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DOUBLE INEQUALITIES OF BOOLE'S QUADRATURE RULE

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Abstract. In this paper double inequalities of Boole's type quadrature rule are given. These inequalities are sharp.

1. Introduction

Recently, a number of authors have considered an error analysis for quadrature rules of Newton-Cotes type ([6], [10], [14]).

N. Ujevic obtained the following double integral inequality (see [14], Theorem 4):

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) and suppose that: $\gamma \le f^{(2)}(x) \le \Gamma$ for all $x \in (a,b)$. Then we have the double inequality:*

$$\frac{3S-\Gamma}{24}(b-a)^2 \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \le \frac{3S-\gamma}{24}(b-a)^2 \tag{1}$$

where $S = \dfrac{f'(b)-f'(a)}{b-a}$.

The same author proved the following double integral inequalities of Simpson type (see [10], Theorem 1):

Theorem 2. *Let $f \in C^4(a,b)$. Then*

$$\begin{aligned}\frac{7\gamma_4-5S_3}{5760}(b-a)^5 &\le \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \int_a^b f(x)dx \\ &\le \frac{7\Gamma_4-5S_3}{5760}(b-a)^5,\end{aligned} \tag{2}$$

where γ_4, Γ_4 are real number such that $\gamma_4 \le f^{(4)}(x) \le \Gamma_4$, $x \in [a,b]$ and

$$S_3 = \frac{f'''(b)-f'''(a)}{b-a}.$$

If γ_4, Γ_4 are given by

$$\Gamma_4 = \min_{x\in[a,b]} f^{(4)}(x), \quad \Gamma_4 = \max_{x\in[a,b]} f^{(4)}(x)$$

then the inequalities (2) are sharp in the usual sense.

In ([6], Theorem 2.1) are given double inequalities of Newton's quadrature rule:

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Theorem 3. *Let $f \in C^4(a,b)$. Then*

$$\begin{aligned} & \frac{23\gamma_4 - 15S_3}{51840}(b-a)^5 \\ & \le \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \int_a^b f(x)dx \\ & \le \frac{23\Gamma_4 - 15S_3}{51840}(b-a)^5, \end{aligned} \tag{3}$$

where $\gamma_4, \Gamma_4 \in \mathbb{R}$, $\gamma_4 \le f^{(4)}(x) \le \Gamma_4$, for any $x \in [a,b]$ and

$$S_3 = \frac{f'''(b) - f'''(a)}{b-a}.$$

If

$$\gamma_4 = \min_{x\in[a,b]} f^{(4)}(x), \quad \Gamma_4 = \max_{x\in[a,b]} f^{(4)}(x)$$

then inequalities (3) are sharp.

We consider Boole's quadrature rule from [8]:

$$\int_a^b f(x)dx = \frac{b-a}{90}\{7[f(a) + f(b)] + 32[f(x_1) + f(x_3)] + 12f(x_2)\} + R[f] \tag{4}$$

where $f : [a,b] \to \mathbb{R}$, $x_1 = \dfrac{3a+b}{4}$, $x_2 = \dfrac{a+b}{2}$, $x_3 = \dfrac{a+3b}{4}$.

If $f \in C^6[a,b]$, the error $R[f]$ from the formula (4) is given by:

$$R[f] = -\int_a^b \varphi(x) f^{(6)}(x)dx$$

where the function φ has the form:

$$\varphi(x) = \begin{cases} \frac{(x-a)^6}{6!} - \frac{7(b-a)}{90}\frac{(x-a)^5}{5!}, & x \in [a, x_1] \\ \frac{(x-a)^6}{6!} - \frac{7(b-a)}{90}\frac{(x-a)^5}{5!} - \frac{32(b-a)}{90}\frac{(x-x_1)^5}{5!}, & x \in [x_1, x_2] \\ \frac{(x-a)^6}{6!} - \frac{7(b-a)}{90}\frac{(x-a)^5}{5!} - \frac{32(b-a)}{90}\frac{(x-x_1)^5}{5!} & \\ \quad - \frac{12(b-a)}{90}\frac{(x-x_2)^5}{5!}, & x \in [x_2, x_3] \\ \frac{(x-a)^6}{6!} - \frac{7(b-a)}{90}\frac{(x-a)^5}{5!} - \frac{32(b-a)}{90}\frac{(x-x_1)^5}{5!} & \\ \quad - \frac{12(b-a)}{90}\frac{(x-x_2)^5}{5!} - \frac{32(b-a)}{90}\frac{(x-x_3)^5}{5!}, & x \in [x_3, b] \end{cases} \tag{5}$$

In this paper we derive double integral inequalities which give upper and lower error bounds for Boole's quadrature formula. Furthermore, it is shown that these error bounds are sharp.

2. Main result

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$, $f \in C^6[a,b]$ and real numbers γ_6, Γ_6 such that $\gamma_6 \le f^{(6)}(x) \le \Gamma_6$, $x \in (a,b)$. Then:*

$$\begin{aligned} & \frac{152\gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7 \\ & \le \frac{b-a}{90}\{7[f(a) + f(b)] + 32[f(x_1) + f(x_3)] + 12f(x_2)\} - \int_a^b f(x)dx \\ & \le \frac{152\Gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7 \end{aligned} \tag{6}$$

where

$$S_5 = \frac{f^{(5)}(b) - f^{(5)}(a)}{b-a}.$$

Moreover

$$\gamma_6 = \min_{x\in[a,b]} f^{(6)}(x), \quad \Gamma_6 = \max_{x\in[a,b]} f^{(6)}(x)$$

the inequalities (5) are sharp.

Proof. Integrating by parts, we have:

$$\int_a^b \varphi(x) f^{(6)}(x)dx \tag{7}$$

$$= \int_a^b f(x)dx - \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\}.$$

It is easy to see [8] that we get the equality:

$$\int_a^b \varphi(x)dx = -\frac{8}{945}\left(\frac{b-a}{4}\right)^7. \tag{8}$$

From (7) and (8) we get the equalities:

$$\int_a^b [f^{(6)}(x) - \gamma_6]\varphi(x)dx = \int_a^b f(x)dx \tag{9}$$

$$-\frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} + \frac{8\gamma_6}{945}\left(\frac{b-a}{4}\right)^7$$

and

$$\int_a^b [\Gamma_6 - f^{(6)}(x)]\varphi(x)dx = -\int_a^b f(x)dx \tag{10}$$

$$+\frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} - \frac{8\Gamma_6}{945}\left(\frac{b-a}{4}\right)^7.$$

On the other hand:

$$\int_a^b [f^{(6)}(x) - \gamma_6]\varphi(x)dx \le \max_{x\in[a,b]} |\varphi(x)| \int_a^b |f^{(6)}(x) - \gamma_6|dx \tag{11}$$

and

$$\max_{x\in[a,b]} |\varphi(x)| = \frac{(b-a)^6}{691200} \tag{12}$$

$$\begin{aligned}\int_a^b |f^{(6)}(x) - \gamma_6|dx &= \int_a^b (f^{(6)}(x) - \gamma_6)dx \\ &= f^{(5)}(b) - f^{(5)}(a) - \gamma_6(b-a) \\ &= (S_5 - \gamma_6)(b-a)\end{aligned} \tag{13}$$

From the relations (9), (11), (12) and (13) it follows:

$$\int_a^b f(x)dx - \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} \tag{14}$$

$$\le \frac{(b-a)^7}{691200}(S_5 - \gamma_6) - \frac{8\gamma_6}{945}\left(\frac{b-a}{4}\right)^7 = \frac{112S_5 - 152\gamma_6}{4725}\left(\frac{b-a}{4}\right)^7.$$

On the other hand:

$$\int_a^b [\Gamma_6 - f^{(6)}(x)]\varphi(x)dx = \max_{x\in[a,b]} |\varphi(x)| \int_a^b |\Gamma_6 - f^{(6)}(x)|dx \tag{15}$$

and

$$\begin{aligned}\int_a^b |\Gamma_6 - f^{(6)}(x)|dx &= \int_a^b (\Gamma_6 - f^{(6)}(x))dx \\ &= \Gamma_6(b-a) - f^{(5)}(b) + f^{(5)}(b) \\ &= (\Gamma_6 - S_5)(b-a)\end{aligned} \tag{16}$$

From (10), (12), (15) and (16) we get:

$$\begin{aligned}&-\int_a^b f(x) + \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} \\ &\le \frac{(b-a)^7}{691200}(\Gamma_6 - S_5) + \frac{8\Gamma_6}{945}\left(\frac{b-a}{4}\right)^7 \\ &= \frac{152\Gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7.\end{aligned} \tag{17}$$

From (14) and (17) it follows (5).

We now show that inequalities given by (5) are sharp. For that purpose, we define the function

$$f(x) = (x-a)^6.$$

It is easy to see that the equalities $f^{(6)}(x) = 720$ and $\gamma_6 = \Gamma_6 = 720$,

$$S_5 = \frac{f^{(5)}(b) - f^{(5)}(a)}{b-a} = 720$$

are obtained. Calculating the three members of the inequality (7) we notice that these have the common value given by the expression $\frac{128}{21}\left(\frac{b-a}{4}\right)^7$. Hence, we deduce that the inequality (5) is sharp. The complete proof. □

Theorem 5. *Under the assumptions of Theorem 1 we have:*

$$\begin{aligned}&\frac{72\gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7 \\ &\le \int_a^b f(x)dx - \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} \\ &\le \frac{72\Gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7.\end{aligned} \tag{18}$$

If

$$\gamma_6 = \min_{x\in[a,b]} f^{(6)}(x), \quad \Gamma_6 = \max_{x\in[a,b]} f^{(6)}(x)$$

then the inequalities (18) are sharp.

Proof. From (9), (11) and (12) we have:

$$-\int_a^b f(x)dx + \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} \tag{19}$$

$$= -\int_a^b [f^{(6)}(x) - \gamma_6]\varphi(x)dx + \frac{8\gamma_6}{945}\left(\frac{b-a}{4}\right)^7$$

$$\int_a^b [\gamma_6 - f^{(6)}(x)]\varphi(x)dx \leq \max_{x\in[a,b]} |\varphi(x)| \int_a^b |\gamma_6 - f^{(6)}(x)|dx$$

$$= \frac{(b-a)^6}{691200}\int_a^b -(\gamma_6 - f^{(6)}(x))dx = -\frac{(b-a)^6}{691260}[\gamma_6(b-a) - (f^{(5)}(b) - f^{(5)}(a))]$$

$$= \frac{(b-a)^6}{691200}[f^{(5)}(b) - f^{(5)}(a) - \gamma_6(b-a)] = \frac{(b-a)^7}{691200}(S_5 - \gamma_6)$$

$$-\int_a^b f(x)dx + \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\}$$

$$\leq \frac{8\gamma_6}{945}\left(\frac{b-a}{4}\right)^7 + \frac{(b-a)^7}{691200}(S_5 - \gamma_6) = \frac{40\gamma_6 + 112S_5 - 112\gamma_6}{4725}\left(\frac{b-a}{4}\right)^7$$

$$= \frac{112S_5 - 72\gamma_6}{4725}\left(\frac{b-a}{4}\right)^7$$

$$\int_a^b f(x)dx - \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} \tag{20}$$

$$\geq \frac{72\gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7.$$

By analog from (10), (12) and (15) we have:

$$-\int_a^b [\Gamma_6 - f^{(6)}(x)]\varphi(x)dx = \int_a^b f(x)dx - \frac{b-a}{90}\{7[f(a)+f(b)]$$

$$+32[f(x_1)+f(x_3)] + 12f(x_2)\} + \frac{8\Gamma_6}{945}\left(\frac{b-a}{4}\right)^7$$

$$-\int_a^b [\Gamma_6 - f^{(6)}(x)]\varphi(x)dx = \int_a^b [f^{(6)}(x) - \Gamma_6]\varphi(x)dx$$

$$\leq \max_{x\in[a,b]} |\varphi(x)| \int_a^b |f^{(6)}(x) - \Gamma_6|dx$$

$$= \frac{(b-a)^6}{691200}\int_a^b [\Gamma_6 - f^{(6)}(x)]dx = \frac{(b-a)^7}{691200}(\Gamma_6 - S_5)$$

$$\int_a^b f(x)dx - \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} \tag{21}$$

$$\leq \frac{(b-a)^7}{691200}(\Gamma_6 - S_5) - \frac{8\Gamma_6}{945}\left(\frac{b-a}{4}\right)^7 = \frac{112\Gamma_6 - 112S_6 - 40\Gamma_6}{4725}\left(\frac{b-a}{4}\right)^7$$

$$= \frac{72\Gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7.$$

From (20) and (21) we will have immediately the inequalities:

$$\frac{72\gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7 \leq -\frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\}$$

$$+\int_a^b f(x)dx \le \frac{72\Gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7.$$

A proof of sharpness is similar to the proof of sharpness in Theorem 4 (we use the same function $f(x) = (x-a)^6$). □

The next theorem offers us inequalities which do not depend upon S_5.

Theorem 6. *Under the assumptions of Theorem 4 we have:*

$$\frac{76\gamma_6 - 36\Gamma_6}{4725}\left(\frac{b-a}{4}\right)^7 \le \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} \tag{22}$$

$$-\int_a^b f(x)dx \le \frac{76\Gamma_6 - 36\gamma_6}{4725}\left(\frac{b-a}{4}\right)^7.$$

If

$$\gamma_6 = \min_{x\in[a,b]} f^{(6)}(x), \quad \Gamma_6 = \max_{x\in[a,b]} f^{(6)}(x)$$

then the inequalities (24) are sharp.

Proof. We have:

$$\frac{152\gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7 \le \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} \tag{23}$$

$$-\int_a^b f(x)dx \le \frac{152\Gamma_6 - 112S_5}{4725}\left(\frac{b-a}{4}\right)^7.$$

If we multiply (18) by -1 we get:

$$\frac{-72\Gamma_6 + 112S_5}{4725}\left(\frac{b-a}{4}\right)^7 \tag{24}$$

$$\le \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\} - \int_a^b f(x)dx$$

$$\le \frac{-72\gamma_6 + 112S_5}{4725}\left(\frac{b-a}{4}\right)^7.$$

From (23) and (24) we see that:

$$\frac{152\gamma_6 - 72\Gamma_6}{4725}\left(\frac{b-a}{4}\right)^7 \tag{25}$$

$$\le 2\left\{\frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\}\right.$$

$$\left.-\int_a^b f(x)dx \le \frac{152\Gamma_6 - 72\gamma_6}{4725}\left(\frac{b-a}{4}\right)^7\right\} \Big| \cdot \frac{1}{2}$$

$$\frac{76\gamma_6 - 36\Gamma_6}{4725}\left(\frac{b-a}{4}\right)^7 \le \frac{b-a}{90}\{7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] + 12f(x_2)\}$$

$$-\int_a^b f(x)dx \le \frac{76\Gamma_6 - 36\gamma_6}{4725}\left(\frac{b-a}{4}\right)^7.$$

We again use the function $f(x) = (x-a)^6$ to prove that (22) are sharp. □

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