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ON SOME INEQUALITIES OF SIMPSON–TYPE VIA QUASI–CONVEX FUNCTIONS AND APPLICATIONS

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ABSTRACT. Some inequalities of Simpson's type for quasi–convex functions are introduced. In the literature the error estimates for the midpoint rule is $|E_{mid}(f,d)| \leq \frac{K}{24}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3$, in this paper we restrict the conditions on f to get better error estimates than the original.

1. INTRODUCTION

Suppose $f:[a,b]\to\mathbb{R}$ is fourth times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty := \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$. The following inequality

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4 \qquad (1)$$

holds, and it is well known in the literature as Simpson's inequality.

It is well known that if the mapping f is neither four times differentiable nor is the fourth derivative $f^{(4)}$ bounded on (a,b), then we cannot apply the classical Simpson quadrature formula. In recent years many authors were established an error estimations for the Simpson's inequality, for refinements, counterparts, generalizations and new Simpson's-type inequalities see [4]–[12] and [14]–[18].

The notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f:[a,b]\to\mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(\lambda x+(1-\lambda)y) \leq \sup\{f(x), f(y)\},$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex, (see [13]). On the other hand, a quasi-convex function may be neither convex nor continuous. For example, the *floor function* $f_{loor}(x)=\lfloor x\rfloor$, is the largest integer not greater than x, is an example of a monotonic increasing function which is quasi-convex but it is neither convex nor continuous.

For recent results and generalizations concerning quasi-convex functions see [1]–[3] and [13].

The aim of this paper is to establish Simpson's type inequalities based on quasi-convexity. We will show that our results can be used in order to give estimates for the approximation error of the integral $\int_a^b f(x)\,dx$ in the Simpson's formula without going through its higher derivatives which may not exists, not bounded or may be hard to find. A restriction made on a quasi–convex function to deduce new error estimates for the midpoint rule.

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2. Inequalities of Simpson's type for quasi-convex functions

In order to prove our main theorems, we need the following lemma (see [16]):

Lemma 1. *Assume $f' : I \subseteq \mathbb{R} \to \mathbb{R}$ be a absolutely continuous mapping on I° and $f'' \in L[a,b]$ for some $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = (b-a)^2 \int_0^1 p(t)\, f''(tb + (1-t)a)\,dt \qquad (2)$$

where,

$$p(t) = \begin{cases} \frac{1}{6}t(3t-1), & t \in \left[0, \frac{1}{2}\right] \\ \frac{1}{6}(t-1)(3t-2), & t \in \left(\frac{1}{2}, 1\right] \end{cases}$$

Proof. We note that

$$I = \int_0^1 p(t)\, f''(tb + (1-t)a)\,dt = \frac{1}{6}\int_0^{1/2} t(3t-1) f''(tb + (1-t)a)\,dt + \frac{1}{6}\int_{1/2}^1 (t-1)(3t-2) f''(tb + (1-t)a)\,dt.$$

Integrating by parts, we get

$$\begin{aligned} I &= \frac{1}{6}t(3t-1)\frac{f'(tb + (1-t)a)}{b-a}\bigg|_0^{1/2} - \left[\frac{1}{2}t + \frac{1}{6}(3t-1)\right]\frac{f(tb + (1-t)a)}{(b-a)^2}\bigg|_0^{1/2} \\ &\quad + \int_0^{1/2}\frac{f(tb + (1-t)a)}{(b-a)^2}dt + \frac{1}{6}(t-1)(3t-2)\frac{f'(tb + (1-t)a)}{b-a}\bigg|_{1/2}^1 \\ &\quad - \left[\frac{1}{2}(t-1) + \frac{1}{6}(3t-2)\right]\frac{f(tb + (1-t)a)}{(b-a)^2}\bigg|_{1/2}^1 + \int_{1/2}^1 \frac{f(tb + (1-t)a)}{(b-a)^2}dt \\ &= \frac{1}{24}\frac{f'\left(\frac{a+b}{2}\right)}{b-a} - \frac{1}{3}\frac{f\left(\frac{a+b}{2}\right)}{(b-a)^2} - \frac{1}{6}\frac{f(a)}{(b-a)^2} + \int_0^{1/2}\frac{f(tb + (1-t)a)}{(b-a)^2}dt \\ &\quad - \frac{1}{6}\frac{f(b)}{(b-a)^2} - \frac{1}{24}\frac{f'\left(\frac{a+b}{2}\right)}{b-a} - \frac{1}{3}\frac{f\left(\frac{a+b}{2}\right)}{(b-a)^2} + \int_{1/2}^1 \frac{f(tb + (1-t)a)}{(b-a)^2}dt \\ &= \frac{1}{(b-a)^2}\int_0^1 f(tb + (1-t)a)\,dt - \frac{1}{6(b-a)^2}\left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)\right]. \end{aligned}$$

Setting $x = tb + (1-t)a$, and $dx = (b-a)dt$, gives

$$(b-a)^2 \cdot I = \frac{1}{(b-a)}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

which gives the desired representation (2). □

The next theorem gives a new refinement of the Simpson's inequality for quasi-convex functions.

Theorem 1. *Let $f' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$, such that $f'' \in L[a, b]$. If $|f''|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{162}\cdot\left[\sup\left\{|f''(a)|, \left|f''\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|, |f''(b)|\right\}\right]. \quad (3)$$

Proof. By Lemma 1 and since $|f''|$ is quasi-convex, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{6}\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb + (1-t)a)|\,dt$$
$$+ \frac{(b-a)^2}{6}\int_{\frac{1}{2}}^1 |t-1|\,|3t-2|\,|f''(tb+(1-t)a)|\,dt$$
$$\leq \frac{(b-a)^2}{6}\cdot\sup\left\{|f''(a)|, \left|f''\left(\frac{a+b}{2}\right)\right|\right\}\left(\int_0^{\frac{1}{3}} t(1-3t)\,dt + \int_{\frac{1}{3}}^{\frac{1}{2}} t(3t-1)\,dt\right)$$
$$+ \frac{(b-a)^2}{6}\cdot\sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|, |f''(b)|\right\}\left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-t)(2-3t)\,dt\right.$$
$$\left. + \int_{\frac{2}{3}}^1 (1-t)(3t-2)\,dt\right)$$
$$\leq \frac{(b-a)^2}{162}\cdot\left[\sup\left\{|f''(a)|, \left|f''\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|, |f''(b)|\right\}\right],$$

which completes the proof. □

Corollary 1. *In Theorem 1, Additionally, if*

(1) $|f''|$ is increasing, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{162}\cdot\left[\left|f''\left(\frac{a+b}{2}\right)\right| + |f''(b)|\right], \quad (4)$$

(2) $|f''|$ is decreasing, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{162}\cdot\left[|f''(a)| + \left|f''\left(\frac{a+b}{2}\right)\right|\right]. \quad (5)$$

The corresponding version for powers of the absolute value of the second derivative is incorporated in the following result:

Theorem 2. *Let $f' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$, such that $f'' \in L[a, b]$. If $|f''|^{p/(p-1)}$ is quasi-convex on $[a, b]$, for some fixed $p > 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$

$$\leq \frac{(b-a)^2}{6}\cdot\left(3^{-p-1}\beta(p+1,p+1) + \frac{4(3)^{-p} + 3(2)^{-p}(p-1)}{12(2+3p+p^2)}\right)^{\frac{1}{p}}$$
$$\left[\left(\sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f''(b)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right.$$
$$\left. + \left(\sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, |f''(a)|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right]. \quad (6)$$

for $p > 1$, where, where $\beta(x, y)$ is the Beta function of Euler type.

Proof. Suppose that $p > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{6}\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb+(1-t)a)|\,dt$$
$$+ \frac{(b-a)^2}{6}\int_{\frac{1}{2}}^1 |t-1|\,|3t-2|\,|f''(tb+(1-t)a)|\,dt$$
$$\leq \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}} (t\,|3t-1|)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} |f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-a)^2}{6}\left(\int_{\frac{1}{2}}^1 (|t-1|\,|3t-2|)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 |f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{3}} t^p(1-3t)^p\,dt + \int_{\frac{1}{3}}^{\frac{1}{2}} t^p(3t-1)^p\,dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^{\frac{1}{2}} |f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$

$$+ \frac{(b-a)^2}{6}\left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-t)^p(2-3t)^p\,dt + \int_{\frac{2}{3}}^1 (1-t)^p(3t-2)^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\frac{1}{2}}^{1}\left|f''\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}$$

Since f is quasi-convex, we have

$$\int_{0}^{1/2}\left|f''\left(tb+(1-t)a\right)\right|^{q}dt\leq\sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{q},\left|f''(a)\right|^{q}\right\},\tag{7}$$

and

$$\int_{1/2}^{1}\left|f''\left(tb+(1-t)a\right)\right|^{q}dt\leq\sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{q},\left|f''(b)\right|^{q}\right\}.\tag{8}$$

Therefore,

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|$$
$$\leq\frac{(b-a)^{2}}{6}\cdot\left(3^{-p-1}\beta(p+1,p+1)+\frac{4(3)^{-p}+3(2)^{-p}(p-1)}{12(2+3p+p^{2})}\right)^{\frac{1}{p}}$$
$$\left[\left(\sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},\left|f''(b)\right|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right.$$
$$\left.+\left(\sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{p/(p-1)},\left|f''(a)\right|^{p/(p-1)}\right\}\right)^{\frac{p-1}{p}}\right],$$

for $p>1$, where we have used the fact that

$$\int_{0}^{\frac{1}{3}}t^{p}(1-3t)^{p}dt=\int_{\frac{2}{3}}^{1}(1-t)^{p}(3t-2)^{p}dt=3^{-p-1}\beta(p+1,p+1),$$

and

$$\int_{\frac{1}{3}}^{\frac{1}{2}}t^{p}(3t-1)^{p}dt=\int_{\frac{1}{2}}^{\frac{2}{3}}(1-t)^{p}(2-3t)^{p}dt=\frac{4(3)^{-p}+3(2)^{-p}(p-1)}{12(2+3p+p^{2})},$$

which completes the proof. □

Corollary 2. *Let f be as in Theorem 2. Additionally, if*

(1) $|f''|$ is increasing, then we have

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq\frac{(b-a)^{2}}{6}\cdot\left(3^{-p-1}\beta(p+1,p+1)+\frac{4(3)^{-p}+3(2)^{-p}(p-1)}{12(2+3p+p^{2})}\right)^{\frac{1}{p}}$$
$$\times\left(\left|f''\left(\frac{a+b}{2}\right)\right|+\left|f''(b)\right|\right),\tag{9}$$

(2) $|f''|$ is decreasing, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\le \frac{(b-a)^2}{6}\cdot\left(3^{-p-1}\beta(p+1,p+1)+\frac{4(3)^{-p}+3(2)^{-p}(p-1)}{12(2+3p+p^2)}\right)^{\frac{1}{p}}$$
$$\times\left(|f''(a)|+\left|f''\left(\frac{a+b}{2}\right)\right|\right), \quad (10)$$

for $p>1$, where, where $\beta(x,y)$ is the Beta function of Euler type.

Proof. It follows directly by Theorem 2. □

A generalization of (3) is given in the following theorem:

Theorem 3. *Let $f': I\subseteq\mathbb{R}\to\mathbb{R}$ be an absolutely continuous function on I° and $a,b\in I$ with $a<b$, such that $f''\in L[a,b]$. If $|f''|^q$ is quasi-convex on $[a,b]$, $q\ge 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\le \frac{(b-a)^2}{162}\left[\left(\sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right\}\right)^{\frac{1}{q}}\right]. \quad (11)$$

Proof. Suppose that $q\ge 1$. From Lemma 1 and using the power mean inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\le \frac{(b-a)^2}{6}\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb+(1-t)a)|\,dt$$
$$+\frac{(b-a)^2}{6}\int_{\frac{1}{2}}^1 |t-1|\,|3t-2|\,|f''(tb+(1-t)a)|\,dt$$

$$\le \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}} t\,|3t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-a)^2}{6}\left(\int_{\frac{1}{2}}^1 |t-1|\,|3t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 |t-1|\,|3t-1|\,|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$

$$= \frac{(b-a)^2}{6} \left(\int_0^{\frac{1}{3}} t(1-3t)\,dt + \int_{\frac{1}{3}}^{\frac{1}{2}} t(3t-1)\,dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb+(1-t)a)|^q\,dt \right)^{\frac{1}{q}}$$

$$+ \frac{(b-a)^2}{6} \left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-t)(2-3t)\,dt + \int_{\frac{2}{3}}^{1} (1-t)(3t-2)\,dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_{\frac{1}{2}}^{1} |t-1|\,|3t-2|\,|f''(tb+(1-t)a)|^q\,dt \right)^{\frac{1}{q}}$$

Since f is quasi-convex, we have

$$\int_0^{\frac{1}{2}} t\,|3t-1|\,|f''(tb+(1-t)a)|^q\,dt = \frac{1}{27} \sup\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \tag{12}$$

and

$$\int_{1/2}^{1} |t-1|\,|3t-2|\,|f''(tb+(1-t)a)|^q\,dt = \frac{1}{27} \sup\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \tag{13}$$

where, we used the fact

$$\int_0^{1/2} t\,|3t-1|\,dt = \int_{1/2}^{1} |t-1|\,|3t-2|\,dt = \frac{1}{27}. \tag{14}$$

Combination of (12), (13) and (14), gives the required result which completes the proof. □

Corollary 3. *Let f be as in Theorem 3. Additionally, if*

(1) $|f''|$ is increasing, then the inequality (4).
(2) $|f''|$ is decreasing, then the inequality (5).

Proof. It follows directly by Theorem 3. □

Remark 1. *For*

$$h(p) = \left(3^{-p-1}\beta(p+1,p+1) + \frac{4(3)^{-p} + 3(2)^{-p}(p-1)}{12(2+3p+p^2)} \right)^{\frac{1}{p}}, \quad p > 1,$$

we have

$$\lim_{p \to 1^+} h(p) = \frac{1}{27},$$

using the fact

$$\sum_{i=1}^{n} (a_i+b_i)^r \le \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r,$$

for $0<r<1$, $a_1, a_2, ..., a_n \ge 0$ and $b_1, b_2, ..., b_n \ge 0$, we obtain

$$\lim_{p\to\infty} h(p) \le \lim_{p\to\infty} 3^{-1-\frac{1}{p}} \beta^{\frac{1}{p}}(p+1,p+1) + \lim_{p\to\infty} \frac{4^{\frac{1}{p}}(3)^{-1} + 3^{\frac{1}{p}}(2)^{-1}(p-1)^{\frac{1}{p}}}{(12)^{\frac{1}{p}}(2+3p+p^2)^{\frac{1}{p}}}$$
$$= \frac{1}{3}\lim_{p\to\infty} \beta^{\frac{1}{p}}(p+1,p+1) + 1,$$

also, Stirling's approximation gives the asymptotic formula

$$\beta(x,y) \simeq \sqrt{2\pi}\frac{x^{x-\frac{1}{2}}y^{y-\frac{1}{2}}}{(x+y)^{x+y-\frac{1}{2}}},$$

$$\lim_{p\to\infty} \beta^{\frac{1}{p}}(p+1,p+1) \cong \sqrt{2\pi}\lim_{p\to\infty}\frac{(p+1)^{2p+1}}{(2p+2)^{2p+\frac{3}{2}}} = \lim_{p\to\infty}\frac{\sqrt{2\pi}}{(2)^{2p+\frac{3}{2}}}\frac{1}{(p+1)^{\frac{1}{2}}} \to 0,$$

so that, $\lim_{p\to\infty} h(p) \to 1$, *therefore* $h(p)$ *satisfies*

$$\frac{1}{27} \le h(p) \le 1.$$

Hence, we observe that the inequality (11) is better than the inequality (2) meaning that the approach via power mean inequality is a better approach than the one through Hölder's inequality.

3. Applications to Some Numerical Quadrature Rules

Let d be a division of the interval $[a,b]$, i.e., $d : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$, $h_i = (x_{i+1} - x_i)/2$ and consider the Simpson's formula

$$S(f,d) = \sum_{i=0}^{n-1} \frac{f(x_i) + 4f(x_i+h_i) + f(x_{i+1})}{6}(x_{i+1}-x_i). \tag{15}$$

It is well known that if the mapping $f : [a,b] \to \mathbb{R}$, is differentiable such that $f^{(4)}(x)$ exists on (a,b) and $M = \max_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$, then

$$I = \int_a^b f(x)\,dx = S(f,d) + E_S(f,d), \tag{16}$$

where the approximation error $E_S(f,d)$ of the integral I by the Simpson's formula $S(f,d)$ satisfies

$$|E_S(f,d)| \le \frac{M}{2880}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^5. \tag{17}$$

It is clear that if the mapping f is not fourth differentiable or the fourth derivative is not bounded on (a,b), then (16) cannot be applied. In the following we give many different estimations for the remainder term $E(f,d)$ in terms of the second derivative.

Proposition 1. *Let* $f' : I \subseteq \mathbb{R} \to \mathbb{R}$ *be an absolutely continuous function on* I° *and* $a, b \in I$ *with* $a < b$, *such that* $f'' \in L[a,b]$. *If* $|f''|$ *is quasi-convex on* $[a,b]$, *then in (16),* *for every division d of $[a,b]$, the following holds:*

$$|E_S(f,d)| \leq \frac{1}{162}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left[\sup\left\{\left|f''\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f''(x_{i+1})|\right\}\right.$$
$$\left.+\sup\left\{\left|f''\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f''(x_i)|\right\}\right]. \tag{18}$$

Proof. Applying Theorem 1 on the subintervals $[x_i, x_{i+1}]$, $(i = 0, 1, ..., n-1)$ of the division d, we get

$$\left|\frac{f(x_i)+4f(x_i+h_i)+f(x_{i+1})}{6}(x_{i+1}-x_i)-\int_{x_i}^{x_{i+1}} f(x)\,dx\right|$$
$$\leq (x_{i+1}-x_i)\left[\sup\left\{\left|f''\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f''(x_{i+1})|\right\}\right.$$
$$\left.+\sup\left\{\left|f''\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f''(x_i)|\right\}\right].$$

Summing over i from 0 to $n-1$ and taking into account that $|f'|$ is quasi-convex, we deduce that

$$\left|S(f,d)-\int_a^b f(x)\,dx\right| \leq \frac{1}{162}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left[\sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_{i+1})|\right\}\right.$$
$$\left.+\sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, |f'(x_i)|\right\}\right],$$

which completes the proof. □

Remark 2. *It is well known that, if the mapping $f : [a,b] \to \mathbb{R}$, is twice differentiable such that $f''(x)$ exists on (a,b) and $K = \sup_{x\in(a,b)} |f''(x)| < \infty$, then*

$$I = \int_a^b f(x)\,dx = M(f,d) + E_{mid}(f,d), \tag{19}$$

where the approximation error $E_{mid}(f,d)$ of the integral I by the midpoint formula $M(f,d)$ satisfies

$$|E_{mid}(f,d)| \leq \frac{K}{24}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3. \tag{20}$$

In the following, we introduce a best error estimate for the midpoint inequality with the assumptions that:

In Theorem 1, Additionally, if $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, then we have,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right|$$
$$\leqslant \frac{(b-a)^2}{162}\cdot\left[\sup\left\{|f''(a)|, \left|f''\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f''\left(\frac{a+b}{2}\right)\right|, |f''(b)|\right\}\right] \tag{21}$$

For instance, for $K > 0$, if $|f''(x)| < K$, for all $x \in [a,b]$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{81}K. \tag{22}$$

Therefore, the error E_{mid} can be estimated, such as:

$$|E_{mid}(f,d)| \leq \frac{K}{81}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3. \tag{23}$$

and so that, the error estimates in (23) is best than the original in (20).

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