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A NONLINEAR MIXED TYPE VOLTERRA-FREDHOLM FUNCTIONAL INTEGRAL EQUATION VIA PEROV'S THEOREM

MARCEL-ADRIAN ŞERBAN

ABSTRACT. In this paper we study the following mixed type Volterra-Fredholm functional integral equation

$$x\left(t\right)=F\left(t,x(t),\int_{a_1}^{t_1}\ldots\int_{a_m}^{t_m}K(t,s,x(s))ds,\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}H(t,s,x(s))ds\right).$$

Using the Perov's Theorem and the Picard operator technique we establish existence, uniqueness, data dependence and Gronwall results for the solutions.

1. Introduction

The theory of integral equations is an important chapter of nonlinear analysis and the most used tool for proving the existence of the solution is the fixed point technique (see [2], [3], [7], [8], [14], [16], etc.)

In this paper we consider the following mixed type Volterra-Fredholm functional nonlinear integral equation:

$$x\left(t\right)=F\left(t,x(t),\int_{a_1}^{t_1}\ldots\int_{a_m}^{t_m}K(t,s,x(s))ds,\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}H(t,s,x(s))ds\right),\qquad(1)$$

where $[a_1;b_1]\times\ldots\times[a_m;b_m]$ be an interval in $\mathbb{R}^m$, $K,H:[a_1;b_1]\times\ldots\times[a_m;b_m]\times[a_1;b_1]\times\ldots\times[a_m;b_m]\times\mathbb{B}\to\mathbb{B}$ continuous functions and $F:[a_1;b_1]\times\ldots\times[a_m;b_m]\times\mathbb{B}^3\to\mathbb{B}$, where $(\mathbb{B},|\cdot|)$ a Banach space. The mixed type Volterra-Fredholm integral equations have been studied by many authors (see [1], [4], [13], [15] [25], etc.).

We apply the technique from Sz. András [1] by using the Perov's Theorem and Picard operators technique to prove the existence and the uniqueness, data dependence and comparison results for the solutions of (1). We use the terminologies and notations from [17], [21] and [23]. For the convenience of the reader we recall some of them.

Let (X,d) be a metric space and $A:X\to X$ an operator. We denote by $A^0:=1_X$, $A^1:=A$, $A^{n+1}:=A^n\circ A$, $n\in\mathbb{N}$ the iterate operators of the operator A. We also have:

$$P\left(X\right):=\{Y\subseteq X\mid Y\neq\emptyset\}$$

$$F_A:=\{x\in X\mid A(x)=x\}$$

$$I\left(A\right):=\{Y\in P\left(X\right)\mid A\left(Y\right)\subseteq Y\}$$

Definition 1. *$A:X\to X$ is called a Picard operator (briefly PO) if:*

(i) $F_A=\{x^*\}$;

(ii) $A^n(x)\to x^*$ *as* $n\to\infty$, *for all* $x\in X$.

The operator A is Picard if and only if the discrete dynamical system generated by A has an equilibrium state which is globally asymptotically stable.

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Definition 2. *$A : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.*

If $A : X \to X$ is a WPO, then we may define the operator $A^\infty : X \to X$ by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Obviously, $A^\infty(X) = F_A$. Moreover, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

Also, in this paper we study the following integral inequalities

$$x(t) \le F\left(t, x(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, x(s))ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, x(s))ds\right) \tag{2}$$

$$x(t) \ge F\left(t, x(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, x(s))ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, x(s))ds\right) \tag{3}$$

using the Picard operators technique and Abstract Gronwall Lemma (I.A. Rus [20], [17]).

2. Existence and uniqueness

We prove the existence and uniqueness for the solution of integral equation (1) using the Perov's Theorem as in Sz. András [1], for standard techniques, when it is used the Banach contraction principle see [4], [5], [6], [10].

Theorem 1. *(Perov). Let (X, d), with $d(x, y) \in \mathbb{R}^m$ be a complete generalized metric space and $A : X \longrightarrow X$ an operator. We suppose that there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$ such that*

(i) $d(A(x), A(y)) \le Qd(x, y)$, for all $x, y \in X$;
(ii) $Q^n \longrightarrow 0$ as $n \longrightarrow \infty$.

Then

(a) $F_A = \{x^*\}$;
(b) $A^n(x) \longrightarrow x^*$ as $n \longrightarrow \infty$ and

$$d(A^n(x), x^*) \le Q^n(I - Q)^{-1}d(x_0, A(x_0)).$$

We have the following result:

Theorem 2. *We assume that:*

(i) $K, H \in C([a_1, b_1] \times \dots \times [a_m, b_m] \times [a_1, b_1] \times \dots \times [a_m, b_m] \times \mathbb{B}, \mathbb{B})$;
(ii) $F \in C([a_1, b_1] \times \dots \times [a_m, b_m] \times \mathbb{B}^3, \mathbb{B})$;
(iii) there exist α, β, γ nonnegative constants such that:

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \le \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a_1, b_1] \times \dots \times [a_m, b_m], u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{B}$;
(iv) there exist L_K and L_H nonnegative constants such that:

$$|K(t, s, u) - K(t, s, v)| \le L_K|u - v|,$$

$$|H(t, s, u) - H(t, s, v)| \le L_H|u - v|,$$

for all $t, s \in [a_1, b_1] \times \dots \times [a_m, b_m]$, $u, v \in \mathbb{B}$;

(v) there exists $\tau > 0$ such that the matrix

$$Q = \begin{pmatrix} \frac{\beta L_K}{\tau^m} & \alpha + \gamma L_H P_1 \\ \frac{\beta L_K}{\tau^m} P_2(\tau) & \alpha + \gamma L_H P_1 \end{pmatrix} \tag{4}$$

is convergent to 0, i.e $Q^n \longrightarrow 0$ as $n \longrightarrow \infty$, where

$$\begin{aligned} P_2(\tau) &= \prod_{i=1}^{m} \left(e^{\tau(b_i - a_i)} - 1 \right), \\ P_1 &= \prod_{i=1}^{m} (b_i - a_i). \end{aligned}$$

Then, the equation (1) has a unique solution $x^* \in C([a_1, b_1] \times \cdots \times [a_m, b_m], \mathbb{B})$.

Proof. We consider the Banach spaces $X_1 = (C([a_1, b_1] \times \cdots \times [a_m, b_m], \mathbb{B}), \|\cdot\|_B)$ and $X_2 = (C([a_1, b_1] \times \cdots \times [a_m, b_m], \mathbb{B}), \|\cdot\|_C)$, where $\|\cdot\|_B$ is the Bielecki's norm and $\|\cdot\|_C$ is the Cebyshev's norm

$$\begin{aligned} \|x\|_B &= \max_{[a_1,b_1]\times\cdots\times[a_m,b_m]} |x(t)| \cdot \prod_{i=1}^{m} e^{-\tau(t_i - a_i)}, \ \tau > 0, \\ \|x\|_C &= \max_{[a_1,b_1]\times\cdots\times[a_m,b_m]} |x(t)|, \end{aligned}$$

the complete generalized metric space (X, d), where

$$\begin{aligned} X &= X_1 \times X_2, \\ d((x_1, x_2), (y_1, y_2)) &= \begin{pmatrix} \|x_1 - y_1\|_B \\ \|x_2 - y_2\|_C \end{pmatrix}, \end{aligned} \tag{5}$$

and the operator $A = (A_1, A_2) : X \to X$, where $A_j : X_1 \times X_2 \to X_j$, $j \in \{1, 2\}$,

$$A_j(x_1, x_2)(t) = F\left(t, x_2(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x_1(s))ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x_2(s))ds\right) \tag{6}$$

It is easy to see that if $(x_1^*, x_2^*) \in F_A$ then $x_1^* = x_2^* = x^*$ and x^* is a solution of *(1)*.
Conditions (iii) and (iv) imply that:

$$\begin{aligned} & |A_j(x_1, x_2)(t) - A_j(y_1, y_2)(t)| \leq \\ \leq \ & \alpha|x_2(t) - y_2(t)| + \beta \left| \textstyle\int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} (K(t, s, x_1(s)) - K(t, s, y_1(s)))ds \right| + \\ & +\gamma \left| \textstyle\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} (K(t, s, x_2(s)) - H(t, s, y_2(s)))ds \right| \leq \\ \leq \ & \alpha|x_2(t) - y_2(t)| + \beta \textstyle\int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} L_K |x_1(s) - y_1(s)| \cdot \prod\limits_{i=1}^{m} e^{-\tau(s_i - a_i)} \cdot \prod\limits_{i=1}^{m} e^{\tau(s_i - a_i)} ds + \\ & +\gamma \textstyle\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} L_H |x_2(s) - y_2(s)| ds \\ \leq \ & \frac{\beta L_K}{\tau^m} \prod\limits_{i=1}^{m} \left(e^{\tau(t_i - a_i)} - 1 \right) \|x_1 - y_1\|_B + (\alpha + \gamma L_H P_1) \|x_2 - y_2\|_C \end{aligned}$$

therefore:

$$\|A_1(x_1, x_2) - A_1(y_1, y_2)\|_B \leq \frac{\beta L_K}{\tau^m} \|x_1 - y_1\|_B + (\alpha + \gamma L_H P_1) \|x_2 - y_2\|_C$$

and

$$\|A_2(x_1, x_2) - A_2(y_1, y_2)\|_C \leq \frac{\beta L_K}{\tau^m} P_2(\tau) \|x_1 - y_1\|_B + (\alpha + \gamma L_H P_1) \|x_2 - y_2\|_C,$$

which means

$$\left(\begin{array}{c} \|A_1(x_1,x_2)-A_1(y_1,y_2)\|_B \\ \|A_2(x_1,x_2)-A_2(y_1,y_2)\|_C \end{array}\right) \leq Q \cdot \left(\begin{array}{c} \|x_1-y_1\|_B \\ \|x_2-y_2\|_C \end{array}\right)$$

for all (x_1,x_2), $(y_1,y_2) \in X$. Using (v), we get that the operator $A : X \to X$ is a $Q-$contraction, so $F_A = \{(x_1^*, x_2^*)\}$, thus $x_1^* = x_2^* = x^*$ and x^* is the unique solution of *(1)*. □

Remark 1. *In the conditions of the Theorem 2, the operator A, given by (6), is PO.*

Remark 2. *Condition* (v) *is equivalent with the condition*

(v') there exists $\tau > 0$ *such that*

$$\varphi(\tau) := \tau^m (1-\alpha-\gamma L_H P_1) - \beta L_K \left[1 + (\alpha + \gamma L_H P_1)(P_2(\tau) - 1)\right] > 0,$$

where $\alpha + \gamma L_H P_1 < 1$.

Proof. A matrix

$$Q = \left(\begin{array}{cc} \alpha_1 & L_1 \\ L_2 & \alpha_2 \end{array}\right)$$

Q is convergent to 0, i.e $Q^n \longrightarrow 0$ as $n \longrightarrow \infty$, iff all eigenvalues of Q are in the unit disc, which is equivalent with the conditions

$$\alpha_1 < 1,\ \alpha_2 < 1,\ \frac{L_1 L_2}{(1-\alpha_1)(1-\alpha_2)} < 1,$$

see M.A. Şerban [26]. In the case of Q from Theorem 2, these conditions imply $\alpha + \gamma L_H P_1 < 1$ and there exists $\tau > 0$ such that $\varphi(\tau) > 0$. □

Example 1. *Let consider the integral equation*

$$x(t) = F\left(t, x(t), \int_a^t K(t,s,x(s))ds, \int_a^b H(t,s,x(s))ds\right), \tag{7}$$

under the following hypothesis:

(i) $F \in C([a,b]\times[a,b]\times\mathbb{R}^3)$, $K,\ H \in C([a,b]\times[a,b]\times\mathbb{R})$;

(ii) there exist α, β, γ *nonnegative constants such that:*

$$|F(t,u_1,v_1,w_1) - F(t,u_2,v_2,w_2)| \leq \alpha|u_1-u_2| + \beta|v_1-v_2| + \gamma|w_1-w_2|,$$

for all $t \in [a,b], u_1,u_2,v_1,v_2,w_1,w_2 \in \mathbb{R}$;

(iii) there exist L_K *and* L_H *nonnegative constants such that:*

$$|K(t,s,u) - K(t,s,v)| \leq L_K|u-v|,$$

$$|H(t,s,u) - H(t,s,v)| \leq L_H|u-v|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$;

(iv) there exists $\tau > 0$ *such that the matrix*

$$Q = \left(\begin{array}{cc} \frac{\beta L_K}{\tau} & \alpha + \gamma L_H(b-a) \\ \frac{\beta L_K}{\tau}\left(e^{\tau(b-a)} - 1\right) & \alpha + \gamma L_H P_1 (b-a) \end{array}\right)$$

is convergent to 0.

Then, the equation (7) has a unique solution $x^* \in C([a,b])$.

Proof. We apply Theorem 2 in particular case of $\mathbb{B} = \mathbb{R}$ and $m = 1$. □

The equation (7) is a general case of equations considered in [1].

Example 2. *Let consider the integral equation*

$$x(t) = f(t, x(t)) + \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x(s))ds + \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x(s))ds, \tag{8}$$

under the following hypothesis:

(i) $f \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R})$, K, $H \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R})$;

(ii) there exists $\alpha > 0$ *such that:*

$$|f(t, u_1) - f(t, u_2)| \le \alpha |u_1 - u_2|,$$

for all $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $u_1, u_2 \in \mathbb{R}$;

(iii) there exist L_K *and* L_H *nonnegative constants such that:*

$$|K(t, s, u) - K(t, s, v)| \le L_K |u - v|,$$

$$|H(t, s, u) - H(t, s, v)| \le L_H |u - v|,$$

for all $t, s \in [a, b]$, $u, v \in \mathbb{R}$;

(iv) there exists $\tau > 0$ *such that the matrix*

$$Q = \begin{pmatrix} \frac{L_K}{\tau^m} & \alpha + L_H P_1 \\ \frac{L_K}{\tau^m} P_2(\tau) & \alpha + L_H P_1 \end{pmatrix}$$

is convergent to 0*, where*

$$\begin{aligned} P_2(\tau) &= \prod_{i=1}^{m} \left(e^{\tau(b_i - a_i)} - 1 \right), \\ P_1 &= \prod_{i=1}^{m} (b_i - a_i). \end{aligned}$$

Then, the equation (8) has a unique solution $x^* \in C([a_1, b_1] \times \cdots \times [a_m, b_m])$.

Proof. This is the special case when F is linear with respect to the last two variables and $\mathbb{B} = \mathbb{R}$. We apply Theorem 2 for $F : [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R}^3 \to \mathbb{R}$,

$$F(t, u, v, w) = f(t, u) + v + w.$$

In this case $\beta = \gamma = 1$. □

Example 3. *Let consider the Darboux problem*

$$\begin{cases} x_{t_1 t_2}(t_1, t_2) &= f(t_1, t_2, x(t_1, t_2)), \quad (t_1, t_2) \in [a_1; b_1] \times [a_2; b_2] \\ x(t_1, a_2) &= \varphi(t_1), \quad t_1 \in [a_1; b_1] \\ x(a_1, t_2) &= \psi(t_2), \quad t_2 \in [a_2; b_2], \ \varphi(a_1) = \psi(a_2) \end{cases} \tag{9}$$

under the following hypothesis:

(i) $f \in C([a_1; b_1] \times [a_2; b_2] \times \mathbb{R})$, $\varphi \in C([a_1; b_1])$, $\psi \in C([a_2; b_2])$;

(ii) there exists $L_f > 0$ *such that:*

$$|f(t_1, t_2, u_1) - f(t_1, t_2, u_2)| \le L_f \cdot |u_1 - u_2|,$$

for all $(t_1, t_2) \in [a_1; b_1] \times [a_2; b_2]$, $u_1, u_2 \in \mathbb{R}$.

Then, the equation Darboux problem (9) has a unique solution $x^* \in C([a_1; b_1] \times [a_2; b_2])$.

Proof. $x \in C([a_1; b_1] \times [a_2; b_2])$ is a solution of *Darboux problem (9) iff it is a solution of the integral equation*

$$x(t_1, t_2) = \varphi(t_1) + \psi(t_2) - \varphi(a_1) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} f(\xi_1, \xi_2, x(\xi_1, \xi_2)) \, d\xi_1 d\xi_2. \tag{10}$$

So, we apply Theorem 2 in particular case of $m = 2$, $\mathbb{B} = \mathbb{R}$ and $F : [a_1, b_1] \times [a_2, b_2] \times \mathbb{R}^3 \to \mathbb{R}$,

$$F(t_1, t_2, u, v, w) = \varphi(t_1) + \psi(t_2) - \varphi(a_1) + v.$$

In this case we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $L_K = L_f$ and $L_H = 0$. Also, the condition (v') from Remark 2 is satisfied: there exists $\tau > 0$ such that $\frac{L_f}{\tau^2} < 1$, for example we can choose $\tau = L_f + 1$. □

Example 4. *We consider the following infinite system of integral equation*

$$x_n(t) = f_n(t) + \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} k(t, s) x_{n+1}(s) ds + \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} h(t, s) x_{n+2}(s) ds, \ n \in \mathbb{N}, \tag{11}$$

under the following hypothesis:

(i) $f_n \in C([a_1, b_1] \times \dots \times [a_m, b_m])$, $n \in \mathbb{N}$, $f_n(t) \to 0$, $n \to +\infty$, *for every* $t \in [a_1, b_1] \times \dots \times [a_m, b_m]$;
(ii) k, $h \in C([a_1, b_1] \times \dots \times [a_m, b_m] \times [a_1, b_1] \times \dots \times [a_m, b_m])$;
(iii) there exists $\tau > 0$ *such that the matrix*

$$Q = \begin{pmatrix} \frac{M_k}{\tau^m} & M_h P_1 \\ \frac{M_k}{\tau^m} P_2(\tau) & M_h P_1 \end{pmatrix}$$

is convergent to 0, *where*

$$\begin{aligned} P_2(\tau) &= \prod_{i=1}^{m} \left(e^{\tau(b_i - a_i)} - 1 \right), \\ P_1 &= \prod_{i=1}^{m} (b_i - a_i). \end{aligned}$$

$$\begin{aligned} M_k &= \max_{t,s \in [a_1, b_1] \times \dots \times [a_m, b_m]} |k(t, s)|, \\ M_h &= \max_{t,s \in [a_1, b_1] \times \dots \times [a_m, b_m]} |h(t, s)|. \end{aligned}$$

Then, the equation (11) has a unique solution such that

$$x_n(t) \to 0 \ as \ n \to +\infty.$$

Proof. Let $(\mathbb{B}, \|\cdot\|)$ the Banach space, where

$$\mathbb{B} = c_0 = \{\mathbf{u} = (u_0, u_1, \dots, u_n, \dots) \in s(\mathbb{R}) : u_n \to 0\}$$

and

$$\|\mathbf{u}\| = \max_{n \in \mathbb{N}} |u_n|.$$

Let $\mathbf{u} = (u_0, u_1, \dots, u_n, \dots) \in \mathbb{B}$. We denote by

$$\begin{aligned} \mathbf{f} &= (f_0, f_1, \dots, f_n, \dots), \\ \mathbf{K} &= (K_0, K_1, \dots, K_n, \dots), \\ \mathbf{H} &= (H_0, H_1, \dots, H_n, \dots), \end{aligned}$$

where

$$K_n\,(t,s,\mathbf{u}) = k(t,s)u_{n+1},$$
$$H_n\,(t,s,\mathbf{u}) = h(t,s)u_{n+2},$$

$t,s \in [a_1,b_1]\times\cdots\times[a_m,b_m]$. From (i) and (ii) we have that $\mathbf{f} \in C([a_1,b_1]\times\cdots\times[a_m,b_m],\mathbb{B})$ and $\mathbf{K},\mathbf{H} \in C([a_1,b_1]\times\cdots\times[a_m,b_m]\times[a_1,b_1]\times\cdots\times[a_m,b_m],\mathbb{B})$. Also,

$$\|\mathbf{K}\,(t,s,\mathbf{u}) - \mathbf{K}\,(t,s,\mathbf{v})\| \le m_k\,\|\mathbf{u}-\mathbf{v}\|\,,$$
$$\|\mathbf{H}\,(t,s,\mathbf{u}) - \mathbf{H}\,(t,s,\mathbf{v})\| \le m_h\,\|\mathbf{u}-\mathbf{v}\|\,,$$

for all $t,s \in [a_1,b_1]\times\cdots\times[a_m,b_m]$ and $\mathbf{u},\mathbf{v} \in \mathbb{B}$. All the conditions of Theorem 2 are satisfied, therefore we get that the equation (11) has a unique solution
$\mathbf{x}^* = (x_0^*, x_1^*, \ldots, x_n^*, \ldots) \in C([a_1,b_1]\times\cdots\times[a_m,b_m],\mathbb{B})$. □

3. Data dependence: continuity

In this section we prove the continuous dependence of the solution for integral equation (1) using the following Abstract Data Dependence Lemma

Lemma 1. *(I. A. Rus, A Petruşel, G. Petruşel* [24]*)(General data dependence theorem). Let (X,d) be a complete generalized metric space and $A,B : X \longrightarrow X$ two operators. We suppose that:*

(i) A is an Q-contraction (Q converges to zero) and $F_A = \{x^*\}$;
(ii) $x_B^* \in F_B$;
(iii) there exists $\eta \in \mathbb{R}_+^m$ such that

$$d(A(x),B(x)) < \eta$$

for all $x \in X$.

In these conditions we have

$$d(x_A^*, x_B^*) \le (I-Q)^{-1}\eta\,.$$

We consider the following equations:

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K_1(t,s,x(s))ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H_1(t,s,x(s))ds\right) \tag{12}$$

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K_2(t,s,x(s))ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H_2(t,s,x(s))ds\right), \tag{13}$$

where $K_i, H_i \in C([a_1,b_1]\times\cdots\times[a_m,b_m]\times[a_1,b_1]\times\cdots\times[a_m,b_m]\times\mathbb{B},\mathbb{B})$, $i=1,2$.

Theorem 3. *We assume that:*

(i) F, K_1, H_1 satisfy the conditions from Theorem 2;
(ii) there exists a nonnegative constant η_1 such that:

$$|K_1(t,s,u) - K_2(t,s,u)| \le \eta_1,\ \forall\ t,s \in [a_1,b_1]\times\cdots\times[a_m,b_m],\ \forall\ u \in \mathbb{R};$$

(iii) there exists a nonnegative constant η_2 such that:

$$|H_1(t,s,u) - H_2(t,s,u)| \le \eta_2,\ \forall\ t,s \in [a_1,b_1]\times\cdots\times[a_m,b_m],\ \forall\ u \in \mathbb{R}.$$

If x_2^* is a solution of the corresponding equation (13) then:

$$\begin{pmatrix} \|x_1^* - x_2^*\|_B \\ \|x_1^* - x_2^*\|_C \end{pmatrix} \leq (I-Q)^{-1}\eta,$$

where x_1^* is the unique solution of the corresponding equation (12), Q is defined by (4) and $\eta \in \mathbb{R}^2$,

$$\eta = \begin{pmatrix} (\beta\eta_1 + \gamma\eta_2)P_1 \\ (\beta\eta_1 + \gamma\eta_2)P_1 \end{pmatrix}. \tag{14}$$

Proof. We consider the complete generalized metric space (X, d) defined by (5) and the operators $A = (A_1, A_2)$, $B = (B_1, B_2)$, $A, B : X \to X$, where $A_j : X_1 \times X_2 \to X_j$, $j \in \{1, 2\}$,

$$A_j(x_1, x_2)(t) = F\left(t, x_2(t), \int_{a_1}^{t_1} \ldots \int_{a_m}^{t_m} K_1(t, s, x_1(s))ds, \int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} H_1(t, s, x_2(s))ds\right) \tag{15}$$

and $B_j : X_1 \times X_2 \to X_j$, $j \in \{1, 2\}$,

$$B_j(x_1, x_2)(t) = F\left(t, x_2(t), \int_{a_1}^{t_1} \ldots \int_{a_m}^{t_m} K_2(t, s, x_1(s))ds, \int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} H_2(t, s, x_2(s))ds\right) \tag{16}$$

From condition (i) we have that the operator A is a $Q-$contraction with Q defined by (4), (see Theorem 2).

From (ii) and (iii) we get:

$$|A_j(x_1, x_2)(t) - B_j(x_1, x_2)(t)| \leq (\beta\eta_1 + \gamma\eta_2)P_1,$$

for $\forall\ t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $\forall\ (x_1, x_2) \in X$, $j \in \{1, 2\}$, which implies that

$$\begin{pmatrix} \|A(x_1, x_2) - B(x_1, x_2)\|_B \\ \|A(x_1, x_2) - B(x_1, x_2)\|_C \end{pmatrix} \leq \begin{pmatrix} (\beta\eta_1 + \gamma\eta_2)P_1 \\ (\beta\eta_1 + \gamma\eta_2)P_1 \end{pmatrix}$$

for $\forall\ (x_1, x_2) \in X$. The conclusion is obtained from Lemma 1 for $\eta \in \mathbb{R}^2$ defined by (14). $\square$

4. Data dependence: comparison results

In this section we prove a comparison result for the solution of integral equation (1) using the following Abstract Comparison Lemma

Lemma 2. *(I.A. Rus* [17], [21], [23]*) (Comparison lemma) Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ operators such that:*

(i) $A \leq B \leq C$;
(ii) A, B, C are WPOs;
(iii) the operator B is increasing.

Then

$$x \leq y \leq z \Longrightarrow A^{\infty}(x) \leq B^{\infty}(y) \leq C^{\infty}(z).$$

We consider the nonlinear integral equations:

$$x(t) = F_i\left(t, x(t), \int_{a_1}^{t_1} \ldots \int_{a_m}^{t_m} K_i(t, s, x(s))ds, \int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} H_i(t, s, x(s))ds\right) \tag{17}$$

where $K_i, H_i \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{B}, \mathbb{B})$, $i \in \{1, 2, 3\}$.

Theorem 4. *We assume that:*

(i) $\mathbb{B}$ *is an ordered Banach space;*
(ii) F_i, K_i, H_i *satisfy the conditions from Theorem 2 for* $i = \{1, 2, 3\}$*;*
(iii) *the functions* $F_2(t, \cdot, \cdot, \cdot)$, $K_2(t, s, \cdot)$ *and* $H_2(t, s, \cdot)$ *are increasing;*
(iv) $F_1 \le F_2 \le F_3$, $K_1 \le K_2 \le K_3$ *and* $H_1 \le H_2 \le H_3$*;*

If x_i^* *is the solution of the equation (17) corresponding to* F_i, K_i, H_i, $i \in \{1, 2, 3\}$, *then:*

$$x_1^* \le x_2^* \le x_3^*.$$

Proof. We consider the complete generalized ordered metric space (X, d) defined by (5) and operators $A, B, C : X \to X$ defined by (6) corresponding to F_i, K_i, H_i, $i \in \{1, 2, 3\}$. From condition (ii) we have that A, B, C are PO, therefore $F_A = \{x_A^*\}$, $x_A^* = (x_1^*, x_1^*)$, $F_B = \{x_B^*\}$, $x_B^* = (x_2^*, x_2^*)$, $F_C = \{x_C^*\}$, $x_C^* = (x_3^*, x_3^*)$, where x_i^* is the solution of the equation (17) corresponding to F_i, K_i, H_i, $i \in \{1, 2, 3\}$. Condition (iii) implies that B is an increasing operator and condition (iv) implies that $A \le B \le C$.

Let $x \in X$ and we denote by $u_1 = A(x)$, $u_2 = B(x)$, $u_3 = C(x)$. It is obvious that:

$$u_1 \le u_2 \le u_3$$

and

$$x_A^* = A^\infty(u_1), \ x_B^* = B^\infty(u_2), \ x_C^* = B^\infty(u_3)$$

thus, from Lemma 2, we have:

$$u_1 \le u_2 \le u_3 \Longrightarrow A^\infty(u_1) \le B^\infty(u_2) \le C^\infty(u_3)$$

Hence, the conclusion follows. □

5. Gronwall lemmas

In this section we study the integral inequalities (2) and (3) using the Abstract Gronwall Lemma and Abstract Gronwall-comparison Lemma.

Lemma 3. *(I.A. Rus* [17], [21], [23]*) (Abstract Gronwall lemma) Let* $(X, d, \le)$ *be an ordered metric space and* $A : X \to X$ *be an operator. We assume that:*

(i) A *is a PO;*
(ii) A *is increasing.*

If we denote by x_A^* *the unique fixed point of* A*, then:*

(a) $x \le A(x) \Longrightarrow x \le x_A^*$*;*
(b) $x \ge A(x) \Longrightarrow x \ge x_A^*$.

Lemma 4. *(I.A. Rus* [17], [21], [23]*) (Abstract Gronwall-comparison lemma) Let* $(X, \to, \le)$ *be an ordered metric space and* $A_1, A_2 : X \to X$ *be two operators. We assume that:*

(i) A *is increasing;*
(ii) A *and* B *are a POs.*
(iii) $A \le B$

If we denote by x_B^* *the unique fixed point of* B*, then*

$$x \le A(x) \Longrightarrow x \le x_B^*.$$

Theorem 5. *We consider the equation (1). We assume that:*

(i) $\mathbb{B}$ *is an ordered Banach space;*
(ii) F, K, H *satisfy the conditions from Theorem 2;*
(iii) $K(t, s, \cdot), H(t, s, \cdot) : \mathbb{B} \to \mathbb{B}$ *are increasing functions for all* $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$*;*
(iv) $F(t, \cdot, \cdot, \cdot) : \mathbb{B}^3 \to \mathbb{B}$ *is increasing, for all* $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$.

Then we have:

(a) If x is a solution of (2) then $x \leq x^*$, where x^* is the unique solution of (1);
(b) If x is a solution of (3) then $x \geq x^*$,where x^* is the unique solution of (1).

Proof. We consider the operator A defined by (6). From Theorem 2 we have that A is PO, if x^* is the unique solution of (1) then $F_A = \{(x^*, x^*)\}$. Conditions (ii) and (iii) imply that A is increasing. In terms of the operator A the integral inequality (2) means

$$(x, x) \leq A(x, x),$$

for all $x \in C([a_1, b_1] \times \cdots \times [a_m, b_m], \mathbb{B})$ and the integral inequality (3) means

$$(x, x) \geq A(x, x),$$

for all $x \in C([a_1, b_1] \times \cdots \times [a_m, b_m], \mathbb{B})$. The conclusion is obtained from Abstract Gronwall Lemma, Lemma 3. □

Remark 3. *To have an effective Gronwall Lemma we need to "construct" x^*, which is usually a very difficult problem.*

In this direction, if we use the Abstract Gronwall-comparison lemma, Lemma 4, we obtain the following result:

Theorem 6. *We consider the integral equation (2) corresponding to F_i, K_i, H_i for $i = \{1, 2\}$. We assume that:*

(i) $\mathbb{B}$ is an ordered Banach space;
(ii) F_i, K_i, H_i satisfy the conditions from Theorem 2 for $i = \{1, 2\}$;
(iii) $F_1(t, \cdot, \cdot, \cdot)$, $K_1(t, s, \cdot)$ and $H_1(t, s, \cdot)$ are increasing functions for all $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$;
(iv) $F_1 \leq F_2$, $K_1 \leq K_2$ and $H_1 \leq H_2$.

If x is a solution of (2) corresponding to F_1, K_1, H_1 then $x \leq x_2^*$, where x_2^* is the unique solution of (1) corresponding to F_2, K_2, H_2.

Proof. We consider the operator A, B defined by (6), corresponding to F_1, K_1, H_1 and F_2, K_2, H_2. From Theorem 2 we have that A and B are POs. If we denote by x_1^* the unique solution of (1) corresponding to F_1, K_1, H_1 and x_2^* the unique solution of (1) corresponding to F_2, K_2, H_2 then $x_A^* = (x_1^*, x_1^*)$ is the unique fixed point of operator A and $x_B^* = (x_2^*, x_2^*)$ is the unique fixed point of operator B. Condition (iii) implies that A is increasing and condition (iv) implies that $A \leq B$. If x is a solution of (2) corresponding to F_1, K_1, H_1 then

$$(x, x) \leq A(x, x).$$

The conclusion is obtained from Abstract Gronwall-comparison Lemma, Lemma 4. □

Example 5. *(Wendroff type inequality* [12]*) If*

$$x(t_1, t_2) \leq u(t_1, t_2) + \int_0^{t_1} \int_0^{t_2} v(\xi_1, \xi_2)\, x(\xi_1, \xi_2)\, d\xi_1 d\xi_2 \tag{18}$$

where $u(t_1, t_2) > 0$, $u_{t_1}(t_1, t_2)$, $u_{t_2}(t_1, t_2) \geq 0$, $x(t_1, t_2)$, $v(t_1, t_2) \geq 0$, then:

$$x(t_1, t_2) \leq x^*(t_1, t_2) \leq \frac{u(0, t_2) \cdot u(t_1, 0)}{u(0, 0)} \cdot e^{\int_0^{t_1} \int_0^{t_2} v(\xi_1, \xi_2) d\xi_1 d\xi_2}, \tag{19}$$

where $x^*(t_1, t_2)$ is the solution of the Darboux problem

$$\begin{cases} x_{t_1 t_2}(t_1, t_2) & = x(t_1, t_2)\, v(t_1, t_2), \quad (t_1, t_2) \in [0; b_1] \times [0; b_2] \\ x(t_1, 0) & = u(t_1, 0), \quad t_1 \in [0; b_1] \\ x(0, t_2) & = u(0, t_2), \quad t_2 \in [0; b_2] \end{cases} \tag{20}$$

Proof. Let $b_1 > 0$ and $b_2 > 0$ and we consider the Banach space $X = (C([0;b_1] \times [0;b_2], \mathbb{R}_+), \|\cdot\|_B)$, where

$$\|x\|_B = \max_{[0;b_1]\times[0;b_2]} |x(t_1,t_2)| \cdot e^{-\tau(t_1+t_2)}, \ \tau > 0.$$

We define the operators $A_1, A_2 : X \to X$

$$A(x)(t_1,t_2) = u(t_1,0) + u(0,t_2) - u(0,0) + \int_0^{t_1}\int_0^{t_2} v(\xi_1,\xi_2)\, x(\xi_1,\xi_2)\, d\xi_1 d\xi_2, \tag{21}$$

$$B(x)(t_1,t_2) = u(0,t_2) + \int_0^{t_1} \left(\frac{u_{t_1}(\xi_1,0)}{u(\xi_1,0)} + \int_0^{t_2} v(\xi_1,\xi_2)\, d\xi_2 \right) \cdot x(\xi_1,t_2)\, d\xi_1, \tag{22}$$

It is clear that any solution of the Darboux problem (20) is a fixed point of operator A. We have that $A : X \to X$ is PO (see Example 3), thus $F_A = \{x^*\}$. Also, A is an increasing operator and from Abstract Gronwall Lemma we have

$$x \le A(x) \Longrightarrow x \le x^*,$$

which means that any x satisfying (18) will satisfy the inequality $x \le x^*$.

The function

$$w^*(t_1,t_2) = \frac{u(0,t_2) \cdot u(t_1,0)}{u(0,0)} \cdot e^{\int_0^{t_1}\int_0^{t_2} v(\xi_1,\xi_2) d\xi_1 d\xi_2} \tag{23}$$

is not the solution of (20), w^* is the solution of the problem

$$\begin{cases} w_{t_1}(t_1,t_2) &= \left(\frac{u_{t_1}(t_1,0)}{u(t_1,0)} + \int_0^{t_2} v(t_1,\xi_2)\, d\xi_2 \right) \cdot w(t_1,t_2) \\ w(0,t_2) &= u(0,t_2) \end{cases} \tag{24}$$

or

$$\begin{cases} \frac{\partial}{\partial t_2} \left(\frac{w_{t_1}(t_1,t_2)}{w(t_1,t_2)} \right) &= v(t_1,t_2) \\ w(t_1,0) &= u(t_1,0) \\ w(0,t_2) &= u(0,t_2) \end{cases} \tag{25}$$

In order to prove (19) we will apply the Abstract Gronwall-comparison lemma, Lemma 4.

We consider the set

$$Y = \{x \in X : x_{t_1} \ge 0, \ x_{t_2} \ge 0, \ x(t_1,0) = u(t_1,0)\}.$$

It is clear that $Y \subseteq X$ is a closed subset, so it is a complete metric space. Moreover, Y is an invariant set of A and $A : Y \to Y$ is a contraction, so $x^* \in Y$ and $A : Y \to Y$ is PO. It is easy to check that also $B : Y \to Y$ is a contraction, so it is PO and $F_B = \{w^*\}$. Now we prove that $A(x) \le B(x)$ for all $x \in Y$.

We have:

$$\begin{aligned} A(x)(t_1,t_2) &= u(0,t_2) + \int_0^{t_1} \left(u_{t_1}(\xi_1,0) + \int_0^{t_2} v(\xi_1,\xi_2)\, x(\xi_1,\xi_2)\, d\xi_2 \right) d\xi_1 \le \\ &\le u(0,t_2) + \int_0^{t_1} \left(\frac{u_{t_1}(\xi_1,0)}{u(\xi_1,0)} + \int_0^{t_2} v(\xi_1,\xi_2)\, d\xi_2 \right) x(\xi_1,t_2)\, d\xi_1 = \\ &= B(x)(t_1,t_2), \end{aligned}$$

since

$$u(\xi_1, 0) = x(\xi_1, 0) \le x(\xi_1, \xi_2), \ \forall (\xi_1, \xi_2) \in [0; b_1] \times [0; b_2]$$

and

$$x(\xi_1, \xi_2) \le x(\xi_1, t_2), \ \forall \xi_1 \in [0; b_1], \ 0 \le \xi_2 \le t_2 \le b_2.$$

All the conditions of the Abstract Gronwall-comparison lemma are satisfied, therefore

$$x \le A(x) \le B(x) \Longrightarrow x \le x^* \le w^*$$

and the proof is complete. □

If we consider the case of $u(t_1, t_2) = a(t_1) + b(t_2) \equiv c$, $c \in \mathbb{R}_+$, we obtain the results from C. Crăciun, N. Lungu [5], N. Lungu [9]. In some particular cases for $v(t_1, t_2)$ we can find the expression of x^* from (19), for example, if $a(t_1)+b(t_2) \equiv c$ and $v(t_1, t_2) \equiv \alpha^2$ then $x^*(t_1, t_2) = cJ_0\left(2\alpha\sqrt{t_1 t_2}\right)$, where $J_0\left(2\alpha\sqrt{t_1 t_2}\right)$ is the Bessel function (see C. Crăciun, N. Lungu [5]).

For other applications of Abstract Gronwall lemma and Abstract Gronwall-comparison lemma see N. Lungu [10], N. Lungu, I.A. Rus [11], I.A. Rus [17], [20].

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Babeş-Bolyai University of Cluj-Napoca
Department of Applied Mathematics
1 M. Kogălniceanu, 400048-Cluj-Napoca, Romania,
E-mail address: mserban@math.ubbcluj.ro