and below let $G_l\,(x,\xi)$ and $G_r\,(x,\xi)$ denote the expressions for the Green's function to the left of the action of the unit force, i.e. at $x_1 \le \xi_1$ and the right of its action, i.e. at $x_1 \ge \xi_1$, respectively. Now we are at the point to show that the infinite series can be summated. So, by making use of the known sum [9, 11]

$$\sum_{n=1}^{\infty}\frac{p^n}{n}\cos n\alpha = -\ln\sqrt{1-2p\cos\alpha+p^2}; \ p^2<1, \ 0\le\alpha<2\pi \text{ or } p^2\le 1, \ 0<\alpha<2\pi \tag{48}$$

we come to the above-mentioned statement. In addition, to take the sum of the series in the expression for the $G\,(x,\xi)$ first it is necessary to use the trigonometric formula

$$\sin\nu_1 x_2 \sin\nu_1\xi_2 = \frac{1}{2}\left[\cos\nu_1\,(x_2-\xi_2) - \cos\nu_1\,(x_2+\xi_2)\right] . \tag{49}$$

After some computations one can get the final expression for the Green's function of the stated BVP for the strip

$$G_2\,(x,\xi) = \begin{cases} G_{2l}\,(x,\xi) = \\ \dfrac{2}{a_2}\displaystyle\sum_{m,n=1}^{\infty}\frac{1}{2\nu_1}e^{\nu_1(x_1-\xi_1)}\frac{1}{2}\left[\cos\nu_1\,(x_2-\xi_2)-\cos\nu_1\,(x_2+\xi_2)\right], \ x_1 \leqslant \xi_1; \\ G_{2r}\,(x,\xi) = \\ \dfrac{2}{a_2}\displaystyle\sum_{m,n=1}^{\infty}\frac{1}{2\nu_1}e^{-\nu_1(x_1-\xi_1)}\frac{1}{2}\left[\cos\nu_1\,(x_2-\xi_2)-\cos\nu_1\,(x_2+\xi_2)\right], \ x_1 \geqslant \xi_1. \end{cases} \tag{50}$$

in the form of infinite series, and

$$G_2 = -\,(2\pi)^{-1}\ln EE_2^{-1}, \tag{51}$$

in terms of elementary functions E, E_2 that are determined by the expressions

$$\begin{aligned} E &= \sqrt{1-2e^{\pi a_2^{-1}(x_1-\xi_1)}\cos\pi a_2^{-1}\,(x_2-\xi_2)+e^{2\pi a_2^{-1}(x_1-\xi_1)}}; \\ E_2 &= \sqrt{1-2e^{\pi a_2^{-1}(x_1-\xi_1)}\cos\pi a_2^{-1}\,(x_2+\xi_2)+e^{2\pi a_2^{-1}(x_1-\xi_1)}}. \end{aligned} \tag{52}$$

In analogical way we can obtain the Green's function for Poisson's equation for the strip with the Neumann boundary conditions:

$$\begin{aligned} \frac{\partial G_1}{\partial x_2} &= 0 ; \qquad x_2 = 0, \ a_2; \ -\infty \le x_1 \le \infty; \\ \frac{\partial G_1}{\partial x_1} &= 0 ; \qquad x_1 = \pm\infty; \ 0 \le x_2 \le a_2. \end{aligned} \tag{53}$$

The expression of this Green's function is:

$$G = b - (2\pi)^{-1}\ln EE_2; \quad b = \text{const}. \tag{54}$$

4. Final expressions for the Green's matrix and Green's integral formulas for an elastic strip

4.1. **Final expressions for the Green's matrix for an elastic strip.** We have now the constructive formulas for the Green's matrix $U_i^{(k)}(x,\xi)$ in terms of known already expressions for Green's functions $G_1(x,\xi) = G_\Theta(x,\xi) = b-(2\pi)^{-1}\ln EE_2$ and $G_2(x,\xi) = -(2\pi)^{-1}\ln EE_2^{-1}$:

$$U_1^{(k)}(x,\xi) = AB\delta_{1k}b - \frac{A}{2\pi}\left(B\delta_{1k} + (x_1-\xi_1)\left(\delta_{1k}\frac{\partial}{\partial\xi_1} + \delta_{2k}\frac{\partial}{\partial\xi_2}\right)\right)(\ln EE_2)\,. \tag{55}$$

Taking into account Eqs (30)–(31) integral representations for $U_2^{(k)}(x,\xi)$ components can be written in the following form:

$$\begin{gathered} U_2^{(k)}(x,\xi) = A\left[\left(B\delta_{2k} - \xi_2\frac{\partial}{\partial\xi_k}\right)G_2(x,\xi) + x_2\frac{\partial}{\partial\xi_k}G_\Theta(x,\xi)\right] - \\ Aa_2\int\limits_{-\infty}^{\infty}\left(\frac{\partial G_2(y_1,a_2;x)}{\partial y_2}\right)\frac{\partial}{\partial\xi_k}G_\Theta(y_1,a_2;\xi)\,dy_1 \end{gathered} \tag{56}$$

The integral in Eq. (56) can be calculated in the following way:

$$\begin{gathered} Aa_2\int\limits_{-\infty}^{\infty}\left(\frac{\partial G_2(y_1,a_2;x)}{\partial y_2}\right)\frac{\partial}{\partial\xi_k}G_\Theta(y_1,a_2;\xi)\,dy_1 = \\ Aa_2\int\limits_{-\infty}^{\infty}\left(\frac{\partial}{\partial\xi_2}\left((\pi)^{-1}\ln E(y_1,a_2;x)\right)\right)\frac{\partial}{\partial\xi_k}\left[-(\pi)^{-1}\ln E(y_1,a_2;\xi)\right]dy_1 = \\ Ax_2\frac{\partial}{\partial\xi_k}(2\pi)^{-1}(-\ln EE_2) - A\xi_2\frac{\partial}{\partial\xi_k}(2\pi)^{-1}\left(-\ln EE_2^{-1}\right) - \\ A(x_1-\xi_1)\int\left(\frac{\partial^2}{\partial\xi_k\partial x_2}(2\pi)^{-1}(-\ln EE_2)\right)dx_1, \end{gathered} \tag{57}$$

Father the last integral in Eq. (57) can be computed in the following way:

$$\begin{gathered} \int\frac{\partial^2}{\partial\xi_k\partial x_2}(2\pi)^{-1}(-\ln EE_2)\,dx_1 = \\ \int\left(\delta_{1k}\frac{\partial^2}{\partial\xi_1\partial x_2} + \delta_{2k}\frac{\partial^2}{\partial\xi_2\partial x_2}\right)(2\pi)^{-1}(-\ln EE_2)\,dx_1 = \\ \int\left(\delta_{1k}\frac{\partial^2}{\partial x_1\partial\xi_2} - \delta_{2k}\frac{\partial^2}{\partial x_2^2}\right)(2\pi)^{-1}\left(-\ln EE_2^{-1}\right)dx_1 = \\ \delta_{1k}\frac{\partial}{\partial\xi_2}(2\pi)^{-1}\left(-\ln EE_2^{-1}\right) + \delta_{2k}\int\frac{\partial^2}{\partial x_1^2}(2\pi)^{-1}\left(-\ln EE_2^{-1}\right)dx_1 = \\ \left(\delta_{1k}\frac{\partial}{\partial\xi_2} - \delta_{2k}\frac{\partial}{\partial\xi_1}\right)(2\pi)^{-1}\left(-\ln EE_2^{-1}\right). \end{gathered} \tag{58}$$

Substituting (58) into (57), then into (56) we obtain:

$$U_2^{(k)}(x,\xi) = A\left[\left(B\delta_{2k} - \xi_2\frac{\partial}{\partial\xi_k}\right)(2\pi)^{-1}\left(-\ln EE_2^{-1}\right) + x_2\frac{\partial}{\partial\xi_k}(2\pi)^{-1}(-\ln EE_2)\right] -$$

$$\begin{array}{c} Ax_2 \dfrac{\partial}{\partial \xi_k} (2\pi)^{-1} \left(-\ln EE_2\right) + A\xi_2 \dfrac{\partial}{\partial \xi_k} (2\pi)^{-1} \left(-\ln EE_2^{-1}\right) + \\ A\left(x_1 - \xi_1\right) \left(\delta_{1k} \dfrac{\partial}{\partial \xi_2} - \delta_{2k} \dfrac{\partial}{\partial \xi_1}\right) (2\pi)^{-1} \left(-\ln EE_2^{-1}\right) \ . \end{array} \tag{59}$$

So, from Eqs. (55) and (59) we can write the following final expressions for the Green's matrix components $U_i^{(k)}(x,\xi)$ for the elastic strip in elementary functions:

$$\begin{array}{c} U_1^{(k)}(x,\xi) = AB\delta_{1k} b - \frac{A}{2\pi} \left(B\delta_{1k} + (x_1 - \xi_1)\left(\delta_{1k}\frac{\partial}{\partial \xi_1} + \delta_{2k}\frac{\partial}{\partial \xi_2}\right)\right)(\ln EE_2)\ ; \\ U_2^{(k)}(x,\xi) = -\frac{A}{2\pi}\left(B\delta_{2k} + (x_1 - \xi_1)\left(\delta_{1k}\frac{\partial}{\partial \xi_2} - \delta_{2k}\frac{\partial}{\partial \xi_1}\right)\right)\left(\ln EE_2^{-1}\right), \qquad (k = 1, 2)\,, \end{array} \tag{60}$$

where

$$\begin{array}{c} E = \sqrt{1 - 2e^{\pi a_2^{-1}(x_1 - \xi_1)} \cos \pi a_2^{-1}\left(x_2 - \xi_2\right) + e^{2\pi a_2^{-1}(x_1 - \xi_1)}}; \\ E_2 = \sqrt{1 - 2e^{\pi a_2^{-1}(x_1 - \xi_1)} \cos \pi a_2^{-1}\left(x_2 + \xi_2\right) + e^{2\pi a_2^{-1}(x_1 - \xi_1)}}; \end{array} \tag{61}$$

$$\left.\begin{array}{c} \frac{\partial}{\partial \xi_1}(\ln E) = \pi a_2^{-1} \frac{e^{\pi a_2^{-1}(x_1-\xi_1)} \cos \pi a_2^{-1}(x_2-\xi_2) - e^{2\pi a_2^{-1}(x_1-\xi_1)}}{1-2e^{\pi a_2^{-1}(x_1-\xi_1)} \cos \pi a_2^{-1}(x_2-\xi_2) + e^{\frac{2\pi}{a_2}(x_1-\xi_1)}}; \\ \frac{\partial}{\partial \xi_1}(\ln E_2) = \pi a_2^{-1} \frac{e^{\pi a_2^{-1}(x_1-\xi_1)} \cos \pi a_2^{-1}(x_2+\xi_2) - e^{2\pi a_2^{-1}(x_1-\xi_1)}}{1-2e^{\pi a_2^{-1}(x_1-\xi_1)} \cos \pi a_2^{-1}(x_2+\xi_2) + e^{2\pi a_2^{-1}(x_1-\xi_1)}} \end{array}\right]; \tag{62}$$

$$\left.\begin{array}{c} \frac{\partial}{\partial \xi_2}(\ln E) = \frac{-\pi a_2^{-1} e^{\pi a_2^{-1}(x_1-\xi_1)} \sin \pi a_2^{-1}(x_2-\xi_2)}{1-2e^{\pi a_2^{-1}(x_1-\xi_1)} \cos \pi a_2^{-1}(x_2-\xi_2) + e^{2\pi a_2^{-1}(x_1-\xi_1)}}; \\ \frac{\partial}{\partial \xi_2}(\ln E_2) = \frac{\pi a_2^{-1} e^{\pi a_2^{-1}(x_1-\xi_1)} \sin \pi a_2^{-1}(x_2+\xi_2)}{1-2e^{\pi a_2^{-1}(x_1-\xi_1)} \cos \pi a_2^{-1}(x_2+\xi_2) + e^{2\pi a_2^{-1}(x_1-\xi_1)}} \end{array}\right]. \tag{63}$$

So, the expressions (60)–(63) satisfy to all conditions for Green's matrix for the BVP for an elastic strip:

Lame's equations (9), boundary conditions (10) on the straight lines Γ_{20} and Γ_{21}, conditions (11) at infinity and Maxwell's theorem of reciprocity of displacements $U_i^{(k)}(x,\xi) = U_k^{(i)}(\xi,x)$.

4.2. **The Green's integral formula for a BVP for an elastic strip.** Let we need to solve the following BVP for the elastic strip V that consists from Lame's equations:

$$\mu\nabla^2 u_k(\xi) + (\lambda + \mu)\,\theta_{,k}(\xi) + F(\xi) = 0 \tag{64}$$

and boundary conditions

$$\left.\begin{array}{l} p_1(y_1, 0) = s_1(y_1, 0)\,;\ u_2(y_1, 0) = g_2(y_1, 0)\,; \\ p_1(y_1, a_2) = s_1(y_1, a_2)\,;\ u_2(y_1, a_2) = g_2(y_1, a_2) \end{array}\right\};\ -\infty < y_1 < \infty, \tag{65}$$

where $F(\xi)$ are body forces, $p_1(y_1, 0)$, $g_2(y_1, 0)$ and $p_1(y_1, 0)$, $g_2(y_1, 0)$ are tractions and displacements given on the boundary straight lines Γ_{20} and Γ_{21}, respectively. Then, using the derived Green's matrix in Eqs. (60)–(63) and Green's general integral formula [7] we can write the solution of the BVP (64) and (65) for the strip in the following integral form:

$$u_k(\xi) = \int_{-\infty}^{\infty}\int_{0}^{a_2} F_j(x)\, U_j^{(k)}(x,\xi)\, dx_1 dx_2 +$$

$$\int_{-\infty}^{\infty} \left[s_1(y_1,0)\, U_1^{(k)}(y_1,0;\xi) - g_2(y_1,0)\, P_2^{(k)}(y_1,0;\xi) \right] dy_1 + \tag{66}$$

$$\int_{-\infty}^{\infty} \left[s_1(y_1,a_2)\, U_1^{(k)}(y_1,a_2;\xi) - g_2(y_1,a_2)\, P_2^{(k)}(y_1,a_2;\xi) \right] dy_1; \quad (j,k=1,2),$$

where

$$U_1^{(k)}(y_1,0;\xi) = AB\delta_{1k}b - 2A\left(B\delta_{1k} + (y_1-\xi_1)\left(\delta_{1k}\frac{\partial}{\partial\xi_1} + \delta_{2k}\frac{\partial}{\partial\xi_2} \right) \right) \ln E(y_1,0;\xi); \tag{67}$$

$$U_1^{(k)}(y_1,a_2;\xi) = AB\delta_{1k}b - 2A\left(B\delta_{1k} + (y_1-\xi_1)\left(\delta_{1k}\frac{\partial}{\partial\xi_1} + \delta_{2k}\frac{\partial}{\partial\xi_2} \right) \right) \ln E(y_1,a_2;\xi); \tag{68}$$

$$\begin{gathered} E(y_1,0;\xi) = \sqrt{1 - 2e^{\pi a_2^{-1}(y_1-\xi_1)}\cos \pi a_2^{-1}\xi_2 + e^{2\pi a_2^{-1}(y_1-\xi_1)}}; \\ \frac{\partial}{\partial\xi_1}\ln E(y_1,0;\xi) = \pi a_2^{-1}\frac{e^{\pi a_2^{-1}(y_1-\xi_1)}\cos \pi a_2^{-1}\xi_2 - e^{2\pi a_2^{-1}(y_1-\xi_1)}}{1 - 2e^{\pi a_2^{-1}(y_1-\xi_1)}\cos \pi a_2^{-1}\xi_2 + e^{2\pi a_2^{-1}(y_1-\xi_1)}}; \end{gathered} \tag{69}$$

$$\frac{\partial}{\partial\xi_2}\ln E(y_1,0;\xi) = \frac{\pi a_2^{-1} e^{\pi a_2^{-1}(y_1-\xi_1)}\sin \pi a_2^{-1}\xi_2}{1 - 2e^{\pi a_2^{-1}(y_1-\xi_1)}\cos \pi a_2^{-1}\xi_2 + e^{2\pi a_2^{-1}(y_1-\xi_1)}}; \tag{70}$$

$$\begin{gathered} E(y_1,a_2;\xi) = \sqrt{1 + 2e^{\pi a_2^{-1}(y_1-\xi_1)}\cos \pi a_2^{-1}\xi_2 + e^{2\pi a_2^{-1}(y_1-\xi_1)}}; \\ \frac{\partial}{\partial\xi_2}\ln E(y_1,a_2;\xi) = -\frac{\pi a_2^{-1} e^{\pi a_2^{-1}(y_1-\xi_1)}\sin \pi a_2^{-1}\xi_2}{1 + 2e^{\pi a_2^{-1}(y_1-\xi_1)}\cos \pi a_2^{-1}\xi_2 + e^{2\pi a_2^{-1}(y_1-\xi_1)}}; \end{gathered} \tag{71}$$

$$\frac{\partial}{\partial\xi_1}\ln E(y_1,a_2;\xi) = -\pi a_2^{-1}\frac{e^{\pi a_2^{-1}(y_1-\xi_1)}\cos \pi a_2^{-1}\xi_2 + e^{2\pi a_2^{-1}(y_1-\xi_1)}}{1 + 2e^{\pi a_2^{-1}(y_1-\xi_1)}\cos \pi a_2^{-1}\xi_2 + e^{2\pi a_2^{-1}(y_1-\xi_1)}}. \tag{72}$$

The tractions on the boundary straight line $\Gamma_{20} \equiv (-\infty < y_1 < \infty,\ y_2 = 0)$ in the formula (66), created by the Green's matrix components $U_i^{(k)}$ are determined by the expression

$$\begin{gathered} P_2^{(k)}(y_1,0;\xi) = -\sigma_{22}^{(k)}(y_1,0;\xi) = -\left[2\mu U_{2,2}^{(k)} + \lambda\Theta^{(k)} \right]_{x\to y\equiv(y_1,0)} = \\ -\left\{ 2\mu\tfrac{A}{2\pi}\left[B\delta_{2k} + (x_1-\xi_1)\left(\delta_{1k}\tfrac{\partial}{\partial\xi_2} - \delta_{2k}\tfrac{\partial}{\partial\xi_1} \right) \right]\tfrac{\partial}{\partial\xi_2} + \tfrac{\lambda}{2\pi(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_k} \right\} (\ln EE_2)\,\big|_{x\to y\equiv(y_1,0)} = \\ -2\left\{ 2\mu\tfrac{A}{2\pi}\left[B\delta_{2k} + (y_1-\xi_1)\left(\delta_{1k}\tfrac{\partial}{\partial\xi_2} - \delta_{2k}\tfrac{\partial}{\partial\xi_1} \right) \right]\tfrac{\partial}{\partial\xi_2} + \tfrac{\lambda}{2\pi(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_k} \right\} \ln E(y_1,0;\xi) = \\ -2\left\{ 2\mu\tfrac{A}{2\pi}\left[B\delta_{2k}\tfrac{\partial}{\partial\xi_2} + (y_1-\xi_1)\left(-\delta_{1k}\tfrac{\partial}{\partial y_1} + \delta_{2k}\tfrac{\partial}{\partial\xi_2} \right)\tfrac{\partial}{\partial y_1} \right] + \tfrac{\lambda}{2\pi(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_k} \right\} \ln E(y_1,0;\xi) \end{gathered} \tag{73}$$

So, for $P_2^{(1)}(y_1,0;\xi)$ and $P_2^{(2)}(y_1,0;\xi)$ we have the following formulas:

$$\begin{array}{c} P_2^{(1)}(y_1,0;\xi)=\frac{1}{\pi(\lambda+2\mu)}\left[(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial y_1}+\lambda\right]\frac{\partial}{\partial y_1}\ln E(y_1,0;\xi);\\ P_2^{(2)}(y_1,0;\xi)=-\frac{1}{\pi(\lambda+2\mu)}\left[(2\lambda+3\mu)+(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial y_1}\right]\frac{\partial}{\partial \xi_2}\ln E(y_1,0;\xi);\ .\\ E(y_1,0;\xi)=\sqrt{1-2e^{\pi a_2^{-1}(a-\xi_1)}\cos\pi a_2^{-1}\xi_2+e^{2\pi a_2^{-1}(a-\xi_1)}} \end{array} \quad (74)$$

In analogical way we obtain the tractions on the boundary straight line $\Gamma_{21}\equiv(-\infty<y_1<\infty,\ y_2=0)$ in integral formula (66), created by the Green's matrix components $U_i^{(k)}$:

$$\begin{array}{c} P_2^{(k)}(y_1,a_2;\xi)=\sigma_{22}^{(k)}(y_1,a_2;\xi);\\ P_2^{(1)}(y_1,a_2;\xi)=-\frac{1}{\pi(\lambda+2\mu)}\left[(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial y_1}+\lambda\right]\frac{\partial}{\partial y_1}\ln E(y_1,a_2;\xi);\\ P_2^{(2)}(y_1,a_2;\xi)=\frac{1}{\pi(\lambda+2\mu)}\left[(2\lambda+3\mu)+(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial y_1}\right]\frac{\partial}{\partial \xi_2}\ln(y_1,a_2;\xi);\\ E(y_1,a_2;\xi)=\sqrt{1+2e^{\pi a_2^{-1}(a-\xi_1)}\cos\pi a_2^{-1}\xi_2+e^{2\pi a_2^{-1}(a-\xi_1)}} \end{array} \quad (75)$$

We do not discus here the general conditions under which the integrals in Eq. (66) exist. But we can confirm that these integrals exist for many particular given actions. We will restrict here to an example of such kind, it mean to a solution for a particular BVP for an elastic strip.

5. A solution in elementary functions for a particular BVP for an elastic strip

Let we need to solve the particular BVP that consists from Lame's equations (64) (when $F_k(\xi)\equiv 0$) and from boundary following conditions:

$$\begin{array}{c} p_1(y_1,0)=0;\ u_2(y_1,0)=g_2(y_1,0)=\begin{cases} c=\text{const};\ \ -a\le y_1\le a;\\ 0;\ (-\infty<y_1\le -a)\bigcup(a\le y_1<\infty);\end{cases}\\ p_1(y_1,a_2)=0;\ u_2(y_1,a_2)=g_2(y_1,a_2)=0;\ -\infty<y_1<\infty. \end{array} \quad (76)$$

Then we obtain the solution of this BVP in the following Green's integral formula:

$$u_k(\xi)=-\int_{-\infty}^{\infty}g_2(y_1,0)\,P_2^{(k)}(y_1,0;\xi)\,dy_1;\qquad(k=1,2), \quad (77)$$

obtained from Eq. (66) (when $F_k(\xi)\equiv 0$) and boundary conditions (76). Taking into account Eqs. (73), (74) and (76) the integral (77) can be written in the following form:

$$u_k(\xi)=-\int_{-a}^{a}cP_2^{(k)}(y_1,0;\xi)\,dy_1. \quad (78)$$

If we calculate the integrals (77) we obtain the following expressions in elementary functions for the displacements of BVP (64) and (76):

$$u_1(\xi)=-\int_{-a}^{a}cP_2^{(1)}(y_1,0;\xi)\,dy_1=$$

$$-\frac{c}{\pi}\int_{-a}^{a}\frac{1}{(\lambda+2\mu)}\left[(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial y_1}+\lambda\right]\frac{\partial}{\partial y_1}\ln E(y_1,0;\xi)\,dy_1=$$

$$\frac{c}{\pi(\lambda+2\mu)}\left[(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial\xi_1}+\mu\right]\ln E(y_1,0;\xi)\Big|_{y_1=-a}^{y_1=a}=$$
$$\frac{c}{\pi(\lambda+2\mu)}\left[(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial\xi_1}+\mu\right]\ln\frac{E(y_1,a;\xi)}{E(y_1,a;\xi)}=$$
$$\frac{c}{\pi(\lambda+2\mu)}\left[(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial\xi_1}+\mu\right]\ln\sqrt{\frac{1-2e^{\pi a_2^{-1}(a-\xi_1)}\cos\pi a_2^{-1}\xi_2+e^{2\pi a_2^{-1}(a-\xi_1)}}{1-2e^{-\pi a_2^{-1}(a+\xi_1)}\cos\pi a_2^{-1}\xi_2+e^{-2\pi a_2^{-1}(a+\xi_1)}}} \tag{79}$$

for displacements $u_1(\xi)$, and

$$u_2(\xi)=-\int_{-a}^{a}cP_2^{(2)}(y_1,0;\xi)\,dy_1=$$
$$\frac{c}{\pi(\lambda+2\mu)}\int_{-a}^{a}\left[(2\lambda+3\mu)+(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial y_1}\right]\frac{\partial}{\partial\xi_2}\ln E(y_1,0;\xi)\,dy_1=$$
$$\frac{c}{\pi(\lambda+2\mu)}\left[(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial\xi_2}\ln E(y_1,0;\xi)\Big|_{y_1=-a}^{y_1=a}+\right.$$
$$\left.(\lambda+2\mu)\int_{-a}^{a}\frac{\partial}{\partial\xi_2}\ln E(y_1,0;\xi)\,dy_1\right]= \tag{80}$$
$$\frac{c}{\pi(\lambda+2\mu)}\left[(\lambda+\mu)(y_1-\xi_1)\frac{\partial}{\partial\xi_2}\ln E(y_1,0;\xi)+\right.$$
$$\left.(\lambda+2\mu)\,\mathrm{arctg}\left(\frac{e^{\pi a_2^{-1}(a-\xi_1)}-\cos\pi a_2^{-1}\xi_2}{\sin\pi a_2^{-1}\xi_2}\right)\right]_{y_1=-a}^{y_1=a}$$

for displacements $u_2(\xi)$.

So, the final expressions in elementary functions for displacements $u_1(\xi)$ and $u_2(\xi)$ can be written in the following form:

$$u_1(\xi)=\tfrac{c}{\pi(\lambda+2\mu)}\left[(\lambda+\mu)(y_1-\xi_1)\tfrac{\partial}{\partial\xi_1}+\mu\right]\ln\sqrt{\tfrac{1-2e^{\pi a_2^{-1}(a-\xi_1)}\cos\pi a_2^{-1}\xi_2+e^{2\pi a_2^{-1}(a-\xi_1)}}{1-2e^{-\pi a_2^{-1}(a+\xi_1)}\cos\pi a_2^{-1}\xi_2+e^{-2\pi a_2^{-1}(a+\xi_1)}}};$$
$$u_2(\xi)=\tfrac{c}{\pi(\lambda+2\mu)}\left\{(\lambda+\mu)(y_1-\xi_1)\tfrac{\partial}{\partial\xi_2}\ln\sqrt{\tfrac{1-2e^{\pi a_2^{-1}(a-\xi_1)}\cos\pi a_2^{-1}\xi_2+e^{2\pi a_2^{-1}(a-\xi_1)}}{1-2e^{-\pi a_2^{-1}(a+\xi_1)}\cos\pi a_2^{-1}\xi_2+e^{-2\pi a_2^{-1}(a+\xi_1)}}}+\right.$$
$$\left.+(\lambda+2\mu)\left[\mathrm{arctg}\left(\tfrac{e^{\pi a_2^{-1}(a-\xi_1)}-\cos\pi a_2^{-1}\xi_2}{\sin\pi a_2^{-1}\xi_2}\right)-\mathrm{arctg}\left(\tfrac{e^{-\pi a_2^{-1}(a+\xi_1)}-\cos\pi a_2^{-1}\xi_2}{\sin\pi a_2^{-1}\xi_2}\right)\right]\right\}. \tag{81}$$

Our investigations have showed that the obtained in elementary functions solution (81) for the elastic strip satisfy to equations of the BVP (64), (76). Beside, the displacements (81) vanish at infinity, that mean $\lim u_1(\xi)=\lim u_2(\xi)=0$, when $|\xi_1|=\infty$.

6. Conclusions

(1) An exact Green's matrix for a BVP for an elastic strip is obtained in terms of elementary functions
(2) An exact Green's type integral formula for a BVP for an elastic strip is obtained. Kernels in this integral formula are presented in terms of elementary functions. This formula is complete ready for analytical and numerical implementation. To solve BVPs of elasticity for strip, using the obtained Green's type integral formula,

we need to compute some integrals only. A solution in elementary functions for a particular BVP for elastic strip is obtained by computing some integrals.

(3) Derived Green's matrix and Green's type integral formula for a BVP for an elastic strip can be applied in different methods, such as: boundary integral equations method (BIEM), boundary elements method (BEM) etc. Also these results can be applied in elasticity, micromechanics, thermoelasticity, poroelasticity etc to solve BVPs for elastic strip.

(4) These results for elastic strip are obtained using general integral representations for Green's matrices in terms of fundamental F+R solutions for Poisson's equation, suggested before by Victor Şeremet [9]. In this paper needed Green's functions for Poisson's equation for a strip are derived using a special method of separation of variables, presented in [11].

(5) The applied here method can be used for derivation Green's matrices for different canonical Cartesian domains for BVPs different from that presented in [9].

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