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# OSCILLATION OF A CLASS OF TWO-VARIABLES FUNCTIONAL EQUATIONS WITH VARIABLE COEFFICIENTS

WENGUI YANG

ABSTRACT. In this paper we will establish some sufficient conditions of oscillation of a class of two-variables functional equations with variable coefficients. Our results extend Zhang and Zhou's results (B.G. Zhang and Y. Zhou, Comput. Math. Appl. 42 (3-5) (2001) 369-378).

## 1. INTRODUCTION

The study of oscillation of functional equations is a relatively new field, but the qualitative theory of oscillation of functional equations has attracted many experts and mathematical workers. The proliferation of this area has been witnessed by several hundreds of research papers and a number of research monographs, i.e. [1-8].

The main aim of this paper is to establish some sufficient conditions of oscillation of the following two-variables functional equation with variable coefficients

$$\mathbf{a}(x,y)\mathbf{A}(x,y)+\mathbf{b}(x,y)\mathbf{A}(x,\sigma(y))+\mathbf{c}(x,y)\mathbf{A}(\tau(x),y)-\mathbf{d}(x,y)\mathbf{A}(\tau(x),\sigma(y))$$
$$+\mathbf{p}(x,y)\mathbf{A}(\tau^{k+1}(x),\sigma^{l+1}(y))=0,\quad x,y\in I \tag{1}$$

where $I$ is an unbounded subset of $\mathbb{R}^+$, $k\geq 1$ and $l\geq 1$ are positive integers, $\tau,\sigma: I\to I$, $\tau(x)\neq x$, $\sigma(y)\neq y$, and $\lim_{x\to\infty}\tau(x)=\infty$, $\lim_{y\to\infty}\sigma(y)=\infty$, $x,y\in I$. By $\tau^i$ and $\sigma^i$ we denote the $i-$th iterate of the function $\tau$ and $\sigma$, respectively, i.e.

$$\tau^0(x)=x,\quad \tau^{i+1}(x)=\tau(\tau^i(x)),\quad i=0,1,\cdots,\quad x\in I,$$

and

$$\sigma^0(y)=y,\quad \sigma^{i+1}(y)=\sigma(\sigma^i(y)),\quad i=0,1,\cdots,\quad y\in I.$$

We always assume that the following hypotheses holds:
($H_1$) Functions $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ and $\mathbf{p}: I\times I\to\mathbb{R}^+$, $\mathbf{a}(x,y)\geq a$, $\mathbf{b}(x,y)\geq b$, $\mathbf{c}(x,y)\geq c$ and $\mathbf{d}(x,y)\leq d$ for all large $x,y\in I$, where $a,b,c$ and $d$ are positive numbers, $\mathbb{R}^+$ denotes the set of positive real numbers.

Our results extend the recent results in the paper [1].

**Definition 1.** *A solution $\mathbf{A}(x,y)$ of equation (1) is said to be eventually positive if $\mathbf{A}(x,y)>0$ for all large $x$ and $y$, and eventually negative if $\mathbf{A}(x,y)<0$ for all large $x$ and $y$. It is said to oscillatory if it is neither eventually positive nor eventually negative.*

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## 2. Main results

In this section, we give our main results and we assume that the following is satisfied through the paper without furthermore mention:
($H_2$) $\limsup_{I\ni x,y\to\infty} \mathbf{p}(x,y) > 0$.
Define a set $E$ and $I_\alpha$ by

$$E = \{\lambda > 0 | d - \lambda \mathbf{p}(x,y) > 0 \ \text{ eventually}\} \tag{2}$$

and

$$I_\alpha = [\alpha, \infty) \cap I \ \text{ for } \ \alpha \in \mathbb{R}^+ \tag{3}$$

respectively.

**Theorem 1.** *Assume that* ($H_1$) *and* ($H_2$) *hold and there exist* $X, Y \in I$ *such that for* $k > l$

$$\sup_{\lambda\in E, x\in I_X, y\in I_Y} \lambda \prod_{i=1}^{l} (d - \lambda \mathbf{p}(\tau^i(x), \sigma^i(y))) \prod_{j=1}^{k-l} (d - \lambda \mathbf{p}(\tau^{l+j}(x), \sigma^l(y))) < \left(a + \frac{2bc}{d}\right)^l b^{k-l}, \tag{4}$$

*and for* $l > k$

$$\sup_{\lambda\in E, x\in I_X, y\in I_Y} \lambda \prod_{i=1}^{k} (d - \lambda \mathbf{p}(\tau^i(x), \sigma^i(y))) \prod_{j=1}^{l-k} (d - \lambda \mathbf{p}(\tau^{k}(x), \sigma^{k+j}(y))) < \left(a + \frac{2bc}{d}\right)^l c^{l-k}, \tag{5}$$

*Then every solution of equation (1) oscillates.*

*Proof.* Suppose to the contrary, we let $\mathbf{A}(x,y)$ be an eventually positive solution of (1). We define the set $S(A)$ of positive numbers as follows:

$$\begin{aligned} S(A) = \{\lambda > 0 | a\mathbf{A}(x,y) + b\mathbf{A}(x,\sigma(y)) + c\mathbf{A}(\tau(x),y) \\ - (d - \lambda \mathbf{p}(x,y))\mathbf{A}(\tau(x),\sigma(y)) \le 0 \ \text{ eventually}\}. \end{aligned} \tag{6}$$

From (1) and ($H_1$), we have

$$a\mathbf{A}(x,y) + b\mathbf{A}(x,\sigma(y)) + c\mathbf{A}(\tau(x),y) \le d\mathbf{A}(\tau(x),\sigma(y)). \tag{7}$$

If $k > l$, then we obtain

$$\begin{aligned} \mathbf{A}(\tau(x),\sigma(y)) &< \left(\frac{d}{a}\right)^l \mathbf{A}(\tau^{l+1}(x), \sigma^{l+1}(y)) \\ &< \left(\frac{d}{a}\right)^l \left(\frac{d}{b}\right)^{k-l} \mathbf{A}(\tau^{k+1}(x), \sigma^{l+1}(y)). \end{aligned} \tag{8}$$

If $l > k$, then we obtain

$$\begin{aligned} \mathbf{A}(\tau(x),\sigma(y)) &< \left(\frac{d}{a}\right)^k \mathbf{A}(\tau^{k+1}(x), \sigma^{k+1}(y)) \\ &< \left(\frac{d}{a}\right)^k \left(\frac{d}{b}\right)^{l-k} \mathbf{A}(\tau^{k+1}(x), \sigma^{l+1}(y)). \end{aligned} \tag{9}$$

Substituting (7) and (8) into (1), due to ($H_1$), we obtain

$$a\mathbf{A}(x,y)+b\mathbf{A}(x,\sigma(y))+c\mathbf{A}(\tau(x),y)$$
$$-\left(d-\left(\frac{a}{d}\right)^{l}\left(\frac{b}{d}\right)^{k-l}\mathbf{p}(x,y)\right)\mathbf{A}(\tau(x),\sigma(y))\le 0 \tag{10}$$

and

$$a\mathbf{A}(x,y)+b\mathbf{A}(x,\sigma(y))+c\mathbf{A}(\tau(x),y)$$
$$-\left(d-\left(\frac{d}{a}\right)^{l}\left(\frac{d}{b}\right)^{l-k}\mathbf{p}(x,y)\right)\mathbf{A}(\tau(x),\sigma(y))\le 0 \tag{11}$$

respectively. Inequalities (9) and (10) show that $S(A)$ is nonempty. For $\lambda\in S(A)$, we have eventually

$$d-\lambda\mathbf{p}(x,y)>0, \tag{12}$$

which implies that $S(A)\subseteq E$. Due to condition ($H_2$), the set $E$ is bounded, and hence $S(A)$ is also bounded. From (7), we have

$$\mathbf{A}(\tau(x),\sigma(y))\le\frac{d}{b}\mathbf{A}(\tau^2(x),\sigma(y)),\quad \mathbf{A}(\tau(x),\sigma(y))\le\frac{d}{c}\mathbf{A}(\tau(x),\sigma^2(y)). \tag{13}$$

Let $\mu\in S(A)$. Then

$$\begin{aligned}&\left(a+\frac{2bc}{d}\right)\mathbf{A}(\tau(x),\sigma(y))\\ &\le a\mathbf{A}(\tau(x),\sigma(y))+b\mathbf{A}(\tau(x),\sigma^2(y))+c\mathbf{A}(\tau^2(x),\sigma(y))\\ &\le (d-\mu\mathbf{p}(\tau(x),\sigma(y)))\mathbf{A}(\tau^2(x),\sigma^2(y)).\end{aligned}$$

If $k>l$, we have

$$\mathbf{A}(\tau(x),\sigma(y))\le\left(a+\frac{2bc}{d}\right)^{-l}\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^i(x),\sigma^i(y)))\mathbf{A}(\tau^{l+1}(x),\sigma^{l+1}(y)),$$

$$\begin{aligned}&\mathbf{A}(\tau^{l+1}(x),\sigma^{l+1}(y))\\ &\le\frac{1}{b}(d-\mu\mathbf{p}(\tau^{l+1}(x),\sigma^{l}(y)))\mathbf{A}(\tau^{l+2}(x),\sigma^{l+1}(y))\\ &\le\cdots\le\left(\frac{1}{b}\right)^{k-l}\prod_{j=1}^{k-l}(d-\mu\mathbf{p}(\tau^{l+j}(x),\sigma^{l}(y)))\mathbf{A}(\tau^{k+1}(x),\sigma^{l+1}(y)).\end{aligned}$$

Combining the above two inequalities, we obtain

$$\begin{aligned}\mathbf{A}(\tau(x),\sigma(y))\le&\left(a+\frac{2bc}{d}\right)^{-l}b^{l-k}\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^i(x),\sigma^i(y)))\\ &\times\prod_{j=1}^{k-l}(d-\mu\mathbf{p}(\tau^{l+j}(x),\sigma^{l}(y)))\mathbf{A}(\tau^{k+1}(x),\sigma^{l+1}(y)).\end{aligned} \tag{14}$$

Similarly, if $l > k$, we have

$$\mathbf{A}(\tau(x),\sigma(y)) \le \left(a+\frac{2bc}{d}\right)^{-k} c^{k-l}\prod_{i=1}^{k}(d-\mu\mathbf{p}(\tau^i(x),\sigma^i(y))) \times \prod_{j=1}^{l-k}(d-\mu\mathbf{p}(\tau^k(x),\sigma^{k+j}(y)))\mathbf{A}(\tau^{k+1}(x),\sigma^{l+1}(y)). \tag{15}$$

Substituting (14) and (15) into (1), corresponding to condition ($\mathrm{H}_1$), we find respectively, for $k > l$,

$$\begin{aligned} &a\mathbf{A}(x,y)+b\mathbf{A}(x,\sigma(y))+c\mathbf{A}(\tau(x),y)-d\mathbf{A}(\tau(x),\sigma(y)) \\ &+\mathbf{p}(x,y)\mathbf{A}(\tau(x),\sigma(y))\left(a+\frac{2bc}{d}\right)^{l} b^{k-l} \\ &\times\left(\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^i(x),\sigma^i(y)))\prod_{j=1}^{k-l}(d-\mu\mathbf{p}(\tau^{l+j}(x),\sigma^{l}(y)))\right)^{-1}\le 0, \end{aligned} \tag{16}$$

and, for $l > k$,

$$\begin{aligned} &a\mathbf{A}(x,y)+b\mathbf{A}(x,\sigma(y))+c\mathbf{A}(\tau(x),y)-d\mathbf{A}(\tau(x),\sigma(y)) \\ &+\mathbf{p}(x,y)\mathbf{A}(\tau(x),\sigma(y))\left(a+\frac{2bc}{d}\right)^{k} c^{l-k} \\ &\times\left(\prod_{i=1}^{k}(d-\mu\mathbf{p}(\tau^i(x),\sigma^i(y)))\prod_{j=1}^{l-k}(d-\mu\mathbf{p}(\tau^{k}(x),\sigma^{k+j}(y)))\right)^{-1}\le 0. \end{aligned} \tag{17}$$

Hence we have respectively, for $k > l$,

$$\begin{aligned} &a\mathbf{A}(x,y)+b\mathbf{A}(x,\sigma(y))+c\mathbf{A}(\tau(x),y)-\Bigg\{d-\mathbf{p}(x,y)\left(a+\frac{2bc}{d}\right)^{l} b^{k-l} \\ &\times\sup_{x\in I_X,y\in I_Y}\left(\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^i(x),\sigma^i(y)))\prod_{j=1}^{k-l}(d-\mu\mathbf{p}(\tau^{l+j}(x),\sigma^{l}(y)))\right)^{-1}\Bigg\} \\ &\times\mathbf{A}(\tau(x),\sigma(y))\le 0, \end{aligned} \tag{18}$$

and, for $l > k$,

$$\begin{aligned} &a\mathbf{A}(x,y)+b\mathbf{A}(x,\sigma(y))+c\mathbf{A}(\tau(x),y)-\Bigg\{d-\mathbf{p}(x,y)\left(a+\frac{2bc}{d}\right)^{k} c^{l-k} \\ &\times\sup_{x\in I_X,y\in I_Y}\left(\prod_{i=1}^{k}(d-\mu\mathbf{p}(\tau^i(x),\sigma^i(y)))\prod_{j=1}^{l-k}(d-\mu\mathbf{p}(\tau^{k}(x),\sigma^{k+j}(y)))\right)^{-1}\Bigg\} \\ &\times\mathbf{A}(\tau(x),\sigma(y))\le 0. \end{aligned} \tag{19}$$

From (18) and (19) we obtain respectively, for $k > l$,

$$\left(a + \frac{2bc}{d}\right)^l b^{k-l} \sup_{x\in I_X, y\in I_Y} \left(\prod_{i=1}^{l}(d - \mu\mathbf{p}(\tau^i(x), \sigma^i(y))) \times \prod_{j=1}^{k-l}(d - \mu\mathbf{p}(\tau^{l+j}(x), \sigma^l(y)))\right)^{-1} \in S(A), \tag{20}$$

and, for $l > k$,

$$\left(a + \frac{2bc}{d}\right)^k c^{l-k} \sup_{x\in I_X, y\in I_Y} \left(\prod_{i=1}^{k}(d - \mu\mathbf{p}(\tau^i(x), \sigma^i(y))) \times \prod_{j=1}^{l-k}(d - \mu\mathbf{p}(\tau^{k}(x), \sigma^{k+j}(y)))\right)^{-1} \in S(A). \tag{21}$$

On the other hand, (4) implies that there exists $\alpha \in (0, 1)$ such that for $k > l$

$$\sup_{\lambda\in E, x\in I_X, y\in I_Y} \lambda \prod_{i=1}^{l}(d - \lambda\mathbf{p}(\tau^i(x), \sigma^i(y))) \prod_{j=1}^{k-l}(d - \lambda\mathbf{p}(\tau^{l+j}(x), \sigma^l(y))) \le \alpha \left(a + \frac{2bc}{d}\right)^l b^{k-l}, \tag{22}$$

and (5) implies that there exists $\alpha \in (0, 1)$ such that for $l > k$

$$\sup_{\lambda\in E, x\in I_X, y\in I_Y} \lambda \prod_{i=1}^{k}(d - \lambda\mathbf{p}(\tau^i(x), \sigma^i(y))) \prod_{j=1}^{l-k}(d - \lambda\mathbf{p}(\tau^{k}(x), \sigma^{k+j}(y))) \le \alpha \left(a + \frac{2bc}{d}\right)^l c^{l-k}. \tag{23}$$

Hence we have respectively, for $k > l$

$$\sup_{x\in I_X, y\in I_Y} \prod_{i=1}^{l}(d - \mu\mathbf{p}(\tau^i(x), \sigma^i(y))) \prod_{j=1}^{k-l}(d - \mu\mathbf{p}(\tau^{l+j}(x), \sigma^l(y))) \le \frac{\alpha}{\mu} \left(a + \frac{2bc}{d}\right)^l b^{k-l}, \tag{24}$$

and for $l > k$

$$\sup_{x\in I_X, y\in I_Y} \prod_{i=1}^{k}(d - \mu\mathbf{p}(\tau^i(x), \sigma^i(y))) \prod_{j=1}^{l-k}(d - \mu\mathbf{p}(\tau^{k}(x), \sigma^{k+j}(y))) \le \frac{\alpha}{\mu} \left(a + \frac{2bc}{d}\right)^l c^{l-k}. \tag{25}$$

From (20) and (24) for $k > l$, (21) and (25) for $l > k$, we have that $\mu/\alpha \in S(A)$. Repeating the above procedure, we conclude that $\mu/\alpha^r \in S(A)$, $r = 1, 2 \cdots$, which contradicts the boundedness of $S(A)$. The proof is completed. □

**Corollary 1.** *Assume that* ($H_1$) *and* ($H_2$) *hold and for* $k > l$

$$\liminf_{I\ni x,y\to\infty} \mathbf{p}(x,y) = P > \frac{d^{k+1}k^k}{(1+k)^{1+k}}\left(a+\frac{2bc}{d}\right)^{-l} b^{l-k} \tag{26}$$

*and for* $l > k$

$$\liminf_{I\ni x,y\to\infty} \mathbf{p}(x,y) = P > \frac{d^{k+1}l^l}{(1+l)^{1+l}}\left(a+\frac{2bc}{d}\right)^{-k} c^{k-l}. \tag{27}$$

*Then every solution of equation (1) oscillates.*

*Proof.* We see that

$$\max_{d/P>\lambda>0} \lambda(d-\lambda P)^k = \frac{d^{k+1}k^k}{P(1+k)^{1+k}}. \tag{28}$$

Hence (26) and (27) imply that (4) and (5) hold. By Theorem 1, every solution of (1) oscillates. The proof is completed. □

**Theorem 2.** *Assume that* ($H_1$) *and* ($H_2$) *hold and there exist* $X, Y \in I$ *such that for* $k > l$

$$\sup_{\lambda\in E, x\in I_X, y\in I_Y} \lambda\left\{\prod_{j=1}^{k-l}\prod_{i=1}^{l}(d-\lambda\mathbf{p}(\tau^{i+j}(x),\sigma^i(y)))\right\}^{\frac{1}{k-l}} < \left(a+\frac{2bc}{d}\right)^{l}\left(\frac{b}{d}\right)^{k-l},$$

*and for* $l > k$

$$\sup_{\lambda\in E, x\in I_X, y\in I_Y} \lambda\left\{\prod_{j=1}^{l-k}\prod_{i=1}^{k}(d-\lambda\mathbf{p}(\tau^{i}(x),\sigma^{i+j}(y)))\right\}^{\frac{1}{l-k}} < \left(a+\frac{2bc}{d}\right)^{k}\left(\frac{c}{d}\right)^{l-k},$$

*Then every solution of equation (1) oscillates.*

*Proof.* For $k > l$, we have

$$\begin{aligned}&\mathbf{A}(\tau(x),\sigma(y))\\ &\le \left(a+\frac{2bc}{d}\right)^{-l}\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^i(x),\sigma^i(y)))\mathbf{A}(\tau^{l+1}(x),\sigma^{l+1}(y)).\end{aligned} \tag{29}$$

From (13) and (29), then

$$\begin{aligned}&\mathbf{A}(\tau^{j+1}(x),\sigma(y))\\ &\le \left(a+\frac{2bc}{d}\right)^{-l}\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^{i+j}(x),\sigma^i(y)))\mathbf{A}(\tau^{l+j+1}(x),\sigma^{l+1}(y))\\ &\le \left(a+\frac{2bc}{d}\right)^{-l}\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^{i+j}(x),\sigma^i(y)))\\ &\times\left(\frac{d}{b}\right)^{k-l-j}\mathbf{A}(\tau^{k+1}(x),\sigma^{l+1}(y)),\end{aligned} \tag{30}$$

for $j = 1, 2, \cdots, k-l$.

Hence due to (30), we obtain

$$\mathbf{A}^{k-l}(\tau(x),\sigma(y)) \leq \prod_{j=1}^{k-l}\left(\frac{d}{b}\right)^{j}\mathbf{A}(\tau^{j+1}(x),\sigma(y))$$

$$\leq \prod_{j=1}^{k-l}\left\{\left(\frac{d}{b}\right)^{j}\left(a+\frac{2bc}{d}\right)^{-l}\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^{i+j}(x),\sigma^{i}(y)))\right.$$

$$\left.\times\left(\frac{d}{b}\right)^{k-l-j}\mathbf{A}(\tau^{k+1}(x),\sigma^{l+1}(y))\right\}$$

$$=\left(a+\frac{2bc}{d}\right)^{-l(k-l)}\left(\frac{d}{b}\right)^{(k-l)^2}$$

$$\times\left\{\prod_{j=1}^{k-l}\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^{i+j}(x),\sigma^{i}(y)))\right\}\mathbf{A}^{k-l}(\tau^{k+1}(x),\sigma^{l+1}(y)).$$

That is,

$$\mathbf{A}(\tau(x),\sigma(y)) \leq \left(a+\frac{2bc}{d}\right)^{-l}\left(\frac{d}{b}\right)^{k-l}$$

$$\times\left\{\prod_{j=1}^{k-l}\prod_{i=1}^{l}(d-\mu\mathbf{p}(\tau^{i+j}(x),\sigma^{i}(y)))\right\}^{\frac{1}{k-l}}\mathbf{A}(\tau^{k+1}(x),\sigma^{l+1}(y)).$$

Similarly, for $l>k$, then

$$\mathbf{A}(\tau(x),\sigma(y)) \leq \left(a+\frac{2bc}{d}\right)^{-k}\left(\frac{d}{b}\right)^{l-k}$$

$$\times\left\{\prod_{j=1}^{l-k}\prod_{i=1}^{k}(d-\mu\mathbf{p}(\tau^{i}(x),\sigma^{i+j}(y)))\right\}^{\frac{1}{l-k}}\mathbf{A}(\tau^{k+1}(x),\sigma^{l+1}(y)).$$

The rest of the proof is similar to that of Theorem 1 and it is thus omitted. □

Due to equation (28), we have immediately the following result.

**Corollary 2.** *Assume that (*$H_1$*) and (*$H_2$*) hold and for* $k>l$

$$\liminf_{I\ni x,y\to\infty}\frac{1}{(k-l)l}\sum_{j=1}^{k-l}\sum_{i=1}^{l}\mathbf{p}(\tau^{i+j}(x),\sigma^{i}(y))$$

$$=P>\frac{d^{l+1}l^{l}}{(1+l)^{1+l}}\left(a+\frac{2bc}{d}\right)^{-l}\left(\frac{d}{b}\right)^{k-l}$$

*and for* $l>k$

$$\liminf_{I\ni x,y\to\infty}\frac{1}{(l-k)k}\sum_{j=1}^{l-k}\sum_{i=1}^{k}\mathbf{p}(\tau^{i}(x),\sigma^{i+j}(y))$$

$$=P>\frac{d^{k+1}k^{k}}{(1+k)^{1+k}}\left(a+\frac{2bc}{d}\right)^{-k}\left(\frac{d}{b}\right)^{l-k}.$$

*Then every solution of equation (1) oscillates.*

**Theorem 3.** *Assume that* (H$_1$) *and* (H$_2$) *hold and there exist* $X, Y \in I$ *such that for* $k = l$

$$\sup_{\lambda \in E, x \in I_X, y \in I_Y} \lambda \prod_{i=1}^{k} (d - \lambda \mathbf{p}(\tau^i(x), \sigma^i(y))) < \left(a + \frac{2bc}{d}\right)^k.$$

*Then every solution of equation (1) oscillates.*

*Proof.* Let $\mu \in S(A)$. Then, from (13)

$$\mathbf{A}(\tau(x), \sigma(y)) \leq \left(a + \frac{2bc}{d}\right)^k \prod_{i=1}^{k} (d - \lambda \mathbf{p}(\tau^i(x), \sigma^i(y))) \mathbf{A}(\tau^{k+1}(x), \sigma^{k+1}(y)).$$

The rest of proof is similar to that of Theorem 1 and it is thus omitted. □

Due to equation (28), we have immediately the following result.

**Corollary 3.** *Assume that* (H$_2$) *holds and for* $k = l$

$$\liminf_{I \ni x, y \to \infty} \mathbf{p}(x, y) = P > d^{k+1} \left(a + \frac{2bc}{d}\right)^{-l} \frac{k^k}{(1+k)^{1+k}}.$$

*Then every solution of equation (1) oscillates.*

**Theorem 4.** *Assume that* (H$_1$) *and* (H$_2$) *hold and there exist* $X, Y \in I$ *such that for* $k, l > 0$

$$\sup_{\lambda \in E, x \in I_X, y \in I_Y} \lambda \left\{ \prod_{i=1}^{k} \prod_{j=1}^{l} (d - \lambda \mathbf{p}(\tau^i(x), \sigma^j(y))) \right\}^{\frac{1}{k}} < c^l \left(\frac{b}{d}\right)^k,$$

*or*

$$\sup_{\lambda \in E, x \in I_X, y \in I_Y} \lambda \left\{ \prod_{j=1}^{l} \prod_{i=1}^{k} (d - \lambda \mathbf{p}(\tau^i(x), \sigma^j(y))) \right\}^{\frac{1}{l}} < b^k \left(\frac{c}{d}\right)^l,$$

*Then every solution of equation (1) oscillates.*

*Proof.* Let $\mu \in S(A)$. Then eventually

$$b\mathbf{A}(x, \sigma(y)) \leq (d - \lambda \mathbf{p}(x, y)) \mathbf{A}(\tau(x), \sigma(y))$$

and

$$c\mathbf{A}(\tau(x), y) \leq (d - \lambda \mathbf{p}(x, y)) \mathbf{A}(\tau(x), \sigma(y))$$

According to the above inequalities and the course of the proof of Theorem 6.3.5 in [5,p.220], so it is thus omitted. □

Due to equation (28), we have immediately the following result.

**Corollary 4.** *Assume that* (H$_1$) *and* (H$_2$) *hold and*

$$\liminf_{I \ni x, y \to \infty} \frac{1}{kl} \sum_{j=1}^{k} \sum_{i=1}^{l} \mathbf{p}(\tau^i(x), \sigma^j(y)) = P > \frac{c^{-l} l^l}{(1+l)^{1+l}} \left(\frac{d}{b}\right)^k$$

*and for* $l > k$

$$\liminf_{I \ni x, y \to \infty} \frac{1}{lk} \sum_{j=1}^{l} \sum_{i=1}^{k} \mathbf{p}(\tau^i(x), \sigma^j(y)) = P > \frac{b^{-k} k^k}{(1+k)^{1+k}} \left(\frac{d}{b}\right)^l.$$

*Then every solution of equation (1) oscillates.*

**Theorem 5.** *Assume that* ($H_1$) *and* ($H_2$) *hold and there exist* $X, Y \in I$ *such that for* $k, l > 0$

$$\sup_{\lambda \in E, x \in I_X, y \in I_Y} \lambda \prod_{i=1}^{k} (d - \lambda \mathbf{p}(\tau^i(x), y)) \prod_{j=1}^{l} (d - \lambda \mathbf{p}(\tau^k(x), \sigma^j(y))) < b^k c^l.$$

*Then every solution of equation (1) oscillates.*

*Proof.* Let $\mu \in S(A)$. Then

$$\begin{aligned}
&\mathbf{A}(\tau(x), \sigma(y)) \\
&\le \frac{1}{b}(d - \mu \mathbf{p}(\tau(x), y))\mathbf{A}(\tau^2(x), \sigma(y)) \\
&\le \left(\frac{1}{b}\right)^k \prod_{i=1}^{k} (d - \mu \mathbf{p}(\tau^i(x), y))\mathbf{A}(\tau^{k+1}(x), \sigma(y)) \\
&\le \left(\frac{1}{b}\right)^k \left(\frac{1}{c}\right) \prod_{i=1}^{k} (d - \mu \mathbf{p}(\tau^i(x), y))(d - \mu \mathbf{p}(\tau^k(x), \sigma(y)))\mathbf{A}(\tau^{k+1}(x), \sigma^2(y)) \\
&\le \left(\frac{1}{b}\right)^k \left(\frac{1}{c}\right)^l \\
&\times \prod_{i=1}^{k} (d - \mu \mathbf{p}(\tau^i(x), y)) \prod_{i=1}^{l} (d - \mu \mathbf{p}(\tau^k(x), \sigma^j(y)))\mathbf{A}(\tau^{k+1}(x), \sigma^{l+1}(y)).
\end{aligned}$$

The rest of the proof is similar to that of Theorem 1 it is thus omitted. □

Due to equation (28), we have immediately the following result.

**Corollary 5.** *Assume that* ($H_1$) *and* ($H_2$) *hold and for* $k, l > 0$

$$\liminf_{I \ni x, y \to \infty} \mathbf{p}(x, y) = P > \frac{d^{l+k+1}(1+k)^{1+k}}{b^k c^l (1+k+l)^{1+k+l}}.$$

*Then every solution of equation (1) oscillates.*

**Remark 1.** *Results in this section are true for* $\mathbf{a}(x, y) \equiv 0$ *in* (1).

**Example 1.** *Consider following functional equation*

$$\begin{aligned}
&\frac{x+1}{x}\mathbf{A}(x, y) + e\mathbf{A}(x, \sigma(y)) + \frac{y+1}{y}\mathbf{A}(\tau(x), y) \\
&-\mathbf{A}(\tau(x), \sigma(y)) + (1+e)\mathbf{A}(\tau^3(x), \sigma^3(y)) = 0. \qquad (31)
\end{aligned}$$

It is easy to see that $\frac{x+1}{x} > 1 = a$ $(x > 0)$, $b = e$, $\frac{y+1}{y} > 1 = c$ $(y > 0)$, $d = 1$, $\mathbf{p}(x, y) = 1 + e$. Obviously, (31) satisfies the conditions of Corollary 5, so every solution of equation (31) oscillates.

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Sanmenxia Polytechnic
Ministry of Public Education
Xiaoshan Street 42, 472000, Sanmenxia, China
*E-mail address*: yangwg8088@163.com