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ON SOME MULTIPLIER DIFFERENCE SEQUENCE SPACES DEFINED OVER A 2-NORMED LINEAR SPACE

B.SURENDER REDDY AND HEMEN DUTTA

ABSTRACT. In this paper, we introduce a new class of generalized difference sequences with base space, a real linear 2-normed space and by means of a fixed multiplier. We study the spaces of thus constructed classes of sequences for relevant linear topological structures. Further we investigate the spaces for solidity, monotonicity, symmetricity etc. We also obtain some relations between these spaces as well as prove some inclusion results.

1. INTRODUCTION

Let w, ℓ_∞, c and c_0 denote the spaces of *all*, *bounded*, *convergent* and *null* (set of convergent scalar sequences with limit zero) sequences $x = (x_k)$ with complex terms respectively. The zero sequence is denoted by $\theta = (0, 0, \dots)$.

The notion of difference sequence space was introduced by Kizmaz [12], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [5] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [18], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$.

Tripathy, Esi and Tripathy [19] generalized the above notions and unified these as follows:

Let m, n be non-negative integers, then for Z a given sequence space we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\},$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}$$

Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [5]. Taking $n = 1$, we get the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi [18]. Taking $m = n = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [12].

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for E a sequence space, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence Λ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}$$

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [9] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, using the multiplier

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sequences (k^{-1}) and (k) respectively. A multiplier sequence can be used to accelerate the convergence of the sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of convergence of a sequence.

The concept of 2-normed spaces was initially developed by Gähler [7] in the mid of 1960's. Since then, Gunawan and Mashadi [11] and many others have studied this concept and obtained various results.

Let X be a real linear space of dimension greater than one and let $\|\bullet,\bullet\|$ be a real valued function on $X \times X$ satisfying the following conditions
$2N_1$: $\|x,y\| = 0$ if and only if x and y are linearly dependent vectors,
$2N_2$: $\|x,y\| = \|y,x\|$,
$2N_3$: $\|\alpha x,y\| \leq |\alpha|\ \|x,y\|$, for every $\alpha \in R$
$2N_4$: $\|x,y+z\| \leq \|x,y\| + \|x,z\|$
then the function $\|\bullet,\bullet\|$ is called a 2-norm on X and the pair $(X,\|\bullet,\bullet\|)$ is called a 2-normed linear space.

The following inequality will be used throughout the article.

Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G$, $D = \max(1, 2^{G-1})$. Then for all $a_k, b_k \in C$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \leq D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}$$

and for $\lambda \in C$, $|\lambda|^{p_k} \leq \max(1, |\lambda|^G)$.

The studies on paranormed sequence spaces were initiated by Nakano [16] and Simons [17] at the initial stage. Later on it was further studied by Maddox [15], Lascardies [13], Lascardies and Maddox [14], Ghosh and Srivastava [10] and many others.

2. Definitions and preliminaries

A sequence space E is said to be solid (or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$.

A sequence space E is said to be monotone if it contains the canonical preimages of all its step spaces.

A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$, where π is a permutation on N.

A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $y_k = 0$ whenever $x_k = 0$.

A sequence space E is said to be sequence algebra if $(x_k, y_k) \in E$ whenever $(x_k) \in E$ and $(y_k) \in E$.

A sequence (x_k) in a 2-normed space $(X,\|\bullet,\bullet\|)$ is said to converge to some $L \in X$ in the 2-norm if $\lim_{k\to\infty} \|x_k - L, u\| = 0$, for every $u \in X$.

A sequence (x_k) in a 2-normed space $(X,\|\bullet,\bullet\|)$ is said to be Cauchy sequence with respect to the 2-norm if $\lim_{k,l\to\infty} \|x_k - x_l, u\| = 0$, for every $u \in X$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be 2-Banach space.

Let us now consider the following known example of 2-norms.

Example 1. *Let C_0 be the linear space of all sequences of real numbers with only finite number of non-zero terms. For $x = (x_k)$, $y = (y_k)$ in C_0, let us define:*
$\|x,y\| = 0$, *if x, y are linearly dependent,*
$= \sum_{k=1}^{\infty} |x_k|\ |y_k|$, *if x, y are linearly independent.*
Then it is obvious that $\|\bullet,\bullet\|$ is a 2-norm on C_0.

Example 2. *Let us take $X = R^2$ and consider the function $\|\bullet,\bullet\|$ on X defined as:*

$$\|x_1, x_2\|_E = abs\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}\right),$$

where $x_i = (x_{i1}, x_{i2}) \in R^2$ for each $i = 1, 2$.
Then $\|\bullet,\bullet\|_E$ is a 2-norm on X and known as Euclidean 2-norm.

Let $p = (p_k)$ be any bounded sequence of positive real numbers and $\Lambda = (\lambda_k)$ be a sequence of non-zero reals. Let m, n be non-negative integers, then for a real linear 2-normed space $(X, \|\bullet,\bullet\|)$ we define the following sequence spaces:
$c_0(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p) =$

$$\left\{x = (x_k) \in w(X) : \lim_{k\to\infty}\left(\|\Delta^n_{(m)}\lambda_k x_k, z\|\right)^{p_k} = 0, \text{ for every } z \text{ in } X\right\},$$

$c(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p) =$

$$\left\{x = (x_k) \in w(X) : \lim_{k\to\infty}\left(\|\Delta^n_{(m)}\lambda_k x_k - L, z\|\right)^{p_k} = 0, \text{ for every } z \text{ and for some } L \in X\right\}$$

$\ell_\infty(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p) =$

$$\left\{x = (x_k) \in w(X) : \sup_{k\geq 1}\left(\|\Delta^n_{(m)}\lambda_k x_k, z\|\right)^{p_k} < \infty, \text{ for every } z \text{ in } X\right\},$$

where $(\Delta^n_{(m)}\lambda_k x_k) = (\Delta^{n-1}_{(m)}\lambda_k x_k - \Delta^{n-1}_{(m)}\lambda_{k-m}x_{k-m})$ and $\Delta^0_{(m)}\lambda_k x_k = \lambda_k x_k$ for all $k \in N$ and which is equivalent to the binomial representation

$$\Delta^n_{(m)}\lambda_k x_k = \sum_{v=0}^{n}(-1)^v\binom{n}{v}\lambda_{k-mv}x_{k-mv}.$$

In the above expansion we take $x_k = 0$ and $\lambda_k = 0$, for non-positive values of k.

It is obvious that

$$c_0(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p) \subset c(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p) \subset \ell_\infty(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$$

The inclusions are strict follows from the following examples.

Example 3. *Let $m = 2$, $n = 2$ and $p_k = 1$ for all $k \geq 1$. Consider the 2-normed space of Example 1 and let the sequences $\Lambda = (k^8)$ and $x = (\frac{1}{k^6})$. Then $x \in c(\|\bullet,\bullet\|, \Delta^2_{(2)}, \Lambda, p)$, but $x \notin c_0(\|\bullet,\bullet\|, \Delta^2_{(2)}, \Lambda, p)$.*

Example 4. *Let $m = 2$, $n = 2$ and $p_k = 2$ for all k odd and $p_k = 3$ for all k even. Consider the 2-normed space of Example 1 and let the sequences $\Lambda = (1, 1, 1, \dots)$ and $x = \{1, 3, 2, 4, 5, 7, 6, 8, 9, 11, 10, 12, \dots\}$. Then $x \in \ell_\infty(\|\bullet,\bullet\|, \Delta^2_{(2)}, \Lambda, p)$, but $x \notin c(\|\bullet,\bullet\|, \Delta^2_{(2)}, \Lambda, p)$.*

Lemma 1. *If a sequence space E is solid, then E is monotone.*

3. Main results

In this section we prove the main results of this article.

Proposition 1. *The classes of sequences $c_0(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$, $c(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$ and $\ell_\infty(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$ are linear.*

Proof. Proof is easy and so omitted. □

Theorem 1. *For $Z = \ell_\infty$, c and c_0, the spaces $Z(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$ are paranormed sapces, paranormed by*

$$g(x) = \sup_{k\geq 1,\ z\in X}\left(\|\Delta^n_{(m)}\lambda_k x_k, z\|\right)^{\frac{p_k}{H}},\ \text{where } H = max(1, \sup_{k\geq 1} p_k).$$

Proof. Clearly $g(x) = g(-x)$; $x = \theta$ implies $g(\theta) = 0$. Let (x_k) and (y_k) be any two elements belongs to any of the space $c_0(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$. Then we have,

$$g(x+y) = e\sup_{k\geq 1,\ z\in X}\left(\|\Delta^n_{(m)}\lambda_k x_k + \Delta^n_{(m)}\lambda_k y_k, z\|\right)^{\frac{p_k}{H}}$$

$$\leq \sup_{k\geq 1,\ z\in X}\left(\|\Delta^n_{(m)}\lambda_k x_k, z\|\right)^{\frac{p_k}{H}} + \sup_{k\geq 1,\ z\in X}\left(\|\Delta^n_{(m)}\lambda_k y_k, z\|\right)^{\frac{p_k}{H}}$$

$\implies g(x+y) \leq g(x) + g(y)$.
The continuity of the scalar multiplication follows from the following equality:

$$g(\alpha x) = \sup_{k\geq 1,\ z\in X}\left(\|\Delta^n_{(m)}\alpha\lambda_k x_k, z\|\right)^{\frac{p_k}{H}}$$

$$= \sup_{k\geq 1,\ z\in X}\left(|\alpha|\ \|\Delta^n_{(m)}\lambda_k x_k, z\|\right)^{\frac{p_k}{H}}$$

$$\leq \max(\ 1 + [\ |\alpha|\ ]\ )g(x),$$

where $[\ |\alpha|\ ]$ denotes the largest integer contained in $|\alpha|$. Hence the spaces $c_0(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$ is a paranormed space, paranormed by g. The rest of the cases will follow similarly. □

Theorem 2. *If $(X,\|\bullet,\bullet\|)$ is a 2-Banach space, then the spaces $Z(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$, for $Z = \ell_\infty$, c and c_0 are complete paranormed spaces, paranormed by*

$$g(x) = \sup_{k\geq 1,\ z\in X}\left(\|\Delta^n_{(m)}\lambda_k x_k, z\|\right)^{\frac{p_k}{H}},\ \text{where } H = max(1, \sup_{k\geq 1} p_k).$$

Proof. We prove the result for the space $\ell_\infty(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$ and for other spaces it will follow on applying similar arguments.

Let (x^i) be any Cauchy sequence in $\ell_\infty(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$. Let $\varepsilon > 0$ be given, then there exists a positive integer n_0 such that $g(x^i - x^j) < \varepsilon$, for all $i,j \geq n_0$. Using the definition of paranorm, we get

$$\sup_{k\geq 1,\ z\in X}\left(\|\Delta^n_{(m)}\lambda_k x^i_k - \Delta^n_{(m)}\lambda_k x^j_k, z\|\right)^{\frac{p_k}{H}} < \varepsilon \text{ for all } i,j \geq n_0$$

It follows that for every $z \in X$ and $k \geq 1$,

$$\|\Delta^n_{(m)}\lambda_k x^i_k - \Delta^n_{(m)}\lambda_k x^j_k, z\| < \varepsilon,\ \text{for all } i,j \geq n_0.$$

Hence $(\Delta^n_{(m)}\lambda_k x^i_k)$ is a Cauchy sequence in X for all $k \in N$.
$\Rightarrow (\Delta^n_{(m)}\lambda_k x^i_k)$ is convergent in X for all $k \in N$, since X is a 2-Banach space. For simplicity, let $\lim_{i\to\infty}\Delta^n_{(m)}\lambda_k x^i_k = y_k$ for each $k \in N$. Let $k = 1$, we have

$$\lim_{i\to\infty}\Delta^n_{(m)}\lambda_1 x^i_1 = \lim_{i\to\infty}\sum_{v=0}^{n}(-1)^v\binom{n}{v}\lambda_{1-mv}x^i_{1-mv} \tag{1}$$

Similarly we have

$$\lim_{i\to\infty}\lambda_k x^i_k = y_k,\ \text{for } k = 1,2,\ldots,nm \tag{2}$$

Thus from (1) and (2) we have $\lim_{i\to\infty} x^i_{1+nm}$ exists. Let $\lim_{i\to\infty} x^i_{1+nm} = x_{1+nm}$. Proceding in this way inductively, we have $\lim_{i\to\infty} x^i_k = x_k$ exists for each $k \in N$. Now we have for all $i, j \geq n_0$.

$$\sup_{k\geq 1,\ z\in X} \Big(\|\Delta^n_{(m)}\lambda_k x^i_k - \Delta^n_{(m)}\lambda_k x^j_k, z\|\Big)^{\frac{p_k}{H}} < \varepsilon$$

$$\Longrightarrow \lim_{j\to\infty} \left[\sup_{k\geq 1,\ z\in X} \Big(\|\Delta^n_{(m)}\lambda_k x^i_k - \Delta^n_{(m)}\lambda_k x^j_k, z\|\Big)^{\frac{p_k}{H}}\right] < \varepsilon \text{ for all } i \geq n_0$$

$$\Longrightarrow \sup_{k\geq 1,\ z\in X} \Big(\|\Delta^n_{(m)}\lambda_k x^i_k - \Delta^n_{(m)}\lambda_k x_k, z\|\Big)^{\frac{p_k}{H}} < \varepsilon \text{ for all } i \geq n_0$$

It follows that $(x^i - x) \in \ell_\infty(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$.

Since $(x^i) \in \ell_\infty(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$ and $\ell_\infty(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$ is a linear space, so we have $x = x^i - (x^i - x) \in \ell_\infty(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$. This completes the proof of the theorem. □

Theorem 3. *If $0 < p_k \leq q_k < \infty$ for each k, then*

$$Z(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p) \subseteq Z(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, q)$$

for $Z = c_0$ and c.

Proof. We prove the result for the case $Z = c_0$ and for the other case it will follow on applying similar arguments.
Let $(x_k) \in c_0(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$. Then we have

$$\lim_{k\to\infty} \Big(\|\Delta^n_{(m)}\lambda_k x_k, z\|\Big)^{p_k} = 0$$

This implies that $\Big(\|\Delta^n_{(m)}\lambda_k x_k, z\|\Big)^{p_k} < \varepsilon$, $(0 < \varepsilon \leq 1)$ for sufficiently large k. Hence we get

$$\lim_{k\to\infty} \Big(\|\Delta^n_{(m)}\lambda_k x_k, z\|\Big)^{q_k} \leq \lim_{k\to\infty} \Big(\|\Delta^n_{(m)}\lambda_k x_k, z\|\Big)^{p_k} = 0$$

$\Longrightarrow$ $(x_k) \in c_0(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, q)$. Thus

$$c_0(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p) \subseteq c_0(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, q)$$

Similarly, $c(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p) \subseteq c(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, q)$. This completes the proof. □

The following result is a consequence of Theorem 3.

Corollary 1. *(a) If $0 < \inf p_k \leq p_k \leq 1$, for each k, then*

$$Z(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p) \subseteq Z(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda), \text{ for } Z = c_0 \text{ and } c.$$

(b) If $1 \leq p_k \leq \sup p_k < \infty$, for each k, then

$$Z(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda) \subseteq Z(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p), \quad \text{for } Z = c_0 \text{ and } c.$$

Theorem 4. $Z(\|\bullet,\bullet\|, \Delta^{n-1}_{(m)}, \Lambda, p) \subset Z(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$
(in general $Z(\|\bullet,\bullet\|, \Delta^i_{(m)}, \Lambda, p) \subset Z(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$, for $i = 1, 2, \ldots, n-1$), for $Z = \ell_\infty$, c and c_0.

Proof. Here we prove the result for $Z = c_0$ and for the other cases it will follow on applying similar arguments.

Let $x = (x_k) \in c_0(\|\bullet,\bullet\|, \Delta^{n-1}_{(m)}, \Lambda, p)$. Then we have

$$\lim_{k\to\infty} \left(\|\Delta^{n-1}_{(m)}\lambda_k x_k, z\|\right)^{p_k} = 0 \tag{3}$$

Now we have

$$\|\Delta^{n}_{(m)}\lambda_k x_k, z\| \leq \|\Delta^{n-1}_{(m)}\lambda_k x_k, z\| + \|\Delta^{n-1}_{(m)}\lambda_{k-m} x_{k-m}, z\|$$

Hence we have

$$\left(\|\Delta^{n}_{(m)}\lambda_k x_k, z\|\right)^{p_k} \leq D\left\{\left(\|\Delta^{n-1}_{(m)}\lambda_k x_k, z\|\right)^{p_k} + \left(\|\Delta^{n-1}_{(m)}\lambda_{k-m} x_{k-m}, z\|\right)^{p_k}\right\}$$

Then using (3), we get

$$\lim_{k\to\infty} \left(\|\Delta^{n}_{(m)}\lambda_k x_k, z\|\right)^{p_k} = 0$$

Thus $c_0(\|\bullet,\bullet\|, \Delta^{n-1}_{(m)}, \Lambda, p) \subset c_0(\|\bullet,\bullet\|, \Delta^{n}_{(m)}, \Lambda, p)$ □

The inclusion is strict follows from the following example.

Example 5. *Let $m = 3$, $n = 2$ and $p_k = 5$ for all k odd and $p_k = 3$ for all k even. Consider the 2-normed space of Example 2. Consider the sequences $\Lambda = \left(\frac{1}{k^3}\right)$ and $x = (x_k) = (k^4, k^4)$. Then $\Delta^2_{(3)}\lambda_k x_k = 0$, for all $k \in N$. Then $x \in c_0(\|\bullet,\bullet\|, \Delta^2_{(3)}, \Lambda, p)$. Again we have $\Delta^1_{(3)}\lambda_k x_k = -3$, for all $k \in N$. Hence $x \notin c_0(\|\bullet,\bullet\|, \Delta^1_{(3)}, \Lambda, p)$. Thus the inclusion is strict.*

Theorem 5. *The spaces $c_0(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$, $c(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$ and $\ell_\infty(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$ are not monotone and as such are not solid in general*

Proof. The proof follows from the following example. □

Example 6. *Let $n = 2$, $m = 3$, $p_k = 1$ for all k odd and $p_k = 2$ for all k even and consider the 2-normed space of Example 1. Then $\Delta^2_{(3)}\lambda_k x_k = \lambda_k x_k - 2\lambda_{k-3}x_{k-3} + \lambda_{k-6}x_{k-6}$, for all $k \in N$. Consider the J^{th} step space of a sequence space E defined as, for (x_k), $(y_k) \in E^J$ implies that $y_k = x_k$ for k odd and $y_k = 0$ for k even. Consider the sequences $\Lambda = (k^3)$ and $x = (\frac{1}{k^2})$. Then $x \in Z(\|\bullet,\bullet\|, \Delta^2_{(3)}, \Lambda, p)$ for $Z = \ell_\infty$, c and c_o, but its J^{th} canonical pre-image does not belong to $Z(\|\bullet,\bullet\|, \Delta^2_{(3)}, \Lambda, p)$ for $Z = \ell_\infty$, c and c_o. Hence the spaces $Z(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$ for $Z = \ell_\infty$, c and c_o are not monotone and as such are not solid in general.*

Theorem 6. *The spaces $c_0(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$, $c(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$ and $\ell_\infty(\|\bullet,\bullet\|, \Delta^n_{(m)}, \Lambda, p)$ are not symmetric in general*

Proof. The proof follows from the following example. □

Example 7. *Let $n = 2$, $m = 2$, $p_k = 2$ for all k odd and $p_k = 3$ for all k even and consider the 2-normed space of Example 1. Then $\Delta^2_{(2)}\lambda_k x_k = \lambda_k x_k - 2\lambda_{k-2}x_{k-2} + \lambda_{k-4}x_{k-4}$, for all $k \in N$. Consider the sequences $\Lambda = (1, 1, 1, \dots)$ and $x = (x_k)$ defined as $x_k = k$ for k odd and $x_k = 0$ for k even. Then $\Delta^2_{(2)}\lambda_k x_k = 0$, for all $k \in N$. Hence $(x_k) \in Z(\|\bullet,\bullet\|, \Delta^2_{(2)}, \Lambda, p)$, for $Z = \ell_\infty$, c and c_o. Consider the rearranged sequence, (y_k) of (x_k) defined as*

$$(y_k) = (x_1, x_3, x_2, x_4, x_5, x_7, x_6, x_8, x_9, x_{11}, x_{10}, x_{12}, \dots)$$

Then $(y_k) \notin Z(\|\bullet,\bullet\|,\Delta^2_{(2)},\Lambda,p)$, *for* $Z=\ell_\infty$, c *and* c_o.
Hence the spaces $Z(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$, *for* $Z=\ell_\infty$, c *and* c_o *are not symmetric in general.*

Theorem 7. *The spaces* $c_0(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$, $c(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$ *and* $\ell_\infty(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$ *are not convergence free in general*

Proof. The proof follows from the following example. □

Example 8. *Let* $m=3$, $n=1$, $p_k=6$ *for all* k *and consider the* 2*-normed space of Example* 2*. Then* $\Delta^1_{(3)}\lambda_k x_k = \lambda_k x_k - \lambda_{k-3}x_{k-3}$, *for all* $k\in N$. *Let* $\Lambda=\left(\frac{7}{k}\right)$ *and consider the sequences* (x_k) *and* (y_k) *defined as* $x_k=\left(\frac{4}{7}k,\frac{4}{7}k\right)$ *for all* $k\in N$ *and* $y_k=\left(\frac{1}{7}k^3,\frac{1}{7}k^3\right)$ *for all* $k\in N$. *Then* $(x_k)\in Z(\|\bullet,\bullet\|,\Delta^1_{(3)},\Lambda,p)$ *but* $(y_k)\notin Z(\|\bullet,\bullet\|,\Delta^1_{(3)},\Lambda,p)$, *for* $Z=\ell_\infty$, c *and* c_o. *Hence the spaces* $Z(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$, *for* $Z=\ell_\infty$, c *and* c_o *are not convergence free in general.*

Theorem 8. *The spaces* $c_0(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$, $c(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$ *and* $\ell_\infty(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$ *are not sequence algebra in general*

Proof. The proof follows from the following example. □

Example 9. *Let* $n=2$, $m=1$, $p_k=1$ *for all* k *and consider the* 2*-normed space of Example* 1*. Then* $\Delta^2_{(1)}\lambda_k x_k = \lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}$, *for all* $k\in N$. *Consider* $\Lambda=\left(\frac{1}{k^4}\right)$ *and let* $x=(k^5)$ *and* $y=(k^6)$. *Then* x, $y\in Z(\|\bullet,\bullet\|,\Delta^2_{(1)},\Lambda,p)$, $Z=\ell_\infty$ *and* c, *but* $x,y\notin Z(\|\bullet,\bullet\|,\Delta^2_{(1)},\Lambda,p)$, *for* $Z=c_o$ *Hence the spaces* $c(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$, $\ell_\infty(\|\bullet,\bullet\|,\Delta^n_{(m)},\Lambda,p)$ *are not sequence algebra in general.*

Example 10. *Let* $n=2$, $m=1$, $p_k=3$ *for all* k *and consider the* 2*-normed space of Example* 2*. Then* $\Delta^2_{(1)}\lambda_k x_k = \lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}$, *for all* $k\in N$. *Consider* $\Lambda=\left(\frac{1}{k^7}\right)$ *and let* $x=(k^8,k^8)$ *and* $y=(k^8,k^8)$. *Then* $x,y\in c_0(\|\bullet,\bullet\|,\Delta^2_{(1)},\Lambda,p)$, *but* $x,y\notin Z(\|\bullet,\bullet\|,\Delta^2_{(1)},\Lambda,p)$, *for* $Z=\ell_\infty$, c. *Hence the space* $c_0(\|\bullet,\bullet\|,\Delta^2_{(1)},\Lambda,p)$ *is not sequence algebra in general.*

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Osmania University
Department of Mathematics
PGCS, Saifabad, Hyderabad - 500004, A.P., India
E-mail address: bsrmathou@yahoo.com

Gauhati University
Department of Mathematics
Kokrajhar Campus, Kokrajhar - 783370, Assam, INDIA
E-mail address: hemen_dutta08@rediffmail.com