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# SOLVING CAUCHY PROBLEM FOR A CLASS OF SIXTH-ORDER HYPERBOLIC EQUATIONS WITH TRIPLE CHARACTERISTICS

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ABSTRACT. In this paper, the Cauchy problem for a class of the homogeneous hyperbolic equations for sixth-order with triple characteristics is considered and can be solved analytically by direct integration techniques. Also, an efficient modification of Adomian decomposition method is proposed for solving this type of problems. We then conduct a comparative study between the ADM and direct method with the help of several illustrative examples.

## 1. Introduction

Partial differential equations of higher order are encountered when studying mathematical models for certain natural and physical processes. The Cauchy problem for general linear hyperbolic differential equations has been studied much more throughly in the case of operators which are strictly hyperbolic, i.e., have simple real characteristic than in the case of operators with multiple characteristics. A. Lax [12] has studied hyperbolic equations with multiple characteristics involving one space variable.
In [16] a class of equations with fourth order partial differential equations with multiple characteristics and dominated low terms is considered. The existence and uniqueness of a Riemann function for this equation is proved.
The reader is referred to [6, 13, 14, 15, 16, 17, 18, 19, 20] for further studies.
This paper is concerned with a direct integration technique for solving the Cauchy problem for a class of linear hyperbolic equation of sixth-order with triple characteristics $x-t=0$ and $x+t=0$. Also, an efficient modification of Adomian decomposition method is proposed for solving this type of problems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

## 2. The Cauchy problem for the homogeneous hyperbolic equation of sixth-order

Consider the following hyperbolic equation of sixth-order with triple real characteristics $t-x=0$ and $t+x=0$

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)^3 u=0,\ x\in\Re \text{ and } t\geq 0, \tag{1}$$

with initial conditions

$$\frac{\partial^i u}{\partial t^i}(0,x)=\varphi_i(x),\ i=0,...,5. \tag{2}$$

Our purpose is to look for the general solution of problem (1)-(2). The basic key of this method is to transform PDEs into integrable PDEs by introducing new variables $w=x+t$ and $z=x-t$.
We shall prove

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**Theorem 1.** *The solution of problem (1)-(2) can be expressed in the form*

$$u(t,x) = \frac{(x+t)^2}{2}\Phi_1(x-t) + (x+t)\Phi_2(x-t) + \Phi_3(x-t)$$
$$+\frac{(x-t)^2}{2}\Psi_1(x+t) + (x-t)\Psi_2(x+t) + \Psi_3(x+t),$$

*where* $\Phi_i \in C^6(\Re)$ *and* $\Psi_i \in C^6(\Re)$, $i = 1,2,3$ *are functions given by*

$$\Psi_1(x) = \frac{1}{32}\int_0^x\int_0^x\int_0^x \left[\varphi_4^{(1)}(x) + \varphi_1^{(4)}(x) - 2\varphi_3^{(2)}(x)\right] dxdxdx$$
$$+\frac{1}{32}\int_0^x\int_0^x\int_0^x \left[-2\varphi_2^{(3)}(x) + \varphi_0^{(5)}(x) + \varphi_5(x)\right] dxdxdx$$
$$+\left[\Psi_1(0) + x\Psi_1^{(1)}(0) + \frac{x^2}{2}\Psi_1^{(2)}(0)\right],$$

$$\Phi_1(x) = \frac{1}{32}\int_0^x\int_0^x\int_0^x \left[\varphi_0^{(5)}(x) - \varphi_5(x) - 2\varphi_2^{(3)}(x) + 2\varphi_3^{(2)}(x)\right] dxdxdx$$
$$+\frac{1}{32}\int_0^x\int_0^x\int_0^x \left[-\varphi_1^{(4)}(x) + \varphi_4^{(1)}(x)\right] dxdxdx$$
$$+\left[\Phi_1(0) + x\Phi_1^{(1)}(0) + \frac{x^2}{2}\Phi_1^{(2)}(0)\right],$$

$$\Psi_2(x) = \int_0^x\int_0^x\int_0^x\int_0^x \left[\Phi_1^{(3)}(x) + x\Phi_1^{(4)}(x) - \Psi_1^{(3)}(x) - x\Psi_1^{(4)}(x)\right] dxdxdxdx$$
$$+\int_0^x\int_0^x\int_0^x\int_0^x \left[\Phi_2^{(4)}(x) + \frac{1}{4}\left(\varphi_1^{(4)}(x) - \varphi_3^{(2)}(x)\right)\right] dxdxdxdx$$
$$+\left[\Psi_2(0) + x\Psi_2^{(1)}(0) + \frac{x^2}{2}\Psi_2^{(2)}(0) + \frac{x^3}{6}\Psi_2^{(3)}(0)\right],$$

$$\Phi_2(x) = \frac{1}{16}\int_0^x\int_0^x\int_0^x\int_0^x \left[\varphi_5(x) - \varphi_1^{(4)}(x) + \varphi_0^{(5)}(x) - \varphi_4^{(1)}(x)\right] dxdxdxdx$$
$$+\frac{1}{16}\int_0^x\int_0^x\int_0^x\int_0^x \left[-16x\Phi_1^{(4)}(x) - 16\Psi_1^{(3)}(x)\right] dxdxdxdx$$
$$+\left[\Phi_2(0) + x\Phi_2^{(1)}(0) + \frac{x^2}{2}\Phi_2^{(2)}(0) + \frac{x^3}{6}\Phi_2^{(3)}(0)\right],$$

$$\Psi_3(x) = \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[-x\Phi_1^{(4)}(x) + 4\Psi_1^{(3)}(x) - \frac{1}{2}x^2\Psi_1^{(5)}(x)\right] dxdxdxdxdx$$
$$+\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[-\Phi_2^{(4)}(x) - x\Psi_2^{(5)}(x)\right] dxdxdxdxdx$$
$$+\frac{1}{2}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[\varphi_3^{(2)}(x) + \varphi_2^{(3)}(x)\right] dxdxdxdxdx$$
$$+\left[\Psi_3(0) + x\Psi_3^{(1)}(0) + \frac{x^2}{2}\Psi_3^{(2)}(0) + \frac{x^3}{6}\Psi_3^{(3)}(0) + \frac{x^4}{24}\Psi_3^{(4)}(0)\right]$$

*and*

$$\begin{aligned}\Phi_3(x) = &-\frac{1}{2}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[20\Phi_1^{(3)}(x) + x^2\Phi_1^{(5)}(x) + 10x\Psi_1^{(4)}(x)\right] dxdxdxdxdx \\ &-\frac{1}{2}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[2x\Phi_2^{(5)}(x) + 10\Psi_2^{(4)}(x)\right] dxdxdxdxdx \\ &+\frac{1}{2}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[\varphi_0^{(5)}(x) - \varphi_5(x)\right] dxdxdxdxdx \\ &+\left[\Phi_3(0) + x\Phi_3^{(1)}(0) + \frac{x^2}{2}\Phi_3^{(2)}(0) + \frac{x^3}{6}\Phi_3^{(3)}(0) + \frac{x^4}{24}\Phi_3^{(4)}(0)\right].\end{aligned}$$

*Proof.* To find the general solution of problem (1)-(2), make the substitutions $w = x + t$ and $z = x - t$ into equation (1), and applying the chain rule to obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} + \frac{\partial u}{\partial z},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial w} - \frac{\partial u}{\partial z},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w\partial z} + \frac{\partial^2 u}{\partial z^2},$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w\partial z} + \frac{\partial^2 u}{\partial z^2},$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial^3 u}{\partial w^3} + 3\frac{\partial^3 u}{\partial w^2\partial z} + 3\frac{\partial^3 u}{\partial w\partial z^2} + \frac{\partial^3 u}{\partial z^3},$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{\partial^4 u}{\partial w^4} + 4\frac{\partial^4 u}{\partial w^3\partial z} + 6\frac{\partial^4 u}{\partial w^2\partial z^2} + 4\frac{\partial^4 u}{\partial w\partial z^3} + \frac{\partial^4 u}{\partial z^4},$$

$$\frac{\partial^4 u}{\partial t^4} = \frac{\partial^4 u}{\partial w^4} - 4\frac{\partial^4 u}{\partial w^3\partial z} + 6\frac{\partial^4 u}{\partial w^2\partial z^2} - 4\frac{\partial^4 u}{\partial w\partial z^3} + \frac{\partial^4 u}{\partial z^4},$$

$$\frac{\partial^4 u}{\partial t^2\partial x^2} = \frac{\partial^4 u}{\partial w^4} - 2\frac{\partial^4 u}{\partial w^2\partial z^2} + \frac{\partial^4 u}{\partial z^4},$$

$$\cdots$$

$$\frac{\partial^6 u}{\partial t^6} = \frac{\partial^6 u}{\partial w^6} - 6\frac{\partial^6 u}{\partial w^5\partial z} + 15\frac{\partial^6 u}{\partial w^4\partial z^2} - 20\frac{\partial^6 u}{\partial w^3\partial z^3} + 15\frac{\partial^6 u}{\partial w^2\partial z^4} - 6\frac{\partial^6 u}{\partial w\partial z^5} + \frac{\partial^6 u}{\partial z^6},$$

$$\frac{\partial^6 u}{\partial x^6} = \frac{\partial^6 u}{\partial w^6} + 6\frac{\partial^6 u}{\partial w^5\partial z} + 15\frac{\partial^6 u}{\partial w^4\partial z^2} + 20\frac{\partial^6 u}{\partial w^3\partial z^3} + 15\frac{\partial^6 u}{\partial w^2\partial z^4} + 6\frac{\partial^6 u}{\partial w\partial z^5} + \frac{\partial^6 u}{\partial z^6},$$

$$\frac{\partial^6 u}{\partial t^2\partial x^4} = \frac{\partial^6 u}{\partial w^6} + 2\frac{\partial^6 u}{\partial w^5\partial z} - \frac{\partial^6 u}{\partial w^4\partial z^2} - 4\frac{\partial^6 u}{\partial w^3\partial z^3} - \frac{\partial^6 u}{\partial w^2\partial z^4} + 2\frac{\partial^6 u}{\partial w\partial z^5} + \frac{\partial^6 u}{\partial z^6}$$

and

$$\frac{\partial^6 u}{\partial t^4\partial x^2} = \frac{\partial^6 u}{\partial w^6} - 2\frac{\partial^6 u}{\partial w^5\partial z} - \frac{\partial^6 u}{\partial w^4\partial z^2} + 4\frac{\partial^6 u}{\partial w^3\partial z^3} - \frac{\partial^6 u}{\partial w^2\partial z^4} - 2\frac{\partial^6 u}{\partial w\partial z^5} + \frac{\partial^6 u}{\partial z^6}.$$

Then equation (1) becomes

$$\frac{\partial^6 u}{\partial w^3\partial z^3} = 0. \tag{3}$$

The successive integrations of equation (3) give us

$$u(t,x) = \frac{w^2}{2}\Phi_1(z) + w\Phi_2(z) + \Phi_3(z) + \frac{z^2}{2}\Psi_1(w) + z\Psi_2(w) + \Psi_3(w). \tag{4}$$

where $\Phi_i$ and $\Psi_i,\ i = 1, ..., 3$ are arbitrary functions.

Now, we return to the original variables $t$ and $x$, we obtain

$$\begin{aligned} u(t,x) = &\frac{(x+t)^2}{2}\Phi_1(x-t) + (x+t)\Phi_2(x-t) + \Phi_3(x-t) \\ &+ \frac{(x-t)^2}{2}\Psi_1(x+t) + (x-t)\Psi_2(x+t) + \Psi_3(x+t). \end{aligned} \tag{5}$$

In other words, if $u$ is a solution of (1), then there exist $\Phi_i \in C^6(\Re)$ and $\Psi_i \in C^6(\Re)$, $i = 1,2,3$ such that (5) holds. Conversely, any functions $\Phi_i \in C^6(\Re)$ and $\Psi_i \in C^6(\Re)$, $i = 1,2,3$ define a solution of (1) via formula (5).

Taking into account the initial conditions (2), and from equation (5) we obtain

$$\frac{x^2}{2}\Phi_1(x) + \frac{x^2}{2}\Psi_1(x) + x\Phi_2(x) + x\Psi_2(x) + \Phi_3(x) + \Psi_3(x) = \varphi_0(x), \tag{6}$$

$$\begin{aligned} &x\Phi_1(x) - \frac{x^2}{2}\Phi_1^{(1)}(x) - x\Psi_1(x) + \frac{x^2}{2}\Psi_1^{(1)}(x) + \Phi_2(x) \\ &- x\Phi_2^{(1)}(x) - \Psi_2(x) + x\Psi_2^{(1)}(x) - \Phi_3^{(1)}(x) + \Psi_3^{(1)}(x) = \varphi_1(x), \end{aligned} \tag{7}$$

$$\begin{aligned} &\Phi_1(x) - 2x\Phi_1^{(1)}(x) + \frac{x^2}{2}\Phi_1^{(2)}(x) + \Psi_1(x) - 2x\Psi_1^{(1)}(x) + \frac{x^2}{2}\Psi_1^{(2)}(x) - 2\Phi_2^{(1)}(x) \\ &+ x\Phi_2^{(2)}(x) - 2\Psi_2^{(1)}(x) + x\Psi_2^{(2)}(x) + \Phi_3^{(2)}(x) + \Psi_3^{(2)}(x) = \varphi_2(x), \end{aligned} \tag{8}$$

$$\begin{aligned} &-3\Phi_1^{(1)}(x) + 3x\Phi_1^{(2)}(x) - \frac{x^2}{2}\Phi_1^{(3)}(x) + 3\Psi_1^{(1)}(x) - 3x\Psi_1^{(2)}(x) + \frac{x^2}{2}\Psi_1^{(3)}(x) \\ &+ 3\Phi_2^{(2)}(x) - x\Phi_2^{(3)}(x) - 3\Psi_2^{(2)}(x) + x\Psi_2^{(3)}(x) - \Phi_3^{(3)}(x) + \Psi_3^{(3)}(x) = \varphi_3(x), \end{aligned} \tag{9}$$

$$\begin{aligned} &6\Phi_1^{(2)}(x) - 4x\Phi_1^{(3)}(x) + \frac{x^2}{2}\Phi_1^{(4)}(x) + 6\Psi_1^{(2)}(x) - 4x\Psi_1^{(3)}(x) + \frac{x^2}{2}\Psi_1^{(4)}(x) - 4\Phi_2^{(3)}(x) \\ &+ x\Phi_2^{(4)}(x) - 4\Psi_2^{(3)}(x) + x\Psi_2^{(4)}(x) + \Phi_3^{(4)}(x) + \Psi_3^{(4)}(x) = \varphi_4(x) \end{aligned} \tag{10}$$

and

$$\begin{aligned} &-10\Phi_1^{(3)}(x) + 5x\Phi_1^{(4)}(x) - \frac{x^2}{2}\Phi_1^{(5)}(x) + 10\Psi_1^{(3)}(x) - 5x\Psi_1^{(4)}(x) + \frac{x^2}{2}\Psi_1^{(5)}(x) \\ &+ 5\Phi_2^{(4)}(x) - x\Phi_2^{(5)}(x) - 5\Psi_2^{(4)}(x) + x\Psi_2^{(5)}(x) - \Phi_3^{(5)}(x) + \Psi_3^{(5)}(x) = \varphi_5(x). \end{aligned} \tag{11}$$

Taking the fifth, fourth, third, second and first derivatives of equations (6)-(10) with respect to $x$ respectively, we obtain

$$\begin{aligned} &10\Phi_1^{(3)}(x) + 5x\Phi_1^{(4)}(x) + \frac{x^2}{2}\Phi_1^{(5)}(x) + 10\Psi_1^{(3)}(x) + 5x\Psi_1^{(4)}(x) \\ &+ \frac{x^2}{2}\Psi_1^{(5)}(x) + 5\Phi_2^{(4)}(x) + x\Phi_2^{(5)}(x) + 5\Psi_2^{(4)}(x) + x\Psi_2^{(5)}(x) \\ &+ \Phi_3^{(5)}(x) + \Psi_3^{(5)}(x) = \varphi_0^{(5)}(x), \end{aligned} \tag{12}$$

$$\begin{aligned} &-2\Phi_1^{(3)}(x) - 3x\Phi_1^{(4)}(x) - \frac{x^2}{2}\Phi_1^{(5)}(x) + 2\Psi_1^{(3)}(x) + 3x\Psi_1^{(4)}(x) \\ &+ \frac{x^2}{2}\Psi_1^{(5)}(x) - 3\Phi_2^{(4)}(x) - x\Phi_2^{(5)}(x) + 3\Psi_2^{(4)}(x) + x\Psi_2^{(5)}(x) \\ &- \Phi_3^{(5)}(x) + \Psi_3^{(5)}(x) = \varphi_1^{(4)}(x), \end{aligned} \tag{13}$$

$$
\begin{aligned}
&-2\Phi_1^{(3)}(x)+x\Phi_1^{(4)}(x)+\frac{x^2}{2}\Phi_1^{(5)}(x)-2\Psi_1^{(3)}(x)+x\Psi_1^{(4)}(x) \\
&\quad+\frac{x^2}{2}\Psi_1^{(5)}(x)+\Phi_2^{(4)}(x)+x\Phi_2^{(5)}(x)+\Psi_2^{(4)}(x)+x\Psi_2^{(5)}(x) \\
&\quad+\Phi_3^{(5)}(x)+\Psi_3^{(5)}(x)=\varphi_2^{(3)}(x),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&2\Phi_1^{(3)}(x)+x\Phi_1^{(4)}(x)-\frac{x^2}{2}\Phi_1^{(5)}(x)-2\Psi_1^{(3)}(x)-x\Psi_1^{(4)}(x) \\
&\quad+\frac{x^2}{2}\Psi_1^{(5)}(x)+\Phi_2^{(4)}(x)-x\Phi_2^{(5)}(x)-\Psi_2^{(4)}(x)+x\Psi_2^{(5)}(x) \\
&\quad-\Phi_3^{(5)}(x)+\Psi_3^{(5)}(x)=\varphi_3^{(2)}(x),
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&2\Phi_1^{(3)}(x)-3x\Phi_1^{(4)}(x)+\frac{x^2}{2}\Phi_1^{(5)}(x)+2\Psi_1^{(3)}(x)-3x\Psi_1^{(4)}(x) \\
&\quad+\frac{x^2}{2}\Psi_1^{(5)}(x)-3\Phi_2^{(4)}(x)+x\Phi_2^{(5)}(x)-3\Psi_2^{(4)}(x)+x\Psi_2^{(5)}(x) \\
&\quad+\Phi_3^{(5)}(x)+\Psi_3^{(5)}(x)=\varphi_4^{(1)}(x).
\end{aligned}
\tag{16}
$$

Now, from equations (11)-(12), (13) with (16) and (14)-(15) respectively, we obtain

$$
\begin{aligned}
&10x\Phi_1^{(4)}(x)+20\Psi_1^{(3)}(x)+x^2\Psi_1^{(5)}(x)+10\Phi_2^{(4)}(x)+2x\Psi_2^{(5)}(x) \\
&\quad+2\Psi_3^{(5)}(x)=\varphi_0^{(5)}(x)+\varphi_5(x),
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&-6x\Phi_1^{(4)}(x)+4\Psi_1^{(3)}(x)+x^2\Psi_1^{(5)}(x)-6\Phi_2^{(4)}(x)+2x\Psi_2^{(5)}(x) \\
&\quad+2\Psi_3^{(5)}(x)=\varphi_4^{(1)}(x)+\varphi_1^{(4)}(x)
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
&2x\Phi_1^{(4)}(x)-4\Psi_1^{(3)}(x)+x^2\Psi_1^{(5)}(x)+2\Phi_2^{(4)}(x)+2x\Psi_2^{(5)}(x) \\
&\quad+2\Psi_3^{(5)}(x)=\varphi_3^{(2)}(x)+\varphi_2^{(3)}(x).
\end{aligned}
\tag{19}
$$

Also, from (17)-(18) and (18)-(19) respectively, we obtain

$$
16x\Phi_1^{(4)}(x)+16\Psi_1^{(3)}(x)+16\Phi_2^{(4)}(x)=\varphi_0^{(5)}(x)+\varphi_5(x)-\varphi_4^{(1)}(x)-\varphi_1^{(4)}(x)
\tag{20}
$$

and

$$
-8x\Phi_1^{(4)}(x)+8\Psi_1^{(3)}(x)-8\Phi_2^{(4)}(x)=\varphi_4^{(1)}(x)+\varphi_1^{(4)}(x)-\varphi_3^{(2)}(x)-\varphi_2^{(3)}(x).
\tag{21}
$$

Now, solving equations (20) and (21) for $\Psi_1$, we obtain

$$
32\Psi_1^{(3)}(x)=\varphi_4^{(1)}(x)+\varphi_1^{(4)}(x)-2\varphi_3^{(2)}(x)-2\varphi_2^{(3)}(x)+\varphi_0^{(5)}(x)+\varphi_5(x).
\tag{22}
$$

Thus, the successive integrations of Eq.(22) give us

$$
\begin{aligned}
\Psi_1(x)=&\frac{1}{32}\int_0^x\int_0^x\int_0^x\left[\varphi_4^{(1)}(x)+\varphi_1^{(4)}(x)-2\varphi_3^{(2)}(x)\right]dxdxdx \\
&+\frac{1}{32}\int_0^x\int_0^x\int_0^x\left[-2\varphi_2^{(3)}(x)+\varphi_0^{(5)}(x)+\varphi_5(x)\right]dxdxdx \\
&+\left[\Psi_1(0)+x\Psi_1^{(1)}(0)+\frac{x^2}{2}\Psi_1^{(2)}(0)\right].
\end{aligned}
\tag{23}
$$

Next, from equations (11)-(12), (13) with (16) and (14)-(15) respectively, we obtain

$$\begin{aligned} &-20\Phi_1^{(3)}(x)-x^2\Phi_1^{(5)}(x)-10x\Psi_1^{(4)}(x)-2x\Phi_2^{(5)}(x)\\ &\quad -10\Psi_2^{(4)}(x)-2\Phi_3^{(5)}(x)=\varphi_5(x)-\varphi_0^{(5)}(x), \end{aligned}\tag{24}$$

$$\begin{aligned} &-4\Phi_1^{(3)}(x)-x^2\Phi_1^{(5)}(x)+6x\Psi_1^{(4)}(x)-2x\Phi_2^{(5)}(x)\\ &\quad +6\Psi_2^{(4)}(x)-2\Phi_3^{(5)}(x)=\varphi_1^{(4)}(x)-\varphi_4^{(1)}(x) \end{aligned}\tag{25}$$

and

$$\begin{aligned} &-4\Phi_1^{(3)}(x)+x^2\Phi_1^{(5)}(x)+2x\Psi_1^{(4)}(x)+2x\Phi_2^{(5)}(x)\\ &\quad +2\Psi_2^{(4)}(x)+2\Phi_3^{(5)}(x)=\varphi_2^{(3)}(x)-\varphi_3^{(2)}(x). \end{aligned}\tag{26}$$

Also, from (24)-(25) and (24) with (26) respectively, we obtain

$$-16\Phi_1^{(3)}(x)-16x\Psi_1^{(4)}(x)-16\Psi_2^{(4)}(x)=\varphi_5(x)-\varphi_0^{(5)}(x)-\varphi_1^{(4)}(x)+\varphi_4^{(1)}(x)\tag{27}$$

and

$$-24\Phi_1^{(3)}(x)-8x\Psi_1^{(4)}(x)-8\Psi_2^{(4)}(x)=\varphi_5(x)-\varphi_0^{(5)}(x)+\varphi_2^{(3)}(x)-\varphi_3^{(2)}(x).\tag{28}$$

Solving equations (27) and (28) for $\Phi_1$, we obtain

$$32\Phi_1^{(3)}(x)=\varphi_0^{(5)}(x)-\varphi_5(x)-2\varphi_2^{(3)}(x)+2\varphi_3^{(2)}(x)-\varphi_1^{(4)}(x)+\varphi_4^{(1)}(x).\tag{29}$$

Thus,

$$\begin{aligned} \Phi_1(x)=&\frac{1}{32}\int_0^x\int_0^x\int_0^x\left[\varphi_0^{(5)}(x)-\varphi_5(x)-2\varphi_2^{(3)}(x)+2\varphi_3^{(2)}(x)\right]dxdxdx\\ &+\frac{1}{32}\int_0^x\int_0^x\int_0^x\left[-\varphi_1^{(4)}(x)+\varphi_4^{(1)}(x)\right]dxdxdx\\ &+\left[\Phi_1(0)+x\Phi_1^{(1)}(0)+\frac{x^2}{2}\Phi_1^{(2)}(0)\right]. \end{aligned}\tag{30}$$

In a similar way, from equations (11) with (13), (12) with (16) and (31) with (32) respectively, we obtain

$$\begin{aligned} &-8\Phi_1^{(3)}(x)+8x\Phi_1^{(4)}(x)+8\Psi_1^{(3)}(x)-8x\Psi_1^{(4)}(x)+8\Phi_2^{(4)}(x)\\ &\quad -8\Psi_2^{(4)}(x)=\varphi_5(x)-\varphi_1^{(4)}(x), \end{aligned}\tag{31}$$

$$\begin{aligned} &8\Phi_1^{(3)}(x)+8x\Phi_1^{(4)}(x)+8\Psi_1^{(3)}(x)+8x\Psi_1^{(4)}(x)+8\Phi_2^{(4)}(x)\\ &\quad +8\Psi_2^{(4)}(x)=\varphi_0^{(5)}(x)-\varphi_4^{(1)}(x), \end{aligned}\tag{32}$$

$$16x\Phi_1^{(4)}(x)+16\Psi_1^{(3)}(x)+16\Phi_2^{(4)}(x)=\varphi_5(x)-\varphi_1^{(4)}(x)+\varphi_0^{(5)}(x)-\varphi_4^{(1)}(x).\tag{33}$$

Thus

$$\Phi_2^{(4)}(x)=\frac{1}{16}\left[\varphi_5(x)-\varphi_1^{(4)}(x)+\varphi_0^{(5)}(x)-\varphi_4^{(1)}(x)-16x\Phi_1^{(4)}(x)-16\Psi_1^{(3)}(x)\right].\tag{34}$$

So that

$$\begin{aligned}\Phi_2(x) = {} & \frac{1}{16}\int_0^x\int_0^x\int_0^x\int_0^x \left[\varphi_5(x) - \varphi_1^{(4)}(x) + \varphi_0^{(5)}(x) - \varphi_4^{(1)}(x)\right] dxdxdxdx \\ & + \frac{1}{16}\int_0^x\int_0^x\int_0^x\int_0^x \left[-16x\Phi_1^{(4)}(x) - 16\Psi_1^{(3)}(x)\right] dxdxdxdx \\ & + \left[\Phi_2(0) + x\Phi_2^{(1)}(0) + \frac{x^2}{2}\Phi_2^{(2)}(0) + \frac{x^3}{6}\Phi_2^{(3)}(0)\right]\end{aligned} \tag{35}$$

To find $\Psi_2(x)$, taking (13) and (15)

$$\begin{aligned}& -4\Phi_1^{(3)}(x) - 4x\Phi_1^{(4)}(x) + 4\Psi_1^{(3)}(x) + 4x\Psi_1^{(4)}(x) - 4\Phi_2^{(4)}(x) \\ & \quad + 4\Psi_2^{(4)}(x) = \varphi_1^{(4)}(x) - \varphi_3^{(2)}(x).\end{aligned} \tag{36}$$

That is

$$\begin{aligned}\Psi_2^{(4)}(x) = {} & \Phi_1^{(3)}(x) + x\Phi_1^{(4)}(x) - \Psi_1^{(3)}(x) - x\Psi_1^{(4)}(x) \\ & + \Phi_2^{(4)}(x) + \frac{1}{4}\left(\varphi_1^{(4)}(x) - \varphi_3^{(2)}(x)\right).\end{aligned} \tag{37}$$

Thus, the successive integrations of Eq.(37) give us

$$\begin{aligned}\Psi_2(x) = {} & \int_0^x\int_0^x\int_0^x\int_0^x \left[\Phi_1^{(3)}(x) + x\Phi_1^{(4)}(x) - \Psi_1^{(3)}(x) - x\Psi_1^{(4)}(x)\right] dxdxdxdx \\ & + \int_0^x\int_0^x\int_0^x\int_0^x \left[\Phi_2^{(4)}(x) + \frac{1}{4}\left(\varphi_1^{(4)}(x) - \varphi_3^{(2)}(x)\right)\right] dxdxdxdx \\ & + \left[\Psi_2(0) + x\Psi_2^{(1)}(0) + \frac{x^2}{2}\Psi_2^{(2)}(0) + \frac{x^3}{6}\Psi_2^{(3)}(0)\right].\end{aligned} \tag{38}$$

Also, from equation (19), we get

$$\begin{aligned}\Psi_3^{(5)}(x) = {} & -x\Phi_1^{(4)}(x) + 4\Psi_1^{(3)}(x) - \frac{1}{2}x^2\Psi_1^{(5)}(x) - \Phi_2^{(4)}(x) \\ & - x\Psi_2^{(5)}(x) + \frac{1}{2}\left(\varphi_3^{(2)}(x) + \varphi_2^{(3)}(x)\right).\end{aligned} \tag{39}$$

The successive integrations of Eq.(39) give us

$$\begin{aligned}\Psi_3(x) = {} & \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[-x\Phi_1^{(4)}(x) + 4\Psi_1^{(3)}(x) - \frac{1}{2}x^2\Psi_1^{(5)}(x)\right] dxdxdxdxdx \\ & + \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[-\Phi_2^{(4)}(x) - x\Psi_2^{(5)}(x)\right] dxdxdxdxdx \\ & + \frac{1}{2}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[\varphi_3^{(2)}(x) + \varphi_2^{(3)}(x)\right] dxdxdxdxdx \\ & + \left[\Psi_3(0) + x\Psi_3^{(1)}(0) + \frac{x^2}{2}\Psi_3^{(2)}(0) + \frac{x^3}{6}\Psi_3^{(3)}(0) + \frac{x^4}{24}\Psi_3^{(4)}(0)\right].\end{aligned} \tag{40}$$

Finally, from equations (24), we get

$$\begin{aligned}& 20\Phi_1^{(3)}(x) + x^2\Phi_1^{(5)}(x) + 10x\Psi_1^{(4)}(x) + 2x\Phi_2^{(5)}(x) \\ & \quad + 10\Psi_2^{(4)}(x) + 2\Phi_3^{(5)}(x) = \varphi_0^{(5)}(x) - \varphi_5(x).\end{aligned} \tag{41}$$

Solving Eq.(41) for $\Phi_3(x)$, we obtain

$$\Phi_3(x) = -\frac{1}{2}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[20\Phi_1^{(3)}(x) + x^2\Phi_1^{(5)}(x) + 10x\Psi_1^{(4)}(x)\right] dxdxdxdxdx \tag{42}$$
$$-\frac{1}{2}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[2x\Phi_2^{(5)}(x) + 10\Psi_2^{(4)}(x)\right] dxdxdxdxdx$$
$$+\frac{1}{2}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left[\varphi_0^{(5)}(x) - \varphi_5(x)\right] dxdxdxdxdx$$
$$+\left[\Phi_3(0) + x\Phi_3^{(1)}(0) + \frac{x^2}{2}\Phi_3^{(2)}(0) + \frac{x^3}{6}\Phi_3^{(3)}(0) + \frac{x^4}{24}\Phi_3^{(4)}(0)\right].$$

□

This completes the proof.

## 3. Adomian decomposition method

Now we shall use the modified Adomian's decomposition method [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] for solving (1)-(2) with a few modifications.

The solution proposed by Adomian is to take $L$ as the highest-ordered derivative of the linear part. Following the Adomian decomposition method, we write Eq.(1) in an operator form as

$$L_{tttttt}u = 3\frac{\partial^6 u}{\partial t^4 \partial x^2} - 3\frac{\partial^6 u}{\partial t^2 \partial x^4} + \frac{\partial^6 u}{\partial x^6}, \tag{43}$$

where $L_{tttttt}(u) = \frac{\partial^6 u}{\partial t^6}$.

Operating with the inverse operator $L^{-1}_{tttttt}(.) = \int_0^t\int_0^t\int_0^t\int_0^t\int_0^t\int_0^t(.)dtdtdtdtdtdt$, we have

$$u(t,x) = \varphi_0(x) + t\varphi_1(x) + \frac{t^2}{2!}\varphi_2(x) + \frac{t^3}{3!}\varphi_3(x) + \frac{t^4}{4!}\varphi_4(x) + \frac{t^5}{5!}\varphi_5(x) \tag{44}$$
$$+ L^{-1}_{tttttt}\left[3\frac{\partial^6 u}{\partial t^4 \partial x^2} - 3\frac{\partial^6 u}{\partial t^2 \partial x^4} + \frac{\partial^6 u}{\partial x^6}\right].$$

The unknown solution $u$ is assumed to be given by a series of the form

$$u = \sum_{n=0}^{\infty} a_n(x)t^n. \tag{45}$$

The substitution of (45) into (44) yields

$$\sum_{n=0}^{\infty} a_n(x)t^n = \sum_{i=0}^{5}\varphi_i(x)\frac{t^i}{i!} + L^{-1}_{tttttt}\left[3\frac{\partial^6}{\partial t^4\partial x^2}[\sum_{n=0}^{\infty} a_n(x)t^n]\right] \tag{46}$$
$$- L^{-1}_{tttttt}\left[3\frac{\partial^6}{\partial t^2\partial x^4}[\sum_{n=0}^{\infty} a_n(x)t^n]\right] + L^{-1}_{tttttt}\left[\frac{\partial^6}{\partial x^6}[\sum_{n=0}^{\infty} a_n(x)t^n]\right],$$

$$\sum_{n=0}^{\infty} a_n(x)t^n = \sum_{i=0}^{5}\varphi_i(x)\frac{t^i}{i!} + L^{-1}_{tttttt}\left[3\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)\frac{\partial^2 a_n(x)}{\partial x^2}t^{n-4}\right] \tag{47}$$
$$- L^{-1}_{tttttt}\left[3\sum_{n=2}^{\infty} n(n-1)\frac{\partial^4 a_n(x)}{\partial x^4}t^{n-2}\right] + L^{-1}_{tttttt}\left[\sum_{n=0}^{\infty}\frac{\partial^6}{\partial x^6}t^n\right],$$

$$\sum_{n=0}^{\infty} a_n(x)t^n = \sum_{i=0}^{5} \varphi_i(x)\frac{t^i}{i!} + L_{tttttt}^{-1}\left[3\sum_{n=0}^{\infty}(n+4)(n+3)(n+2)(n+1)\frac{\partial^2 a_{n+4}(x)}{\partial x^2}t^n\right] \tag{48}$$

$$- L_{tttttt}^{-1}\left[3\sum_{n=0}^{\infty}(n+2)(n+1)\frac{\partial^4 a_{n+2}(x)}{\partial x^4}t^n\right] + L_{tttttt}^{-1}\left[\sum_{n=0}^{\infty}\frac{\partial^6}{\partial x^6}t^n\right].$$

We now carry out the above integrations to write

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(x)t^n = \sum_{i=0}^{5} \varphi_i(x)\frac{t^i}{i!} &+ \left[3\sum_{n=0}^{\infty}\frac{t^{n+6}}{(n+5)(n+6)}\frac{\partial^2 a_{n+4}(x)}{\partial x^2}\right] \\ &- \left[3\sum_{n=0}^{\infty}\frac{t^{n+6}}{(n+3)(n+4)(n+5)(n+6)}\frac{\partial^4 a_{n+2}(x)}{\partial x^4}\right] \\ &+ \left[\sum_{n=0}^{\infty}\frac{t^{n+6}}{(n+1)(n+2)...(n+6)}\frac{\partial^6 a_n(x)}{\partial x^6}\right].\end{aligned} \tag{49}$$

In the summation on the right, $n$ can be replaced by and $n-6$ to write

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(x)t^n = \sum_{i=0}^{5} \varphi_i(x)\frac{t^i}{i!} &+ \left[3\sum_{n=6}^{\infty}\frac{t^{n}}{(n-1)n}\frac{\partial^2 a_{n-2}(x)}{\partial x^2}\right] \\ &- \left[3\sum_{n=6}^{\infty}\frac{t^{n}}{(n-3)(n-2)(n-1)n}\frac{\partial^4 a_{n-4}(x)}{\partial x^4}\right] \\ &+ \left[\sum_{n=6}^{\infty}\frac{t^{n}}{(n-5)(n-4)...(n)}\frac{\partial^6 a_{n-6}(x)}{\partial x^6}\right].\end{aligned} \tag{50}$$

Finally, we can equate coefficients of like powers of $t$ on the left side and on the right side to obtain the recurrence relations for the coefficients. Thus

$$\begin{cases} a_0 = \varphi_0(x), \\ a_1 = \varphi_1(x), \\ a_2 = \dfrac{\varphi_2(x)}{2!}, \\ a_3 = \dfrac{\varphi_3(x)}{3!}, \\ a_4 = \dfrac{\varphi_4(x)}{4!}, \\ a_5 = \dfrac{\varphi_5(x)}{5!}, \\ ... \\ a_n = \dfrac{\frac{\partial^6 a_{n-6}}{\partial x^6} - 3(n-5)(n-4)\frac{\partial^4 a_{n-4}}{\partial x^4} + 3(n-2)(n-3)(n-4)(n-5)\frac{\partial^2 a_{n-2}}{\partial x^2}}{(n-5)(n-4)(n-3)(n-2)(n-1)n}, \ n \geqslant 6. \end{cases}$$

## 4. Applications

In order to demonstrate the feasibility and efficiency of these methods, some examples with a priori known exact solutions are studied in detail.

**Example 1.** *Consider problem (1)-(2) with*

$$\varphi_0(x) = x^6,\ \varphi_1(x) = 0,\ \varphi_2(x) = 6x^4,\ \varphi_3(x) = 0,\ \varphi_4(x) = 72x^2 \text{ and } \varphi_5(x) = 0.$$

*Using Theorem 1, and straightforward computation yields*

$$\Phi_1(x) = \frac{3}{4}x^4,$$

$$\Psi_1(x) = \frac{3}{4}x^4,$$

$$\Phi_2(x) = 0,$$

$$\Psi_2(x) = 0,$$

$$\Phi_3(x) = \frac{1}{8}x^6,$$

*and*

$$\Psi_3(x) = \frac{1}{8}x^6.$$

*Then, the solution of this problem is expressed by (5) as*

$$u(t,x) = x^6 + 3x^4t^2 + 3x^2t^4 + t^6.$$

*Now, by modified decomposition method, the components $a_n,\ n = 0, 1...$ are determined by the recursive algorithm*

$$\begin{cases} a_0 &= x^6, \\ a_1 &= 0, \\ a_2 &= 3x^4, \\ a_3 &= 0, \\ a_4 &= 3x^2, \\ a_5 &= 0, \\ a_6 &= 1, \\ ... & \\ a_n &= 0,\ n \geq 7. \end{cases}$$

*Thus the solution*

$$u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n = x^6 + 3x^4t^2 + 3x^2t^4 + t^6,$$

*which can be verified through substitution to be the exact solution.*

**Example 2.** *Consider equation (1)-(2) with*

$$\varphi_0(x) = 0,\ ,\varphi_1(x) = 2e^x,\ \varphi_2(x) = 0,\ \varphi_3(x) = 2e^x,\ \varphi_4(x) = 0,\ \varphi_5(x) = 2e^x.$$

*Then by Theorem 1, we obtain*

$$\Phi_1(x) = \Phi_1(0) + x\Phi_1^{(1)}(0) + \frac{x^2}{2}\Phi_1^{(2)}(0),$$

$$\Psi_1(x) = \Psi_1(0) + x\Psi_1^{(1)}(0) + \frac{x^2}{2}\Psi_1^{(2)}(0),$$

$$\Phi_2(x) = \Phi_2(0) + x\Phi_2^{(1)}(0) + \frac{x^2}{2}\Phi_2^{(2)}(0) + \frac{x^3}{6}\Phi_2^{(3)}(0),$$

$$\Psi_2(x) = \Psi_2(0) + x\Psi_2^{(1)}(0) + \frac{x^2}{2}\Psi_2^{(2)}(0) + \frac{x^3}{6}\Psi_2^{(3)}(0),$$

$$\Phi_3(x) = \Phi_3(0) + x\Phi_3^{(1)}(0) + \frac{x^2}{2}\Phi_3^{(2)}(0) + \frac{x^3}{6}\Phi_3^{(3)}(0) + \frac{x^4}{24}\Phi_3^{(4)}(0),$$

*and*

$$\Psi_3(x) = \Psi_3(0) + x\Psi_3^{(1)}(0) + \frac{x^2}{2}\Psi_3^{(2)}(0) + \frac{x^3}{6}\Psi_3^{(3)}(0) + \frac{x^4}{24}\Psi_3^{(4)}(0).$$

*Direct calculation produces*

$$\Phi_1(x) = \Psi_1(x) = 0, \ \Phi_2(x) = \Psi_2(x) = 0$$

*and*

$$\Phi_3(x) = -e^x, \ \Psi_3(x) = e^x.$$

*Then, the solution of this problem is expressed as*

$$u(t,x) = e^{x+t} - e^{x-t}.$$

*By modified decomposition method, the components* $a_n, \ n = 0, 1...$ *are determined by the recursive algorithm Now, applying the above recurrent scheme, we obtain*

$$\begin{cases} a_n &= 0, \ n = 0, 2, 4, ..., \\ a_n &= 2\frac{e^x}{n!}, \ n = 1, 3, 5, ..., \end{cases}$$

*and*

$$u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n = \frac{2e^x}{1!}t + \frac{2e^x}{3!}t^3 + \frac{2e^x}{5!}t^5 + ...,$$

*which is the partial sum of the Taylor series of the exact solution* $e^{x+t} - e^{x-t}$.

## 5. Conclusion

The modified decomposition method has been proved to be reliable in handing the initial value problems for linear hyperbolic equations of sixth-order. Some examples with closed form solutions are studied, and the results obtained by theorem 1 are just the same as those given from applying the modified decomposition method.

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