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ON ALGEBRAIC PROPERTIES OF THE GENERALIZED CHEBYSHEV POLYNOMIALS

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Abstract. Chebyshev polynomials are of great importance in many areas of mathematics, particularly approximation theory. Numerous articles and books have been written about this topic. Analytical properties of Chebyshev polynomials are well understood, but algebraic properties less so. In this paper, new generalized Chebyshev polynomials of the first and second kinds have been introduced and studied. Many of the properties of these polynomials are proved.

1. Introduction

Diophantus raised the following problem ($[6]$, pp. $179 - 181$): " *To find four numbers such that the product of any two increased by unity is a sequare*", for which he obtained the solution $\frac{1}{16}$, $\frac{33}{16}$, $\frac{68}{16}$, $\frac{105}{16}$.

Fermat ($[5]$, p. 251) found the solution 1, 3, 8, 120.

A. Baker and H. Davenport [1] studied this question and concluded that, in fact, 120 is the unique integer satisfying the problem raised by J.H. van Lint.

In 1977, V.E. Hoggatt and G.E. Bergum [7] observed that 1, 3, 8 are, respectively, the terms F_2, F_4, F_6, of the Fibonacci sequence $(F_n)_{n\geq 0}$, defined by the conditions

$$F_0 = 0,\ \ F_1 = 1 \ \text{ and } \ F_{n+2} = F_{n+1} + F_n,\ n \geq 0,$$

and formulated the problem of finding a positive integer n such that

$$F_{2t}n + 1,\ \ F_{2t+2}n + 1,\ \ F_{2t+4}n + 1$$

be perfect squares.

Hoggatt and Bergum [7] obtained the number

$$n = 4F_{2t+1}F_{2t+2}F_{2t+3},$$

which, for $t = 1$, gives exactly $n = 120$.

In 1984, this result was generalized ($[3]$, p. 443), by showing that the product of any two distinct elements of the set

$$\{F_n, F_{n+2r}, F_{n+4r}, 4F_{n+r}F_{n+2r}F_{n+3r}\},$$

increased by $\pm F_a^2 F_b^2$ (for suitable integers a and b) is a perfect square, i.e., this set is a Fibonacci quadruple.

For easy reference, the definitions of the Chebyshev polynomials are presented. The sequence of Chebyshev polynomials of the first kind is the sequence $(T_n(x))_{n\geq 0}$, where $x \in \mathbb{C}$, defined by the recurrence relation

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x), \tag{1}$$

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with $T_0(x) = 1$ and $T_1(x) = x$. Thus, one has

$$T_2(x) = 2x^2 - 1,\ \ T_3(x) = 4x^3 - 3x,\ \ T_4(x) = 8x^4 - 8x^2 + 1,\ \ \ldots.$$

The sequence of Chebyshev polynomials of the second kind is the sequence $(U_n(x))_{n\geq 0}$, where $x \in \mathbb{C}$, defined by the same recurrence relation

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x),$$

with $U_0(x) = 1$ and $U_1(x) = 2x$. Thus, one has

$$U_2(x) = 4x^2 - 1,\ \ U_3(x) = 8x^3 - 4x,\ \ U_4(x) = 16x^4 - 12x^2 + 1,\ \ \ldots.$$

In [2], Gheorghe Udrea generalizes a result obtained in [3], by showing that, if $(U_n)_{n\geq 0}$ is the sequence of Chebyshev polynomials of the second kind, then the product of any two distinct elements of the set

$$\{U_n, U_{n+2r}, U_{n+4r}, 4U_{n+r}U_{n+2r}U_{n+3r}\},\quad r, n \in \mathbb{N},$$

increased by $U_a^2 U_b^2$, for suitable nonnegative integers a and b, is a perfect sequare.

In 1995, Morgado [4] obtained, for the Chebyshev polynomials of the first kind, a result analogous to that obtained by Gheorghe Udrea [2] for the Chebyshev polynomials of the second kind.

2. Finding a closed form for generalized Chebyshev polynomials of the first and second kind

In this section, a new generalization of the Chebyshev polynomials of the first and second kinds are introduced. It should be noted that the recurrence formula of these polynomials depend on two integral parameters instead of one parameter. Also, in here, the generating functions for the generalized Chebyshev polynomials of the first and second kinds are given.

Definition 1. *For any complex numbers x, t, the nth sequence of generalized Chebyshev polynomials of the first kind, say $\{H_n(x,t)\}_{n\geq 0}$ is defined recurrently by*

$$H_{n+2}(x,t) = 2xH_{n+1}(x,t) - tH_n(x,t), \tag{2}$$

with $H_0(x,t) = 1$ and $H_1(x,t) = x$.

Thus, one has

$$H_2(x,t) = 2x^2 - t,\ \ H_3(x,t) = 4x^3 - 3xt,\ \ H_4(x,t) = 8x^4 - 8tx^2 + t^2,\ \ \ldots.$$

Definition 2. *For any complex numbers x, t, the nth sequence of generalized Chebyshev polynomials of the second kind, say $\{G_n(x,t)\}_{n\geq 0}$ is defined recurrently by*

$$G_{n+2}(x,t) = 2xG_{n+1}(x,t) - tG_n(x,t), \tag{3}$$

with $G_0(x,t) = 1$ and $G_1(x,t) = 2x$.

Thus, one has

$$G_2(x,t) = 4x^2 - t,\ \ G_3(x,t) = 8x^3 - 4xt,\ \ G_4(x,t) = 16x^4 - 12tx^2 + t^2,\ \ \ldots.$$

Sometimes, instead of $H_n(x,t)$ and $G_n(x,t)$, we shall write plainly H_n and G_n, respectively.

Why should one care about the generating function for a sequence? There are several answers, but here is one: if we can find a generating function for a sequence, then we can often find a closed form for the nth coefficient—which can be pretty useful! There are several approaches. For a generating function that is a ratio of polynomials, we can use the method of partial fractions, which we learned in calculus. Just as the terms in a partial fractions expansion are easier to integrate, the coefficients of those terms are easy to compute.

Let us suppose that the generalized Chebyshev polynomials of the first kind are the coefficients of a potential series centered at the origin, and let us consider the corresponding analytic function $g_1(y)$. The function defined in such a way is called the generating function of the generalized Chebyshev polynomials of the first kind. So,

$$g_1(y) = H_0 + H_1 y + H_2 y^2 + \ldots + H_n y^n + \ldots. \tag{4}$$

By taking into account the recurrence relation (1), we are led to consider the reducing polynomial

$$k(y) = 1 - 2xy + ty^2.$$

One has clearly

$$\begin{aligned} g_1(y)\, k(y) &= \left[H_0 + H_1 y + H_2 y^2 + \ldots + H_n y^n + \ldots\right]\left(1 - 2xy + ty^2\right) \\ &= H_0 + [H_1 - 2xH_0]\, y + \ldots \\ &+ \ldots + [H_n - 2xH_{n-1} + tH_{n-2}]\, y^n + \ldots \\ &= 1 - xy, \end{aligned}$$

since, by (2), $H_n - 2xH_{n-1} + tH_{n-2}$ is the zero polynomial for $n \geq 2$. Thus, one obtains the generating function, $g_1(y)$, under a finite form,

$$g_1(y) = \frac{1 - xy}{1 - 2xy + ty^2},$$

which can be written as

$$\begin{aligned} tg_1(y) &= \frac{1 - xy}{\left[y - \left(\frac{x+\sqrt{x^2-t}}{t}\right)\right]\left[y - \left(\frac{x-\sqrt{x^2-t}}{t}\right)\right]} \\ &= \frac{A}{y - \left(\frac{x+\sqrt{x^2-t}}{t}\right)} + \frac{B}{y - \left(\frac{x-\sqrt{x^2-t}}{t}\right)}, \end{aligned}$$

where

$$\begin{cases} A + B = -x \\ A\left(\frac{x-\sqrt{x^2-t}}{t}\right) + B\left(\frac{x+\sqrt{x^2-t}}{t}\right) = -1. \end{cases}$$

From this, it follows (with $x \neq t$) that

$$A = \frac{1}{2\sqrt{x^2 - t}}\left(t - x^2 - x\sqrt{x^2 - t}\right),$$

and

$$B = \frac{1}{2\sqrt{x^2 - t}}\left(x^2 - t - x\sqrt{x^2 - t}\right),$$

and, consequently,

$$\begin{aligned} g_1(y) &= \frac{A}{ty-\left(x+\sqrt{x^2-t}\right)}+\frac{B}{ty-\left(x-\sqrt{x^2-t}\right)} \\ &= \frac{1}{2\sqrt{x^2-t}}\left[\frac{t-x^2-x\sqrt{x^2-t}}{ty-\left(x+\sqrt{x^2-t}\right)}+\frac{x^2-t-x\sqrt{x^2-t}}{ty-\left(x-\sqrt{x^2-t}\right)}\right] \\ &= \frac{1}{2\sqrt{x^2-t}}\left[\frac{-\left(x^2-t\right)-x\sqrt{x^2-t}}{ty-\left(x+\sqrt{x^2-t}\right)}+\frac{x^2-t-x\sqrt{x^2-t}}{ty-\left(x-\sqrt{x^2-t}\right)}\right] \\ &= \frac{1}{2\sqrt{x^2-t}}\left[\frac{-\sqrt{x^2-t}\left(x+\sqrt{x^2-t}\right)}{ty-\left(x+\sqrt{x^2-t}\right)}+\frac{-\sqrt{x^2-t}\left(x-\sqrt{x^2-t}\right)}{ty-\left(x-\sqrt{x^2-t}\right)}\right] \\ &= \frac{1}{2}\left[\frac{-1}{\frac{ty}{x+\sqrt{x^2-t}}-1}+\frac{-1}{\frac{ty}{x-\sqrt{x^2-t}}-1}\right] \\ &= \frac{1}{2}\left[\frac{1}{1-\left(x-\sqrt{x^2-t}\right)y}+\frac{1}{1-\left(x+\sqrt{x^2-t}\right)y}\right] \\ &= \frac{1}{2}\left[1+\left(x+\sqrt{x^2-t}\right)y+\left(x+\sqrt{x^2-t}\right)^2y^2+\right. \\ &\left.+\ldots+\left(x+\sqrt{x^2-t}\right)^ny^n+\ldots\right] \\ &+\frac{1}{2}\left[1+\left(x-\sqrt{x^2-t}\right)y+\left(x-\sqrt{x^2-t}\right)^2y^2+\right. \\ &\left.+\ldots+\left(x-\sqrt{x^2-t}\right)^ny^n+\ldots\right]. \end{aligned}$$

Since, by (4) is the coefficient of y^n, one concludes that

$$H_n=\frac{1}{2}\left[\left(x+\sqrt{x^2-t}\right)^n+\left(x-\sqrt{x^2-t}\right)^n\right]. \tag{5}$$

For the generalized Chebyshev polynomials of the second kind, one finds, by a similar way, the corresponding generating function, under a finite form (with $x\neq t$):

$$g_2(y)=\frac{1}{1-2xy+ty^2}$$

which can be written as

$$tg_2(y)=\frac{C}{y-\left(\frac{x+\sqrt{x^2-t}}{t}\right)}+\frac{D}{y-\left(\frac{x-\sqrt{x^2-t}}{t}\right)}$$

where

$$\begin{cases} C+D=0 \\ C\left(\frac{x-\sqrt{x^2-t}}{t}\right)+D\left(\frac{x+\sqrt{x^2-t}}{t}\right)=-1. \end{cases}$$

From this, it follows that

$$C=\frac{t}{2\sqrt{x^2-t}} \quad \text{and} \quad D=-\frac{t}{2\sqrt{x^2-t}},$$

and, consequently,

$$\begin{aligned} g_2(y) &= \frac{t}{2\sqrt{x^2-t}}\left[\frac{1}{ty-\left(x+\sqrt{x^2-t}\right)}-\frac{1}{ty-\left(x-\sqrt{x^2-t}\right)}\right] \\ &= \frac{t}{2\sqrt{x^2-t}}\left[\frac{\frac{1}{x+\sqrt{x^2-t}}}{\frac{ty}{x+\sqrt{x^2-t}}-1}-\frac{\frac{1}{x-\sqrt{x^2-t}}}{\frac{ty}{x-\sqrt{x^2-t}}-1}\right] \\ &= \frac{t}{2\sqrt{x^2-t}}\left[\frac{\frac{x-\sqrt{x^2-t}}{t}}{\left(x-\sqrt{x^2-t}\right)y-1}-\frac{\frac{x+\sqrt{x^2-t}}{t}}{\left(x+\sqrt{x^2-t}\right)y-1}\right] \\ &= \frac{1}{2\sqrt{x^2-t}}\left[\frac{x+\sqrt{x^2-t}}{1-\left(x+\sqrt{x^2-t}\right)y}-\frac{x-\sqrt{x^2-t}}{1-\left(x-\sqrt{x^2-t}\right)y}\right], \end{aligned}$$

and one obtains, after the developments in power series of

$$\frac{x+\sqrt{x^2-t}}{1-\left(x+\sqrt{x^2-t}\right)y} \quad \text{and} \quad \frac{x-\sqrt{x^2-t}}{1-\left(x-\sqrt{x^2-t}\right)y},$$

$$G_n = \frac{1}{2\sqrt{x^2-t}}\left[\left(x+\sqrt{x^2-t}\right)^{n+1}-\left(x-\sqrt{x^2-t}\right)^{n+1}\right]. \tag{6}$$

Since, for $x \in \mathbb{C}$, there is some $\theta \in \mathbb{C}$ such that $x = \sqrt{t}\cos\theta$, one can write

$$H_n\left(\sqrt{t}\cos\theta, t\right) = \frac{t^{n/2}}{2}\left[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\right],$$

or

$$H_n\left(\sqrt{t}\cos\theta, t\right) = t^{n/2}\cos n\theta, \tag{7}$$

and

$$G_n\left(\sqrt{t}\cos\theta, t\right) = \frac{t^{n/2}}{2i\sin\theta}\left[(\cos\theta + i\sin\theta)^{n+1} - (\cos\theta - i\sin\theta)^{n+1}\right],$$

or

$$G_n\left(\sqrt{t}\cos\theta, t\right) = \frac{t^{n/2}\sin(n+1)\theta}{\sin\theta}. \tag{8}$$

By means of the relations (7) and (8), it is easy to see that the following connections, between the two kinds of generalized polynomials, hold:

$$H_n(x,t) = \sqrt{t^n}G_n(x,t) - x\sqrt{t^{n-1}}G_{n-1}(x,t), \;\; n \ge 1, \tag{9}$$

$$\left(t-x^2\right)G_n(x,t) = x\sqrt{t^{n+2}}H_{n+1}(x,t) - \sqrt{t^{n+3}}H_{n+2}(x,t), \;\; n \ge 0, \tag{10}$$

$$G_n - tG_{n-2} = 2H_n, \;\; n \ge 2. \tag{11}$$

By taking into account (7), it is natural to extend the meaning of H_n for $n < 0$: one puts

$$\begin{aligned} H_{-r}(x,t) &= H_{-r}\left(\sqrt{t}\cos\theta, t\right) \\ &= t^{-r/2}\cos(-r)\theta \\ &= t^{-r}H_r(x,t). \end{aligned}$$

3. Some properties of the generalized Chebyshev polynomials of the first kind

Lemma 1. *If $(H_n)_{n\geq 0}$ is the sequence of the generalized Chebyshev polynomials of the first kind, one has:*

$$H_nH_{n+r+s}+\frac{1}{2}t^n\left[t^sH_{r-s}-H_{r+s}\right]=H_{n+r}H_{n+s}. \tag{12}$$

Proof. By setting $x=\sqrt{t}\cos\theta$ (and so $H_n=t^{(n+2)/2}\cos n\theta$), one has

$$\begin{aligned} H_nH_{n+r+s} &= t^{\frac{2n+r+s}{2}}\cos n\theta\cos\left(n+r+s\right)\theta \\ &= \frac{1}{2}t^{\frac{2n+r+s}{2}}\left[\cos\left(2n+r+s\right)\theta+\cos\left(r+s\right)\theta\right], \end{aligned}$$

and

$$\begin{aligned} H_{n+r}H_{n+s} &= t^{\frac{2n+r+s}{2}}\cos\left(n+r\right)\theta\cos\left(n+s\right)\theta \\ &= \frac{1}{2}t^{\frac{2n+r+s}{2}}\left[\cos\left(2n+r+s\right)\theta+\cos\left(r-s\right)\theta\right], \end{aligned}$$

and, consequently,

$$\begin{aligned} H_nH_{n+r+s}-H_{n+r}H_{n+s} &= \frac{1}{2}t^{\frac{2n+r+s}{2}}\left[\cos\left(r+s\right)\theta-\cos\left(r-s\right)\theta\right] \\ &= \frac{1}{2}t^n\left[t^{\frac{r+s}{2}}\cos\left(r+s\right)\theta-t^st^{\frac{r-s}{2}}\cos\left(r-s\right)\theta\right] \\ &= \frac{1}{2}t^n\left[H_{r+s}-t^sH_{r-s}\right]. \end{aligned}$$

Hence,

$$H_nH_{n+r+s}+\frac{1}{2}t^n\left[t^sH_{r-s}-H_{r+s}\right]=H_{n+r}H_{n+s},$$

which proves (12). □

Corollary 1. *If $(H_n)_{n\geq 0}$ is the sequence of the generalized Chebyshev polynomials of the first kind, one has:*

$$\begin{aligned} 4H_nH_{n+r}H_{n+s}H_{n+r+s}+\frac{1}{4}t^{2n+2}\left[t^sH_{r-s}-H_{r+s}\right]^2 &= [H_nH_{n+r+s} \qquad (13)\\ &\quad +H_{n+r}H_{n+s}]^2. \qquad (14) \end{aligned}$$

Proof. From (12), one has clearly

$$\frac{1}{4}t^{2n}\left[t^sH_{r-s}-H_{r+s}\right]^2=H_{n+r}^2H_{n+s}^2+H_n^2H_{n+r+s}^2-2H_nH_{n+r}H_{n+s}H_{n+r+s},$$

and so

$$4H_nH_{n+r}H_{n+s}H_{n+r+s}+\frac{1}{4}t^{2n}\left[t^sH_{r-s}-H_{r+s}\right]^2=\left[H_nH_{n+r+s}+H_{n+r}H_{n+s}\right]^2,$$

which proves (14). □

Now, we are going to state the following

Theorem 1. *If $(H_n(x,t))_{n\geq 0}$, with $t\geq 1$, is the sequence of generalized Chebyshev polynomials of the first kind, then the product of any two distinct elements of the set*

$$\left\{H_n,H_{n+2r},H_{n+4r},4H_{n+r}H_{n+2r}H_{n+3r}\right\},\quad r,n\in\mathbb{N},$$

increased by $\left[\frac{1}{2}t^n\left(t^rH_p - H_q\right)\right]^m$*, where* H_p *and* H_q*, with* $q > p \geq 0$*, are suitable terms of the sequence* $\left(H_n\left(x,t\right)\right)_{n\geq 0}$*, and* m *is* 1 *or* 2*, according to the number of factors* H_n*, in that product, is* 2 *or* 4*, is a perfect sequare.*

Proof. Indeed, if one sets $s = r$, in (12), one obtains

$$H_nH_{n+2r} + \frac{1}{2}t^n\left[t^rH_0 - H_{2r}\right] = H^2_{n+r}. \tag{15}$$

If r is replaced by $2r$, in (15), one gets

$$H_nH_{n+4r} + \frac{1}{2}t^n\left[t^{2r}H_0 - H_{4r}\right] = H^2_{n+2r}. \tag{16}$$

By replacing, in (15) n by $n + 2r$, one obtains

$$H_{n+2r}H_{n+4r} + \frac{1}{2}t^{n+2r}\left[t^rH_0 - H_{2r}\right] = H^2_{n+3r}. \tag{17}$$

If one puts $s = 2r$, in (14), one gets

$$4H_nH_{n+r}H_{n+2r}H_{n+3r} + \frac{1}{4}t^{2n}\left[t^{2r}H_{-r} - H_{3r}\right]^2 = \left[H_nH_{n+3r} + H_{n+r}H_{n+2r}\right]^2,$$

or

$$4H_nH_{n+r}H_{n+2r}H_{n+3r} + \frac{1}{4}t^{2n}\left[t^rH_r - H_{3r}\right]^2 = [H_nH_{n+3r} + H_{n+r}H_{n+2r}]^2. \tag{18}$$

Now, by changing n into $n + r$, in (18), it comes

$$4H_{n+r}H_{n+2r}H_{n+3r}H_{n+4r} + \frac{1}{4}t^{2n+2r}\left[t^rH_r - H_{3r}\right]^2 = [H_{n+r}H_{n+4r} + H_{n+2r}H_{n+3r}]^2. \tag{19}$$

If one replaces n by $n + r$, in (14), and, furthermore, one puts $s = r$, one obtains

$$4H_{n+r}H^2_{n+2r}H_{n+3r} + \frac{1}{4}t^{2n+2r}\left[t^rH_0 - H_{2r}\right]^2 = [H_{n+r}H_{n+3r} + H^2_{n+2r}]^2, \tag{20}$$

which completes the proof of the theorem above. □

4. Conclusions

New generalized Chebyshev polynomials of the first and second kinds have been introduced and studied. Many of the properties of these polynomials are proved.

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