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A NOTE ON BOUNDS FOR THE SPECTRAL NORMS OF CIRCULANT-CAUCHY-TOEPLITZ MATRICES

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Abstract. In this paper, we established lower and upper bounds for the spectral norms of some Circulant-Cauchy-Toeplitz matrices.

1. Introduction

A Toeplitz matrix is an $n \times n$ matrix $T_n = [t_{k,j}; k, j = 0, 1, ..., n-1]$ where $t_{k,j} = t_{k-j}$. This structure is very interesting in itself for all the rich theoretical properties which it involves, but at the same time it is important for the dramatic impact that it has in applications. Many problems involve Toeplitz-like matrices (i.e., Cauchy-Toeplitz, Cauchy-Hankel, Hilbert), or matrices having a displacement structure.

A Cauchy-Toeplitz matrix is a matrix that is both a Cauchy matrix (i.e. $(1/(x_i - y_j))_{i,j=1}^{n}, x_i \neq y_j$) and a Toeplitz matrix (i.e. $(z_{i-j})_{i,j=1}^{n}$) such that

$$T_n(g,h) = \left[\frac{1}{g+(i-j)h}\right]_{i,j=1}^{n},$$

where g and $h \neq 0$ are arbitrary numbers and g/h is not integer.

A Cauchy-Hankel matrix is a matrix that is both a Cauchy matrix (i.e. $(1/(x_i - y_j))_{i,j=1}^{n}, x_i \neq y_j$) and a Hankel matrix (i.e. $(h_{i+j})_{i,j=1}^{n}$) such that

$$H_n(g,h) = \left[\frac{1}{g+(i+j)h}\right]_{i,j=1}^{n},$$

where g and $h \neq 0$ are arbitrary numbers and g/h is not integer.

Moler has computed the singular values of the Cauchy-Toeplitz matrix $T_n(1/2, 1)$. The remarkable result is that most of these singular values (first 20) were equal to $\pi - \varepsilon$ with ε very small. Later, Parter has given a qualitative explanation of this phenomena [9]. Tyrtyshnikov has shown that minimal singular values of the Cauchy-Toeplitz matrix $T_n(1/2, 1)$ converge to zero with increasing n [10]. Civciv and Turkmen [3] gave a lower bound and an upper bound for the ℓ_p norms of Khatri-Rao and Tracy-Singh products of Cauchy-Toeplitz matrices of the form

$$T_n(g,h) = \begin{pmatrix} T_n^{(11)} & T_n^{(12)} \\ T_n^{(21)} & T_n^{(22)} \end{pmatrix},$$

where $T_n^{(ij)}$ is the ijth submatrix of order $p_i \times q_j$ $(p_1 + p_2 = n, q_1 + q_2 = n)$ with $T_n^{(11)} = T_{n-1}(g,h)$, and g, $h \neq 0$ are arbitrary numbers and g/h is not integer such that $0 <$

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$g/h < 1$. Also, Civciv and Turkmen [2] have established a lower bound and an upper bound for the l_p norms of the Khatri-Rao product of Cauchy-Hankel matrices of the form

$$H_n\left(1/2,1\right)=\left(\begin{array}{cc} H_n^{(11)} & H_n^{(12)} \\ H_n^{(21)} & H_n^{(22)} \end{array}\right),$$

where $H_n^{(ij)}$ is the ijth submatrix of order $p_i \times q_j$ $(p_1+p_2=n, q_1+q_2=n)$ with $H_n^{(11)} = H_{n-1}\left(1/2,1\right)$.

For any given $a_0, a_1, ..., a_{n-1} \in \mathbb{C}$, the circulant matrix $B=(b_{ij})_{n\times n}$ is defined by $b_{ij}=a_{j-i(\operatorname{mod} n)}$, i.e.,

$$B=\left[\begin{array}{ccccc} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_1 & a_2 & \cdots & a_0 \end{array}\right].$$

A circulant matrix is a special kind of Toeplitz matrix where each row vector is rotated one element to the right relative to the preceding row vector. Several problems of physics and electromagnetics involve circulant matrices (CM) and block-circulant matrices (BCM). For example, such matrices appear when the method of moments is used to study the electromagnetic behavior of structures having circular periodicity around an axis (e.g., circular arrays of equally spaced dipoles or cylindrical arrays of equally spaced parallel wires). In such problems, the exploitation of the properties of CM and BCM may lead to a considerable reduction of the computational complexity [11, 12]. CM and BCM have been studied extensively and closed-form expressions for their inversion are well known [5, 8]. In [8], the formulas of inversion have been derived in the CM case by writing a matrix as a combination of permutation matrices, which are expressed in terms of their eigenvalues and eigenvectors. In [5], this procedure has been extended to the BCM case.

Now we only include the minimal amount of background necessary to understand this note .

2. Preliminaries

Let A be any $n \times n$ matrix. The Euclidean norm of the matrix A is defined as

$$\|A\|_E=\left(\sum_{i,j=1}^{n}\left|a_{ij}\right|^2\right)^{1/2}.$$

The spectral norm of the matrix A is

$$\|A\|_2=\sqrt{\max_{1\le i\le n}\lambda_i},$$

where λ_i is eigenvalue of $A^H A$ and A^H is conjugate transpose of the matrix A. Between $\|A\|_E$ and $\|A\|_2$ norms is valid the following inequality

$$\|A\|_2 \le \|A\|_E \le \sqrt{n}\,\|A\|_2\,. \tag{1}$$

Define the maximum column length norm $c_1\left(.\right)$ and the maximum row length norm $r_1\left(.\right)$ of any matrix A by

$$c_1(A)=\max_j\left(\sum_i\left|a_{ij}\right|^2\right)^{1/2}, \tag{2}$$

and

$$r_1(A) = \max_i \left(\sum_j |a_{ij}|^2 \right)^{1/2}, \tag{3}$$

respectively.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of the same size, not necessarily square. Then their Hadamard product (also called Schur product) $A \circ B$ is defined by entrywise multiplication: $A \circ B = (a_{ij} b_{ij})$. Let A, B and C be $m \times n$ matrices. If $A \circ B = C$, then

$$\|A\|_2 \leq r_1(B) c_1(C) \tag{4}$$

For more comprehensive treatments on matrices we refer to [6].

The principal solution of

$$\underset{w}{\Delta} F(x \mid w) = \phi(x)$$

for primitives $F(x \mid w)$ given $\phi(x)$, or sum of $\phi(x)$ is

$$F(x \mid w) = \int_a^\infty \phi(t)\, dt - w \sum_{j=0}^\infty \phi(x + jw),$$

where both integral and sum converge.

The notation introduced by Nörlund for the principal solution is

$$F(x \mid w) = S_a^x \phi(z) \underset{w}{\Delta} z$$

and the operation is referred to as "summing $\phi(z)$ from a to x." The notation $F(x)$ is used when $w = 1$. The quantity w is called the "span" of the sum and, unless otherwise stated, is assumed to be positive (for more details [7]).

Examples of evaluation directly from the definition are

$$S_a^x z^{-v} \underset{w}{\Delta} z = -\frac{a^{1-v}}{1-v} - w^{1-v} \zeta\left(v, \frac{x}{v}\right), \quad v > 1 \tag{5}$$

in which

$$\zeta(v, x) = \sum_{j=0}^\infty \frac{1}{(x+j)^v}$$

is the generalized zeta function (for more details [1]), and

$$S_a^x \phi(z) \underset{w}{\Delta} z = \int_a^\infty \phi(t)\, dt - w \sum_{j=0}^\infty \phi(x + jw).$$

In this note, we first define the Circulant-Cauchy-Toeplitz matrix, and then present lower and upper bounds for the spectral norms of some Circulant-Cauchy-Toeplitz matrices.

3. Main Results

Definition 1. *A Circulant-Cauchy-Toeplitz matrix is a matrix that is both a Circulant matrix and a Cauchy-Toeplitz matrix such that*

$$C_n(g, h) = \left[\frac{1}{g + h mod(j - i, n)} \right]_{i,j=1}^n,$$

where g and $h \neq 0$ are arbitrary numbers and g/h is not integer.

Theorem 1. *Let the $n\times n$ Circulant-Cauchy-Toeplitz matrix $C_n(1/g,1)$ be as $C_n(1/g,1) = [c_{ij}]_{n\times n}$ such that*

$$c_{ij} = \frac{1}{(1/g) + mod\,(j-i,n)},$$

with $g \in \mathbb{R}^+$. Then

$$\sqrt{g\left[\zeta\left(2,\frac{1}{g}\right) - \zeta\left(2,\frac{1}{g}+n\right)\right]} \le \|C_n(1/g,1)\|_2 \le g\left[\zeta\left(2,\frac{1}{g}\right) - \zeta\left(2,\frac{1}{g}+n\right)\right], \quad (6)$$

where the symbol $\|.\|_2$ is the spectral norm and $\zeta(v,x)$ is the generalized zeta function.

Proof. The matrix $C_n(1/g,1)$ is of the form

$$C_n(1/g,1) = \begin{bmatrix} g & \frac{1}{(1/g)+1} & \frac{1}{(1/g)+2} & \cdots & \frac{1}{(1/g)+n-1} \\ \frac{1}{(1/g)+n-1} & g & \frac{1}{(1/g)+1} & \cdots & \frac{1}{(1/g)+n-2} \\ \frac{1}{(1/g)+n-2} & \frac{1}{(1/g)+n-1} & g & \cdots & \frac{1}{(1/g)+n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(1/g)+1} & \frac{1}{(1/g)+2} & \frac{1}{(1/g)+3} & \cdots & g \end{bmatrix}.$$

On the other hand, let matrices K_n and L_n be as

$$K_n = (k_{ij}) = \begin{cases} k_{ij} = \frac{1}{(1/g)+\mathrm{mod}(j-i,n)} & i \ge j \\ k_{ij} = 1 & i < j \end{cases}$$

and

$$L_n = (l_{ij}) = \begin{cases} l_{ij} = \frac{1}{(1/g)+\mathrm{mod}(j-i,n)} & i \le j \\ k_{ij} = 1 & i > j \end{cases}$$

such that $C_n(1/g,1) = K_n \circ L_n$. Then,

$$\begin{aligned} r_1(K_n) &= \max_i \sqrt{\sum_j |k_{ij}|^2} \\ &= \sqrt{\sum_{s=0}^{n-1} \frac{1}{((1/g)+s)^2}} \end{aligned}$$

and

$$\begin{aligned} c_1(L_n) &= \max_j \sqrt{\sum_i |l_{ij}|^2} \\ &= \sqrt{\sum_{s=0}^{n-1} \frac{1}{((1/g)+s)^2}}. \end{aligned}$$

From (2), (3), (4) and (5) we have

$$\begin{aligned} \frac{1}{g}\|C_n(1/g,1)\|_2 &\le g\sum_{s=0}^{n-1} \frac{1}{(gs+1)^2} \\ &= S_a^{1+gn} z^{-2} \underset{g}{\Delta} z - S_a^{1} z^{-2} \underset{g}{\Delta} z, \end{aligned}$$

and hence for the spectral norm of $C_n(1/g,1)$ we have an upper bound such that

$$\|C_n(1/g,1)\|_2 \le g\left[\zeta\left(2,\frac{1}{g}\right) - \zeta\left(2,\frac{1}{g}+n\right)\right].$$

On the other hand, the Euclidean norm of the matrix $C_n(1/g,1)$ is computed as

$$\|C_n(1/g,1)\|_E = \sqrt{ng^2\sum_{s=0}^{n-1}\frac{1}{(gs+1)^2}}. \tag{7}$$

From (5) and (7) we get

$$\begin{aligned}\frac{1}{n}\|C_n(1/g,1)\|_E^2 &= g^2\sum_{s=0}^{n-1}\frac{1}{(gs+1)^2}\\ &= g\left[\zeta\left(2,\frac{1}{g}\right)-\zeta\left(2,\frac{1}{g}+n\right)\right]. \end{aligned} \tag{8}$$

From (1) and (8), for the spectral norm of $C_n(1/g,1)$ we have a lower bound such that

$$\sqrt{g\left[\zeta\left(2,\frac{1}{g}\right)-\zeta\left(2,\frac{1}{g}+n\right)\right]} \le \|C_n(1/g,1)\|_2.$$

Thus, the proof is completed. □

Corollary 1. *Let the $n\times n$ Circulant-Cauchy-Hankel matrix $H_n(g)$ be as $H_n(g) = [h_{ij}]_{n\times n}$ such that*

$$h_{ij} = \frac{1}{(1/g)+mod(i+j,n)},$$

with $g\in\mathbb{R}^+$. Then,

$$\sqrt{g\left[\zeta\left(2,\frac{1}{g}\right)-\zeta\left(2,\frac{1}{g}+n\right)\right]} \le \|H_n(g)\|_2 \le g\left[\zeta\left(2,\frac{1}{g}\right)-\zeta\left(2,\frac{1}{g}+n\right)\right].$$

Corollary 2. *For the spectral norm of the matrix $C_n(1/2,1)$*

$$\frac{\pi}{\sqrt{2}} \le \lim_{n\to\infty}\|C_n(1/2,1)\|_2 \le \frac{\pi^2}{2}$$

is valid.

Proof. If we now consider (6), we write

$$\sqrt{2\left[\zeta\left(2,\frac{1}{2}\right)-\zeta\left(2,\frac{1}{2}+n\right)\right]} \le \|C_n(1/2,1)\|_2 \le 2\left[\zeta\left(2,\frac{1}{2}\right)-\zeta\left(2,\frac{1}{2}+n\right)\right].$$

Therefore, since $2\left[\zeta\left(2,\frac{1}{2}\right)-\zeta\left(2,\frac{1}{2}+n\right)\right] = 4\sum_{s=0}^{n-1}\frac{1}{(2s+1)^2}$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|C_n(1/2,1)\|_2 &\le 4\sum_{s=0}^{\infty}\frac{1}{(2s+1)^2}\\ &= 4\left[\zeta(2)-\sum_{s=1}^{\infty}\frac{1}{(2s)^2}\right]\\ &= 4\left(1-\frac{1}{4}\right)\zeta(2)\\ &= \frac{\pi^2}{2},\end{aligned}$$

in which $\zeta(n) = \sum_{k=0}^{\infty} \frac{1}{k^n}$ is the Riemann-Zeta function(for more details [4]). Similarly, we have

$$\frac{\pi}{\sqrt{2}} \leq \lim_{n\to\infty} \|C_n(1/2,1)\|_2,$$

which completes the proof. □

Corollary 3. *For the spectral norm of the matrix $C_n(1/4,1)$*

$$\sqrt{3\pi^2 + 4\pi \log\left(1+\sqrt{2}\right)} \leq \lim_{n\to\infty} \|C_n(1/4,1)\|_2 \leq 3\pi^2 + 4\pi \log\left(1+\sqrt{2}\right)$$

is valid.

Proof. From (6), we write

$$\sqrt{4\left[\zeta\left(2,\frac{1}{4}\right) - \zeta\left(2,\frac{1}{4}+n\right)\right]} \leq \|C_n(1/4,1)\|_2 \leq g\left[\zeta\left(2,\frac{1}{4}\right) - \zeta\left(2,\frac{1}{4}+n\right)\right].$$

Therefore, since $2\left[\zeta\left(2,\frac{1}{4}\right) - \zeta\left(2,\frac{1}{4}+n\right)\right] = 16\sum_{s=0}^{n-1} \frac{1}{(4s+1)^2}$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{16}\|C_n(1/4,1)\|_2 &\leq \sum_{s=0}^{\infty} \frac{1}{(4s+1)^2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)^2} + \sum_{s=0}^{\infty} \frac{1}{(4s+3)^2} \\
&= \frac{1}{2}\left[\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)^2} + \sum_{s=0}^{\infty}\left\{\frac{1}{(4s+3)^2} + \frac{1}{(4s+1)^2}\right\}\right] \\
&= \frac{1}{2}\left[\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)^2} + \sum_{s=0}^{\infty} \frac{1}{(2s+1)^2}\right] \\
&= \frac{1}{2}\left[\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)^2} + \sum_{s=1}^{\infty}\left\{\frac{1}{(2s-1)^2} + \frac{1}{(2s)^2}\right\} - \sum_{s=1}^{\infty} \frac{1}{(2s)^2}\right] \\
&= \frac{1}{2}\left[\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)^2} + \frac{3}{4}\sum_{s=1}^{\infty} \frac{1}{s^2}\right] \\
&= \frac{1}{2}\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)^2} + \frac{1}{16}\pi^2 \\
&= \frac{1}{16}\pi^2 + \frac{1}{2}\sum_{s=0}^{\infty} \binom{-\frac{1}{2}}{s} \frac{1}{2s+1} . \frac{2.4.6...(2s)}{3.5.7...(2s+1)} \\
&= \frac{1}{16}\pi^2 + \frac{1}{2}\sum_{s=0}^{\infty} \binom{-\frac{1}{2}}{s} \frac{1}{2s+1} \int_0^{\pi/2} (\sin x)^{2s+1}\, dx \\
&= \frac{1}{16}\pi^2 + \frac{1}{2}\int_0^{\pi/2}\int_0^{x} \cos y \sum_{s=0}^{\infty} \binom{-\frac{1}{2}}{s} (\sin y)^{2s}\, dy dx \\
&= \frac{1}{16}\pi^2 + \frac{1}{2}\int_0^{\pi/2}\int_0^{x} \frac{\cos y dy}{\sqrt{1+\sin^2 y}} dx
\end{aligned}$$

$$= \frac{1}{16}\pi^2 + \frac{1}{2}\int_0^{\pi/2} \log\left(\sin x + \sqrt{1+\sin^2 x}\right) dx$$
$$= \frac{1}{16}\pi^2 + \frac{1}{2}\left[x \log\left(\sin x + \sqrt{1+\sin^2 x}\right) |_0^{\pi/2} - \int_0^{\pi/2} \frac{x\cos x}{\sqrt{1+\sin^2 x}} dx\right]$$
$$= \frac{1}{16}\pi^2 + \frac{1}{4}\pi \log\left(1+\sqrt{2}\right) - \int_0^1 \frac{\sin^{-1} u}{\sqrt{1+u^2}}, \quad u = \sin x$$
$$= \frac{3}{16}\pi^2 + \frac{1}{4}\pi \log\left(1+\sqrt{2}\right).$$

Thus, from where it is immediately seen that

$$\lim_{n\to\infty} \|C_n(1/4,1)\|_2 \leq 3\pi^2 + 4\pi \log\left(1+\sqrt{2}\right).$$

Similarly, we get

$$\sqrt{3\pi^2 + 4\pi \log\left(1+\sqrt{2}\right)} \leq \lim_{n\to\infty} \|C_n(1/4,1)\|_2 .$$

Thus, the proof is completed. □

Conjecture 1.

$$\lim_{n\to\infty} \frac{1}{n}\left(\left\|C_n\left(\frac{1}{4},1\right)\right\|_E^2 - \left\|C_n\left(\frac{3}{4},1\right)\right\|_E^2\right) = \frac{K}{16},$$

where $K = 0.915965594177$ is Catalan constant.

Conjecture 2. *Let the $n\times n$ Circulant-Cauchy-Toeplitz matrix $C_n\left(\frac{1}{\pi^2},1\right)$ be as $C_n\left(\frac{1}{\pi^2},1\right) = [c_{ij}]_{n\times n}$ such that*

$$c_{ij} = \frac{1}{\left(\frac{1}{\pi^2}\right) + [mod\,(j-i,n)]^2}.$$

Then,

$$\frac{\pi^2\sqrt{-e^4+8e^2-3}}{2(e^2-1)} \leq \lim_{n\to\infty} \left\|C_n\left(\frac{1}{\pi^2},1\right)\right\|_2 \leq \frac{\pi^4\left(-e^4+8e^2-3\right)}{4\left(e^2-1\right)^2}.$$

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