TJMM
2 (2010), No. 1, 37-43

NEW EXPLICIT AND IMPLICIT SOLUTIONS TO ELLIPTIC EQUATIONS WITH TWO SPACE VARIABLES

MOHAMMED A. AL-KADHI

Abstract. We present a direct method for finding a new explicit and implicit solutions of elliptic equations with hyperbolic, trigonometric and exponential nonlinearities.

1. INTRODUCTION

Many solutions have been obtained in different forms to the elliptic equations with two space variables and a nonlinear source of the form [1, 2, 3]:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(u). \quad (1)$$

In particular, when Eq.(1) involving hyperbolic nonlinearities, namely $f(u) = \alpha \sinh u$ and $f(u) = \alpha \sinh u + \beta \sinh 2u$ there are traveling-wave solutions in implicit form and other kind of solutions called solutions with central symmetry about the point $(-C_1, -C_2)$. Also, many functional separable solutions for $f(u) = \alpha \sinh u$ are obtained in explicit form

$$u(x, y) = 4 \tanh^{-1} (g(x)h(y)),$$

where the functions g and h are determined by the first-order ordinary differential equations

$$(g'_x)^2 = Ag^4 + Bg^2 + C \text{ and } (h'_y)^2 = -Cg^4 + (\alpha - B)g^2 - A,$$

where A, B and C are constants. For other exact solutions of this equation with trigonometric and exponential nonlinearities, namely $f(u) = \alpha \sin u$, $f(u) = \alpha \sin u + \beta \sin 2u$ and $f(u) = \alpha e^u$ the reader is referred to [1, 2, 3].

The purpose of this paper is to show that new explicit and implicit solutions of elliptic equations with hyperbolic, trigonometric and exponential nonlinearities can be obtained by analytic techniques.

2. Equations with Hyperbolic Nonlinearities

2.1. Equation of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha \sinh u$.

Consider the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha \sinh u. \quad (2)$$

This is a special case of Eq.(1) with $f(u) = \alpha \sinh u$.

We write Eq.(2) as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\alpha}{2} \left(e^u - e^{-u}\right). \quad (3)$$

Let

$$u(x, y) = \ln v(x, y). \quad (4)$$

2010 *Mathematics Subject Classification.* 35J05.

Key words and phrases. Explicit solutions, Implicit solutions, Elliptic equations.

Hence Eq.(3) may be written as

$$\left[\frac{v_{xx}}{v}-\left(\frac{v_x}{v}\right)^2\right]+\left[\frac{v_{yy}}{v}-\left(\frac{v_y}{v}\right)^2\right]=\frac{\alpha}{2}\left(v-\frac{1}{v}\right), \tag{5}$$

where $v_x=\frac{\partial v}{\partial x},\ v_y=\frac{\partial v}{\partial y},\ v_{xx}=\frac{\partial^2 v}{\partial x^2}$ and $v_{yy}=\frac{\partial^2 v}{\partial y^2}$.

Now setting

$$w(z)=\frac{v_x}{v}+\imath\frac{v_y}{v},\ \text{where } z=x+\imath y \ \text{ and } \imath^2=-1. \tag{6}$$

Thus

$$w'(z)=\left[\frac{v_{xx}}{v}-\left(\frac{v_x}{v}\right)^2\right]+\imath\left[\frac{v_{xy}}{v}-\frac{v_xv_y}{v^2}\right] \tag{7}$$

and

$$w'(z)=\left[\frac{v_{yy}}{v}-\left(\frac{v_y}{v}\right)^2\right]-\imath\left[\frac{v_{xy}}{v}-\frac{v_xv_y}{v^2}\right]. \tag{8}$$

From Eq.(7) and Eq.(8) we get

$$2w'(z)=\left[\frac{v_{xx}}{v}-\left(\frac{v_x}{v}\right)^2\right]+\left[\frac{v_{yy}}{v}-\left(\frac{v_y}{v}\right)^2\right]. \tag{9}$$

Substituting Eq.(9) into Eq.(5) and in view of Eq.(6) we obtain

$$2w'(z)=\frac{\alpha}{2}\left[\frac{v_x+\imath v_y}{w}-\frac{v_x+\imath v_y}{wv^2}\right]. \tag{10}$$

It follows that

$$2w(z)w'(z)=\frac{\alpha}{2}\left[\left(v_x-\frac{v_x}{v^2}\right)+\imath\left(v_y-\frac{v_y}{v^2}\right)\right]. \tag{11}$$

Integrating Eq.(11) with respect to z we obtain

$$2\int w(z)w'(z)dz=\frac{\alpha}{2}\int\left[\frac{\partial}{\partial x}\left(v+\frac{1}{v}\right)+\imath\frac{\partial}{\partial y}\left(v+\frac{1}{v}\right)\right]dz. \tag{12}$$

If we suppose

$$v_x=v_y, \tag{13}$$

then

$$\begin{aligned}2\int w(z)w'(z)dz&=\frac{\alpha}{2}\int\left[\frac{\partial}{\partial x}\left(v+\frac{1}{v}\right)+\imath\frac{\partial}{\partial x}\left(v+\frac{1}{v}\right)\right]dz=\\&=\frac{\alpha}{2}\int d\left[\left(v+\frac{1}{v}\right)+\imath\left(v+\frac{1}{v}\right)\right]\end{aligned} \tag{14}$$

So that

$$w^2=\frac{\alpha}{2}(1+\imath)\left(v+\frac{1}{v}\right)+b, \tag{15}$$

where b is an arbitrary constant of integration.

Now substituting Eq.(6) into Eq.(15) we get

$$2\imath\left(\frac{v_x}{v}\right)^2=\frac{\alpha}{2}(1+\imath)\left(v+\frac{1}{v}\right)+b. \tag{16}$$

Thus

$$\left(\frac{v_x}{v}\right)^2=a(1-\imath)\left(v+\frac{1}{v}\right)+b_1, \tag{17}$$

where $a=\alpha/4$ and $b_1=\frac{b}{2\imath}$. It follows

$$\frac{v_x}{v}=\pm c\sqrt{av+\frac{a}{v}+b_1}, \tag{18}$$

where $c = (1-\imath)^{\frac{1}{2}}$. Hence

$$\frac{v_x}{\sqrt{av^3 + b_1 v^2 + av}} = \pm c. \tag{19}$$

Now, if we choose $b_1 = 2a$ in Eq.(19), then, in view of $dv = v_x dx + v_y dy = v_x(dx+dy)$, we get

$$\frac{dv}{\sqrt{av^3 + 2av^2 + av}} = \pm c(dx+dy). \tag{20}$$

Integrate Eq.(20)

$$\frac{2\tan^{-1}(\sqrt{v})}{\sqrt{a}} = \pm c(x+y) + d.$$

Thus

$$v = \tan^2\left(\pm c\frac{\sqrt{a}}{2}(x+y) + \frac{\sqrt{a}}{2}d\right). \tag{21}$$

Therefore, in view of Eq.(4), we obtain the following result:
For the case $b_1 = 2a$, the functions

$$u(x,y) = \ln\left[\tan^2\left(\pm c\frac{\sqrt{a}}{2}(x+y) + \frac{\sqrt{a}}{2}d\right)\right].$$

where $a = \alpha/4$, $c = (1-\imath)^{\frac{1}{2}}$ and d is an arbitrary constant, are solutions of Eq.(2).

2.2. **Equation of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha \sinh u + \beta \sinh 2u$.** Consider the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha \sinh u + \beta \sinh 2u. \tag{22}$$

As above we write Eq.(22) as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\alpha}{2}\left(e^u - e^{-u}\right) + \frac{\beta}{2}\left(e^{2u} - e^{-2u}\right). \tag{23}$$

Let

$$u(x,y) = \frac{1}{2}\ln v(x,y), \text{ i.e. } u(x,y) = \ln\sqrt{v(x,y)}. \tag{24}$$

Then Eq.(23) may be written as

$$\left[\frac{v_{xx}}{v} - \left(\frac{v_x}{v}\right)^2\right] + \left[\frac{v_{yy}}{v} - \left(\frac{v_y}{v}\right)^2\right] = \alpha\left(\sqrt{v} - \frac{1}{\sqrt{v}}\right) + \beta\left(v - \frac{1}{v}\right). \tag{25}$$

When we substitute Eq.(9) into Eq.(25) and using Eq.(6), we get

$$2w'(z) = \alpha\left(\frac{\sqrt{v_x + \imath v_y}}{\sqrt{w}} - \frac{\sqrt{v_x + \imath v_y}}{v\sqrt{w}}\right) + \beta\left(\frac{v_x + \imath v_y}{w} - \frac{v_x + \imath v_y}{wv^2}\right). \tag{26}$$

It follows that

$$2w'w = \alpha\sqrt{w}\left[\sqrt{v_x + \imath v_y} - \frac{\sqrt{v_x + \imath v_y}}{v}\right] + \beta\left[\left(v_x - \frac{v_x}{v^2}\right) + \imath\left(v_y - \frac{v_y}{v^2}\right)\right]. \tag{27}$$

Inserting Eq.(6) into the RHS of Eq.(27) we obtain

$$2ww' = \alpha\frac{\sqrt{v_x + \imath v_y}}{\sqrt{v}}\left[\sqrt{v_x + \imath v_y} - \frac{\sqrt{v_x + \imath v_y}}{v}\right] + \beta\left[\left(v_x - \frac{v_x}{v^2}\right) + \imath\left(v_y - \frac{v_y}{v^2}\right)\right], \tag{28}$$

that is,

$$2w'w = \alpha\left[\frac{v_x + \imath v_y}{\sqrt{v}} - \frac{v_x + \imath v_y}{v\sqrt{v}}\right] + \beta\left[\left(v_x - \frac{v_x}{v^2}\right) + \imath\left(v_y - \frac{v_y}{v^2}\right)\right], \tag{29}$$

so that

$$2ww' = \alpha\left[\left(\frac{v_x}{\sqrt{v}} - \frac{v_x}{v\sqrt{v}}\right) + \imath\left(\frac{v_y}{\sqrt{v}} - \frac{v_y}{v\sqrt{v}}\right)\right] + \beta\left[\left(v_x - \frac{v_x}{v^2}\right) + \imath\left(v_y - \frac{v_y}{v^2}\right)\right]. \quad (30)$$

Thus

$$2ww' = 2\alpha\left[\frac{\partial}{\partial x}\left(\sqrt{v} + \frac{1}{\sqrt{v}}\right) + \imath\frac{\partial}{\partial y}\left(\sqrt{v} + \frac{1}{\sqrt{v}}\right)\right] + \beta\left[\frac{\partial}{\partial x}\left(v + \frac{1}{v}\right) + \imath\frac{\partial}{\partial y}\left(v + \frac{1}{v}\right)\right]. \quad (31)$$

Integrating Eq.(31) with respect to z and taking into account that $v_x = v_y$, we obtain

$$2\int w(z)w'(z)dz = 2\alpha\int\left[\frac{\partial}{\partial x}\left(\sqrt{v} + \frac{1}{\sqrt{v}}\right) + \imath\frac{\partial}{\partial x}\left(\sqrt{v} + \frac{1}{\sqrt{v}}\right)\right]dz + \beta\int\left[\frac{\partial}{\partial x}\left(v + \frac{1}{v}\right) + \imath\frac{\partial}{\partial x}\left(v + \frac{1}{v}\right)\right]dz. \quad (32)$$

Thus

$$w^2 = 2(1+\imath)\alpha\left(\sqrt{v} + \frac{1}{\sqrt{v}}\right) + (1+\imath)\beta\left(v + \frac{1}{v}\right) + k, \quad (33)$$

where k is an arbitrary constant of integration.

As before, substituting Eq.(6) into Eq.(33) we get Eq.(15) and

$$2\imath\left(\frac{v_x}{v}\right)^2 = 2(1+\imath)\alpha\left(\sqrt{v} + \frac{1}{\sqrt{v}}\right) + (1+\imath)\beta\left(v + \frac{1}{v}\right) + k. \quad (34)$$

Thus

$$\left(\frac{v_x}{v}\right)^2 = (1-\imath)\alpha\left(\sqrt{v} + \frac{1}{\sqrt{v}}\right) + (1-\imath)\frac{\beta}{2}\left(v + \frac{1}{v}\right) + k_1, \quad (35)$$

where $k_1 = \frac{k}{2\imath}$. So that

$$\frac{v_x}{v} = \pm c\sqrt{\alpha\left(\sqrt{v} + \frac{1}{\sqrt{v}}\right) + \frac{\beta}{2}\left(v + \frac{1}{v}\right) + k_1}. \quad (36)$$

Thus

$$\frac{dv}{\sqrt{\alpha(v^2\sqrt{v} + \frac{v^2}{\sqrt{v}}) + \frac{\beta}{2}(v^3 + v) + k_1 v^2}} = \pm c d(x+y). \quad (37)$$

Now, to integrate the separation equation (37), we choose $\beta = 2\alpha$ and $k_1 = 0$ and using *Wolfram Mathematica Integrator,* we get

$$\psi(v) = \pm c\sqrt{\alpha}(x + y + r), \quad (38)$$

where r is a constant and

$$\psi(v) = \frac{2\sqrt{(\sqrt{v}+1)}\sqrt{v - \sqrt{v} + 1}\left[\log(\sqrt{v}+1) - \log(-3\sqrt{v} + 2\sqrt{3}\sqrt{v - \sqrt{v} + 1} + 3)\right]}{\sqrt{3}\sqrt{1 + v\sqrt{v}}}. \quad (39)$$

Thus, for the case $\beta = 2\alpha$ and $k_1 = 0$, the functions

$$u(t, x) = \frac{1}{2}\ln v(t, x), \quad (40)$$

where the implicit solution for v is given by Eq.(38) and Eq.(39), are solutions of Eq.(22).

3. Equations with Trigonometric Nonlinearities

3.1. Equation of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha \sin u$. Consider the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha \sin u. \tag{41}$$

If we introduce the transformation $u(x, y) = \imath v(x, y)$ and substitute this into Eq.(41) and take into account that $\sin(iv) = i \sinh v$, then, Eq.(41) can be transformed to

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \alpha \sinh v. \tag{42}$$

In a manner similar to that of above we obtain

$$v(x, y) = \ln\left[\tan^2\left(\pm c\frac{\sqrt{a}}{2}(x+y) + \frac{\sqrt{a}}{2}d\right)\right].$$

Thus, one solutions of Eq.(41) is given by:

$$u(x, y) = \imath \ln\left[\tan^2\left(\pm c\frac{\sqrt{a}}{2}(x+y) + \frac{\sqrt{a}}{2}d\right)\right].$$

3.2. Equation of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha \sin u + \beta \sin 2u$. Consider the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha \sin u + \beta \sin 2u, \tag{43}$$

To solve this equation, we set $u(x, y) = iv_1(x, y)$. Then

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} = \alpha \sinh v_1 + \beta \sinh 2v_1. \tag{44}$$

In a manner similar to the previous section, we obtain

$$v_1(x, y) = \frac{1}{2} \ln v(x, y). \tag{45}$$

Thus, one solution of Eq.(43) is:

$$u(x, y) = \frac{i}{2} \ln v(x, y), \tag{46}$$

where $v(x, y)$ is given by Eq.(38) and Eq.(39).

4. Equations with Exponential Nonlinearities

4.1. Equation of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha e^u$. Consider the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \alpha e^u. \tag{47}$$

In a manner similar to the previous sections and after using Eq.(4)- Eq.(6) and $v_x = v_y$ we obtain

$$\frac{v_x}{v} = \pm c\sqrt{\alpha v + b}, \tag{48}$$

where b is an arbitrary constant.

Thus

$$\frac{v_x}{\sqrt{\alpha v^2 + bv}} = \pm c, \tag{49}$$

so that

$$\frac{dv}{\sqrt{\alpha v^2 + bv}} = \pm c d(x+y). \tag{50}$$

If we choose $b = \alpha$ and integrate Eq.(50), then

$$\frac{\sqrt{v}\sqrt{v+1}\sinh^{-1} v}{v(v+1)} = \pm c\frac{\sqrt{\alpha}}{2}(x+y) + d. \tag{51}$$

Thus, the functions

$$u(x,y) = \ln v(x,y), \tag{52}$$

where $v(x,y)$ is given by Eq.(51), are solutions of Eq.(47).

5. Conclusion

In fact, the exponential nonlinearity of Eq.(2) immediately suggests the change of dependent variable $u = \ln v$, whose partial derivatives with respect to x and y are $\frac{v_x}{v}$ and $\frac{v_y}{v}$, respectively, which can be complexify to define $w(z)$ as done in this equation. The condition $b_1 = 2a$ is used to obtain a simple solution. Also, other values of a and b_1 result in analytical solutions to Eq.(19). Also, in section 2, we have presented a simple solution when $k = 0$ and $\beta = 2\alpha$ and other values for these three constants lead to a new solutions.

Taking into account the techniques made at the beginning of this method and as indicated in this paper, the cases 3.1 and 3.2 can be obtained from cases 2.1 and 2.2 by first introducing $u = iv$ and then making use of Euler's formula to obtain new solutions. In addition, Eq.(47) has solutions of traveling-wave type for $b = \alpha$ as indicated in section 4 and other solutions for $b \neq \alpha$.

However, all the analytical solutions obtained in [1, 2, 3] for one type of nonlinearity, e.g., the hyperbolic one are in explicit forms

$$u(x,y) = 4\tanh^{-1}\left(g(x)h(y)\right),$$

where the functions g and h are determined by the first-order ordinary differential equations

$$(g'_x)^2 = Ag^4 + Bg^2 + C \text{ and } (h'_y)^2 = -Cg^4 + (\alpha - B)g^2 - A,$$

where $A,\ B$ and C are constants, and there are traveling-wave solutions in implicit form

$$\int \left[D + \frac{2a\cosh(\beta u)}{\beta(A^2+B^2)}\right]^{\frac{-1}{2}} du = Ax + By + C,$$

where $A,\ B,\ C,\ D$ are arbitrary constants. Other kind of solutions can be found in [1, 2, 3] and called solutions with central symmetry about the point $(-C_1, -C_2)$:

$$u = u(\xi),\ \xi = \sqrt{(x+C_1)^2 + (x+C_2)^2},$$

where $C_1,\ C_2$ are arbitrary constants, and the function $u(\xi)$ is determined by an ordinary differential equation. Similarly, for Eq.(47) the solutions obtained in [1, 2, 3] are in implicit forms:

$$u(x,y) = \frac{1}{\beta}\ln\left[\frac{2(A^2+B^2)}{\alpha\beta(Ax+By+C)^2}\right],\ \alpha\beta > 0,$$

$$u(x,y) = \frac{1}{\beta}\ln\left[\frac{2(A^2+B^2)}{\alpha\beta\sinh^2(Ax+By+C)^2}\right],\ \alpha\beta > 0,$$

$$u(x,y) = \frac{1}{\beta}\ln\left[\frac{-2(A^2+B^2)}{\alpha\beta\cosh^2(Ax+By+C)^2}\right],\ \alpha\beta < 0$$

and

$$u(x,y) = \frac{1}{\beta}\ln\left[\frac{2(A^2+B^2)}{\alpha\beta\cos^2(Ax+By+C)^2}\right],\ \alpha\beta > 0.$$

In conclusion, as a complement of the results obtained in the cited references, we have successfully presented, by analytic techniques, a large number of new explicit and implicit solutions to elliptic equations with hyperbolic, trigonometric and exponential nonlinearities which are not found in [1, 2, 3].

References

[1] Bullough, R. K. and Caudrey, P. J. , (Editors) *Solitons,* Springer-Verlag, Berlin, 1980.
[2] Miller, J. (Jr.) and Rubel, L. A., *Functional separation of variables for Laplace equations in two dimensions,* J. Phys. A, Vol. 26, pp. 1901-1913, 1993.
[3] Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Partial Differential Equations,* Chapman and Hall/CRC, Boca Raton, 2004.

Al-Imam Mohammed Ibn Saud University
Faculty of Science
Department of Mathematics
P.O.Box 122037, 11721, Riyadh, Saudi Arabia
E-mail address: malki2@yahoo.com