TJMM
2 (2010), No. 1, 89-95

THE SOLUTION OF A SYSTEM OF NONLINEAR INTEGRAL EQUATIONS FROM PHYSICS

MARIA DOBRIŢOIU

Abstract. Using the Perov's Theorem and the General data dependence Theorem, in this paper, we obtain some conditions concerning the existence and uniqueness of the solution in the Banach space $C([a,b],\mathbb{R}^m)$ and the continuous data dependence of the solution of the following system of nonlinear integral equations from physics:

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t), \ \ t \in [a,b].$$

An example is also given here.

1. Introduction

A nonlinear integral equation with modified argument of the form

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t), \quad t \in [a,b],$$

where $K : [a,b] \times [a,b] \times \mathbb{R}^3 \to \mathbb{R}$, $f : [a,b] \to \mathbb{R}$, appeared in the '70 in some studies of a problem from turbo-reactors industry.

In this paper we consider a system of nonlinear integral equations with modified argument of this type:

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t), \tag{1}$$

where $t \in [a,b]$, $K \in C\left([a,b] \times [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m\right)$, $f \in C([a,b],\mathbb{R}^m)$.

Some properties of the solution of this type of nonlinear integral equations (existence and uniqueness, continuous dependence, differentiability with respect to a parameter, integral inequalities, approximating the solution) have been studied by author in [1], [4], [5], [6], [7], [8], [9] and [10]. Also, we mention that the solutions of other nonlinear integral equations, as a part of the theory of integral equations, have been studied by many authors (see [2], [3], [11], [12], [14], [15], [19], [20]) .

We use the Perov's Theorem and the General data dependence Theorem to study the existence and uniqueness of the solution of the system of integral equations (1), in the Banach space $C([a,b],R^m)$ and also, to study the data dependence of the solution.

In this paper we use the notations and several basic results from [11], [12], [13], [16] and [17], which we present some of them, below.

Let X be a nonempty set, d a metric on X, (X,d) a metric space and $A : X \to X$ an operator.

We denote by

$$F_A := \{x \in X |\ A(x) = x\},$$

2010 *Mathematics Subject Classification.* 45G10, 47H10.
Key words and phrases. System of nonlinear integral equations, modified argument, generalized contraction, Perov's Theorem, data dependence.

the fixed point set of the operator A and by

$$A^0 := 1_X, \quad A^1 := A, \quad A^{n+1} := A^n \circ A, \quad n \in \mathbb{N},$$

the iterate operators of A. Also, we will use the Banach space:

$$C([a,b],\mathbb{R}^m) = \{f : [a,b] \longrightarrow \mathbb{R}^m | f - continuous\ function\},$$

endowed with the generalized Chebyshev norm on $C([a,b],\mathbb{R}^m)$, defined by the relation:

$$\|x\| := \begin{pmatrix} \|x_1\|_C \\ \dots \\ \|x_m\|_C \end{pmatrix}, \tag{2}$$

where

$$\|x_k\|_C := \max_{t\in[a,b]} |x_k(t)| \,. \tag{3}$$

In order to study the existence and uniqueness of the solution of the system of integral equations (1) we need the Perov's Theorem.

Theorem 1. *(Perov). Let (X,d), with $d(x,y) \in \mathbb{R}^m$ be a complete generalized metric space and $A : X \longrightarrow X$ an operator. We suppose that there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$ such that*

(i) $d(A(x),A(y)) \le Qd(x,y)$, for all x, $y \in X$;
(ii) $Q^n \longrightarrow 0$ as $n \longrightarrow \infty$.

Then

(a) $F_A = \{x^*\}$;
(b) $A^n(x) \longrightarrow x^*$ as $n \longrightarrow \infty$ and

$$d(A^n(x),x^*) \le Q^n(I-Q)^{-1}d(x_0,A(x_0)).$$

Definition 1. *(Rus [16]) A matrix $Q \in M_{mm}(\mathbb{R})$ converges to zero if Q^k converges to the zero matrix as $k \longrightarrow \infty$.*

The following theorem has two conditions which are equivalents with the convergence to zero of a matrix $Q \in M_{mm}(\mathbb{R}_+)$. This theorem is useful in the example presented in the last section.

Theorem 2. *(see [16]) Let $Q \in M_{mm}(\mathbb{R}_+)$ be a matrix. The following conditions are equivalents:*

(i) $Q^k \to \infty$ as $k \to \infty$;
(ii) The eigenvalues λ_k , $k = \overline{1,m}$ of the matrix Q, satisfies the condition $|\lambda_k| < 1$, $k = \overline{1,m}$;
(iii) The matrix $I - Q$ is non-singular and

$$(I-Q)^{-1} = I + Q + ... + Q^m + ...$$

In order to study the data dependence of the solution of system (1), the General data dependence Theorem (see [13] and [16]) is useful.

Theorem 3. *(I. A. Rus, A Petruşel, G. Petruşel [17]) (General data dependence theorem) Let (X,d) be a complete generalized metric space and $A,B : X \longrightarrow X$ two operators. We suppose that:*

(i) A is a Q-contraction (Q converges to zero) and $F_A = \{x^*\}$;
(ii) $x_B^* \in F_B$;

(iii) there exists $\eta \in \mathbb{R}^m_+$ *such that*

$$d(A(x), B(x)) < \eta$$

for all $x \in X$.

In these conditions we have

$$d(x^*_A, x^*_B) \leq (I - Q)^{-1}\eta\,.$$

2. Existence and uniqueness in the space $C([a,b],\mathbb{R}^m)$

Using the Perov's Theorem in the study of existence and uniqueness of the solution of the system of integral equations (1) in the Banach space $C([a,b],\mathbb{R}^m)$, the following result was obtained:

Theorem 4. *We suppose that:*

(i) $K \in C\left([a,b]\times[a,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m\right)$;

(ii) $f \in C\left([a,b],\mathbb{R}^m\right)$;

(iii) there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$ *such that*

$$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3) - K_1(t,s,v_1,v_2,v_3)| \\ \dots\dots\dots \\ |K_m(t,s,u_1,u_2,u_3) - K_m(t,s,v_1,v_2,v_3)| \end{pmatrix} \leq$$

$$\leq Q \begin{pmatrix} |u_{11}-v_{11}| + |u_{21}-v_{21}| + |u_{31}-v_{31}| \\ \dots\dots\dots \\ |u_{1m}-v_{1m}| + |u_{2m}-v_{2m}| + |u_{3m}-v_{3m}| \end{pmatrix},$$

for all $t,s \in [a,b]$, $u_i, v_i \in \mathbb{R}^m$, $i = \overline{1,3}$.

(iv) $[3(b-a)Q]^n \longrightarrow 0$, *as* $n \longrightarrow \infty$.

Then the system of integral equations (1) has a unique solution $x^* \in C\left([a,b],\mathbb{R}^m\right)$. *This solution can be obtained by the successive approximations method, starting at any element* $x_0 \in C\left([a,b],\mathbb{R}^m\right)$. *Moreover, if* x_n *is the* n^{th} *successive approximation, then we have the following estimation:*

$$\|x^* - x_n\|_{\mathbb{R}^m} \leq [3(b-a)Q]^n \cdot [I - 3(b-a)Q]^{-1} \cdot \|x_0 - x_1\|_{\mathbb{R}^m}\,. \tag{4}$$

Proof. We consider the operator $A : C([a,b],\mathbb{R}^m) \longrightarrow C([a,b],\mathbb{R}^m)$, defined by the relation:

$$A(x)(t) := \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t),\;\; t \in [a,b]\,. \tag{5}$$

From the conditions (i) and (ii) it results that the operator A is well defined.

The set of the solutions of the system of integral equations (1) coincides with the set of fixed points of the operator A, i.e. with F_A. Therefore, our study is reduced to a fixed point problem.

From the condition (iii) it results that the function K satisfies a Lipschitz condition with respect to the last three arguments, with a matrix $Q \in M_{mm}(\mathbb{R}_+)$.

We have:

$$|A(x)(t) - A(y)(t)| = \begin{pmatrix} |A_1(x)(t) - A_1(y)(t)| \\ \dots \\ |A_m(x)(t) - A_m(y)(t)| \end{pmatrix} =$$

$$= \begin{pmatrix} \left|\int_a^b [K_1(t,s,x(s),x(a),x(b)) - K_1(t,s,y(s),y(a),y(b))]\,ds\right| \\ \dots \\ \left|\int_a^b [K_m(t,s,x(s),x(a),x(b)) - K_m(t,s,y(s),y(a),y(b))]\,ds\right| \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} \int_a^b |K_1(t,s,x(s),x(a),x(b)) - K_1(t,s,y(s),y(a),y(b))|\, ds \\ \dots \\ \int_a^b |K_m(t,s,x(s),x(a),x(b)) - K_m(t,s,y(s),y(a),y(b))|\, ds \end{pmatrix}.$$

Using the condition (iii) and the generalized Chebyshev norm on $C([a,b],\mathbb{R}^m)$, defined by the relations (2) and (3), we obtain

$$\|A(x) - A(y)\|_{\mathbb{R}^m} \leq 3(b-a)Q \cdot \|x-y\|_{\mathbb{R}^m} \ ,$$

and by the condition (iv) it results that the operator A is a contraction with the matrix $3(b-a)Q$, $Q \in M_{mm}(\mathbb{R}_+)$.

Now, by the conditions $(i)-(iv)$ and using the Perov's Theorem, it results that the system of nonlinear integral equations with modified argument (1) has a unique solution $x^* \in C([a,b],\mathbb{R}^m)$, and the proof is complete. □

3. Data dependence

In what follows, we will study the continuous dependence of the solution of the system of integral equations (1), with respect to K and f.

Now, we consider the perturbed system of integral equations

$$y(t) = \int_a^b H(t,s,y(s),y(a),y(b))ds + h(t),\ \ t \in [a,b]\,, \tag{6}$$

where $H \in C\left([a,b]\times[a,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m\right)$, $h \in C([a,b],\mathbb{R}^m)$.

We have the following data dependence theorem:

Theorem 5. *We suppose that:*

(i) the conditions of the Theorem 4 are satisfied and we denote by x^* the unique solution of the system of integral equations (1), in the Banach space $C([a,b],\mathbb{R}^m)$;

(ii) there exists $T_1, T_2 \in M_{m1}(\mathbb{R}_+)$ such that

$$\|K(t,s,u_1,u_2,u_3) - H(t,s,u_1,u_2,u_3)\|_{\mathbb{R}^m} \leq T_1,$$

for all $t,s \in [a,b]$, $u_i \in \mathbb{R}^m$, $i=\overline{1,3}$, and

$$|f(t) - h(t)| \leq T_2\,,$$

for all $t \in [a,b]$.

In these conditions, if $y^* \in C([a,b],\mathbb{R}^m)$ is a solution of the perturbed system of integral equations (6), then we have the following estimation:

$$\|x^* - y^*\|_{\mathbb{R}^m} \leq [I - 3(b-a)Q]^{-1} \cdot [(b-a)T_1 + T_2]\,. \tag{7}$$

Proof. We consider the operator $A : C([a,b],\mathbb{R}^m) \longrightarrow C([a,b],\mathbb{R}^m)$ attached to the system (1) and defined by the relation (5).

Let $B : C([a,b],\mathbb{R}^m) \longrightarrow C([a,b],\mathbb{R}^m)$ be an operator attached to the perturbed system (6) and defined by the relation:

$$B(y)(t) := \int_a^b H(t,s,y(s),y(a),y(b))ds + h(t),\ \ t \in [a,b]\ \ . \tag{8}$$

We have

$$|A(x)(t) - B(x)(t)| = |\int_a^b [K(t,s,x(s),x(a),x(b)) -$$
$$-H(t,s,x(s),x(a),x(b))]ds + [f(t) - h(t)]|\ \leq$$
$$\leq \int_a^b |K(t,s,x(s),x(a),x(b)) - H(t,s,x(s),x(a),x(b))|ds +$$

$$+|f(t) - h(t)|.$$

Using the condition (ii) and the generalized Chebyshev norm on $C([a,b],\mathbb{R}^m)$, defined by the relations (2) and (3), we obtain the estimation:

$$\|A(x) - B(x)\|_{\mathbb{R}^m} \le T_1(b-a) + T_2.$$

Now, the conditions of the *General data dependence Theorem* being satisfied, it results the estimation (7) and the proof is complete. □

4. Example

In what follows, we consider the system of integral equations with modified argument

$$\begin{cases} x_1(t) = \int_0^1 \left[\frac{2t+1}{15}x_1(s) + \frac{t}{5}x_1(0) + \frac{t}{5}x_1(1)\right] ds + 2t + 1 \\ x_2(t) = \int_0^1 \left[\frac{2t+1}{21}x_2(s) + \frac{t}{7}x_2(0) + \frac{t}{7}x_2(1)\right] ds + \sin t \end{cases}, \quad t \in [0,1] \tag{9}$$

where $K \in C\left([0,1]\times[0,1]\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R}^2\right)$,
$K(t,s,u_1,u_2,u_3) = (K_1(t,s,u_1,u_2,u_3), K_2(t,s,u_1,u_2,u_3))$,
$K_1(t,s,u_1,u_2,u_3) = \frac{2t+1}{15}u_{11} + \frac{t}{5}u_{21} + \frac{t}{5}u_{31}$,
$K_2(t,s,u_1,u_2,u_3) = \frac{2t+1}{21}u_{12} + \frac{t}{7}u_{22} + \frac{t}{7}u_{32}$, $t,s \in [0,1]$, $u_1,u_2,u_3 \in \mathbb{R}^2$,
$u_1 = (u_{11},u_{12})$, $u_2 = (u_{21},u_{22})$, $u_3 = (u_{31},u_{32})$,
$f \in C([0,1],\mathbb{R}^2)$, $f(t) = (f_1(t), f_2(t))$, $f_1(t) = 2t+1$, $f_2(t) = \sin t$,
and $x \in C([0,1],\mathbb{R}^2)$, $x(t) = (x_1(t), x_2(t))$.

The operator $A : C([0,1],\mathbb{R}^2) \longrightarrow C([0,1],\mathbb{R}^2)$, $A = (A_1, A_2)$, attached to the system (9), defined by the relation:

$$\begin{cases} A_1(x)(t) := \int_0^1 \left[\frac{2t+1}{15}x_1(s) + \frac{t}{5}x_1(0) + \frac{t}{5}x_1(1)\right] ds + 2t + 1 \\ A_2(x)(t) := \int_0^1 \left[\frac{2t+1}{21}x_2(s) + \frac{t}{7}x_2(0) + \frac{t}{7}x_2(1)\right] ds + \sin t \end{cases}$$

satisfies the Lipschitz condition with the matrix $Q = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/7 \end{pmatrix}$.

From the Theorem 2 it results that the matrix $3(b-a)Q = 3Q = \begin{pmatrix} 3/5 & 0 \\ 0 & 3/7 \end{pmatrix}$ converges to zero and therefore, the operator A is a contraction with this matrix.

The conditions of the Theorem 4 being satisfied, it results that the system of integral equations (9) has a unique solution $x^* \in C([0,1],\mathbb{R}^2)$.

In order to study the data dependence of the solution of the system of integral equations (9), we consider the following perturbed system of integral equations:

$$\begin{cases} y_1(t) = \int_0^1 \left[\frac{s+3}{15}y_1(s) + \frac{t}{5}y_1(0) + \frac{t}{5}y_1(1) - 3t\right] ds + t \\ y_2(t) = \int_0^1 \left[\frac{s+3}{21}y_2(s) + \frac{t}{7}y_2(0) + \frac{t}{7}y_2(1) - t\right] ds + \cos t \end{cases}, \quad t \in [0,1], \tag{10}$$

where $H \in C\left([0,1]\times[0,1]\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R}^2\right)$,
$H(t,s,v_1,v_2,v_3) = (H_1(t,s,v_1,v_2,v_3), H_2(t,s,v_1,v_2,v_3))$,
$H_1(t,s,v_1,v_2,v_3) = \frac{s+3}{15}v_{11} + \frac{t}{5}v_{21} + \frac{t}{5}v_{31} - 3t$,
$H_2(t,s,v_1,v_2,v_3) = \frac{s+3}{21}v_{12} + \frac{t}{7}v_{22} + \frac{t}{7}v_{32} - t$, $t,s \in [0,1]$, $v_1,v_2,v_3 \in \mathbb{R}^2$,
$v_1 = (v_{11},v_{12})$, $v_2 = (v_{21},v_{22})$, $v_3 = (v_{31},v_{32})$,
$h \in C([0,1],\mathbb{R}^2)$, $h(t) = (h_1(t), h_2(t))$, $h_1(t) = t$, $h_2(t) = \cos t$,
and $y \in C([0,1],\mathbb{R}^2)$, $y(t) = (y_1(t), y_2(t))$.

We attach to the system (10), the operator $B : C([0,1],\mathbb{R}^2) \longrightarrow C([0,1],\mathbb{R}^2)$, $B = (B_1, B_2)$, defined by the relation:

$$\begin{cases} B_1(y)(t) := \int_0^1 \left[\frac{s+3}{15}y_1(s) + \frac{t}{5}y_1(0) + \frac{t}{5}y_1(1) - 3t\right] ds + t \\ B_2(y)(t) := \int_0^1 \left[\frac{s+3}{21}y_2(s) + \frac{t}{7}y_2(0) + \frac{t}{7}y_2(1) - t\right] ds + \cos t \end{cases}.$$

Now, we have the following estimations:

$$\|K(t,s,x(s),x(0),x(1)) - H(t,s,x(s),x(0),x(1))\|_{\mathbb{R}^2} \leq \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

for all $t, s \in [0,1]$, and

$$\|f(t) - h(t)\|_{\mathbb{R}^2} \leq \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad for \;\; all \;\; t \in [0,1],$$

and therefore, we have

$$\|A(x) - B(x)\|_{\mathbb{R}^2} \leq \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

In these conditions, if $y^* \in C([0,1], \mathbb{R}^2)$ is a solution of the perturbed system of integral equations (10), then by the Theorem 5 it results the following estimation:

$$\|x^* - y^*\|_{\mathbb{R}^2} \leq \begin{pmatrix} 2/5 & 0 \\ 0 & 4/7 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 25/2 \\ 7/2 \end{pmatrix}.$$

References

[1] Ambro, M., Aproximarea soluţiilor unei ecuaţii integrale cu argument modificat, *Studia Univ. Babeş-Bolyai*, Cluj-Napoca, Mathematica, **2**(1978), 26-32.

[2] András, Sz., *Ecuaţii integrale Fredholm–Volterra*, Editura Didactică şi Pedagogică, Bucureşti, 2005.

[3] Berinde, V., *Generalized contractions and applications*, Cub Press 22, Baia Mare, 1997.

[4] Dobriţoiu, M., An integral equation with modified argument, *Studia Univ. Babeş-Bolyai*, Cluj-Napoca, Mathematica, vol. XLIX, nr. 3/2004, 27-33.

[5] Dobriţoiu, M., Existence and continuous dependence on data of the solution of an integral equation, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM – CVI / 2005.

[6] Dobriţoiu, M., Analysis of an integral equation with modified argument*, Studia Univ. Babeş-Bolyai*, Cluj-Napoca, Mathematica, vol. 51, nr. 1/2006, 81-94.

[7] Dobriţoiu, M., System of integral equations with modified argument, *Carpathian Journal of Mathematics*, Baia Mare, vol. 24 (2008), No. 2, 26-36.

[8] Dobriţoiu, M., Analysis of a Nonlinear Integral Equation with Modified Argument from Physics, *International Journal of Mathematical Models and Methods in Applied Sciences (NAUN Electronic Journal)*, Issue 1, Vol. 2, 2008, http://www.naun.org/journals/m3as/, 403–412.

[9] Dobriţoiu, M., Gronwall Type Integral Inequalities via Picard Operators, Proceedings of the 5th International Conference "Dynamical Systems and Applications", *Ovidius University Annals Series: Civil Engineering*, Vol. 1, Special Issue 11, June 2009, Ovidius University Press, 291-298.

[10] Dobriţoiu, M., *Ecuaţii integrale cu argument modificat*, Presa Universitară Clujeană, Cluj-Napoca, 2009.

[11] Guo, D., Lakshmikantham, V., Liu, X., *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1996.

[12] Precup, R., *Methods in nonlinear integral equation*, Kluwer Academic Publisher, 2002.

[13] Rus, I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979.

[14] Rus, I. A., Generalized contractions, Univ. of Cluj-Napoca, *Preprint* Nr. 3, 1983, 1-130.

[15] Rus, I. A., Picard operators and applications, Babeş-Bolyai University of Cluj-Napoca, *Preprint* Nr. 3, 1996.

[16] Rus, I. A., *Generalized contractions and applications*, Cluj University Press, Cluj–Napoca, 2001.

[17] Rus, I.A., Petruşel, A., Petruşel, G., *Fixed Point Theory*, Cluj University Press, 2008.

[18] Sotomayor, J., Smooth dependence of solution of differential equation on initial data: a simple proof, *Bol. Soc. Brasil*, **4**, 1(1973), 55-59.

[19] Şerban, M.A., Existence and uniqueness theorems for the Chandrasekhar's equation, Acadmie Roumaine Filiale de Cluj-Napoca, *Mathematica*, Tome 41(64), Nr. 1, 1999, 91-103.

[20] Şerban, M.A., *Teoria punctului fix pentru operatori definiţi pe produs cartezian*, Presa Universitară Clujeană, Cluj-Napoca, 2002.

University of Petroşani
Department of Mathematics
Universităţii 20, 332006, Petroşani, Romania
E-mail address: mariadobritoiu@yahoo.com