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ONE DERIVATIVE OF ONE COMPONENT REGULARITY CRITERION FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT. We study the incompressible Navier-Stokes equations in the entire three-dimensional space and we prove that if there exists one derivative of one component of the velocity

$$\partial_3 u_3 \in L_t^s L_x^r, where \frac{2}{s} + \frac{3}{r} \leq \frac{1}{4}, 12 \leq r \leq \infty,$$

then the solution is regular. This extends one result of Patrick Penel, Toulon, Milan Pokorý, Praha [Appl.Math.,49,483-493(2004)] [15].

1. Introduction

In this paper, we consider the Cauchy problem for the incompressible Navier-Stokes equations

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= \nu \triangle u \\ \nabla \cdot u &= 0 \\ u(x,0) &= u_0(x) \end{cases} \tag{1}$$

in which $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the unknown velocity field, $u_0(x)$ is the initial velocity field with $\nabla \cdot u_0 = 0$, and $p(x,t)$ is a scalar pressure. While ν is the kinematic viscosity coefficient, we will assume that $\nu \equiv 1$ for simplicity in this paper. Here we use the classical notion

$$\nabla u = (\partial_1 u, \partial_2 u, \partial_3 u), \triangle u = \sum_{i=1}^{3} \partial_i^2 u, \nabla \cdot u = \sum_{i=1}^{3} \partial_i u_i, u \cdot \nabla u = \sum_{j=1}^{3} u_j \partial_j u.$$

In 1934, Leray has proved that there exists a global weak solution

$$u \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \bigcap L^2(0,\infty; H^1(\mathbb{R}^3)). \tag{2}$$

on the condition that $u_0 \in L^2$ and $\nabla \cdot u_0 = 0$. It is called the Leray-Hopf weak solution which satisfies the energy inequality [1-3]. If $\|u(t)\|_{H^1}$ is continuous, we say $u(t)$ is regular. Up to the present, the regularity and uniqueness of the weak solutions is still open problem. On the other hand, there are already many criterion which ensure that the weak solution is regular. We list some notable criteria in Lebesgue space $L_t^s L_x^r = L^s(0,T; L^r(\mathbb{R}^3))$:

$\langle 1 \rangle$. Concerning u, ∇u or p:

1. If $u \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq 1, 3 \leq r \leq \infty$, then it is regular [4-6];
2. If $\nabla u \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq 2, \frac{3}{2} \leq r \leq \infty$, then it is regular [7,8];
3. If $p \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq 2, \frac{3}{2} \leq r \leq \infty$, then it is regular [9];

$\langle 2 \rangle$. Concerning one component of the velocity:

1. If $u_3 \in L^\infty(\mathbb{R}^3 \times (0,\infty))$, then it is regular[10];
2. If $u_3 \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq \frac{1}{2}, 6 \leq r \leq \infty$, then it is regular [11];

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3. If $u_3 \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq \frac{5}{8}, \frac{24}{5} \leq r \leq \infty$, then it is regular [12].

⟨3⟩. Concerning gradient of one component of the velocity:
1. If $\nabla u_3 \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq 1, 3 \leq r \leq \infty$, then it is regular [13];
2. If $\nabla u_3 \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq \frac{3}{2}, 2 \leq r \leq \infty$, then it is regular [14,15];
3. If $\nabla u_3 \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq \frac{11}{6}, \frac{54}{23} \leq r \leq \frac{18}{5}$, then it is regular [12].

⟨4⟩. Concerning one derivative of the velocity:
1. If $\partial_3 u \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq \frac{3}{2}, 2 \leq r \leq \infty$, then it is regular [15];
2. If $\partial_3 u_3 \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq 1, 3 \leq r \leq \infty$ and $\partial_3 u_i \in L_t^{s_1} L_x^{r_1}$, with $\frac{2}{s_1} + \frac{3}{r_1} \leq 2, \frac{3}{2} \leq r_1 \leq \infty (i = 1, 2)$, then it is regular [15];
3. If $\partial_3 u \in L_t^s L_x^r$, with $\frac{2}{s} + \frac{3}{r} \leq 2, \frac{9}{4} \leq r \leq 3$, then it is regular [16];

⟨5⟩. Concerning one derivative of one component of the velocity:
If $\partial_3 u_3 \in L_t^\infty L_x^\infty$, then it is regular [15].

In this paper we give a regularity criterion concerning one derivative of one component of the velocity. This extends one result ⟨5⟩ of Patrick Penel, Toulon, Milan Pokorý, Praha [Appl.Math., 49, 483-493(2004)].

2. Main Result and Proof

Our main result can be stated as follows.

Theorem 1. *Let*

$$\partial_3 u_3 \in L_t^s L_x^r, \frac{2}{s} + \frac{3}{r} \leq \frac{1}{4}, 12 \leq r \leq \infty, \tag{3}$$

then u is regular.

At first, let us recall the definition of the Leray-Hopf weak solution.

Definition 1. *Leray-Hopf weak solution*

If a measurable vector u satisfies the following properties in $0 \leq T \leq \infty$,
(1) u is weakly continuous in $[0, T) \times L^2(\mathbb{R}^3)$;
(2) u verifies (1) in the sense of distribution,

$$\int_0^T \int_{\mathbb{R}^3} (\frac{\partial \Phi}{\partial t} + u \cdot \nabla \Phi + \triangle \Phi) \cdot u dx dt + \int_{\mathbb{R}^3} u_0 \cdot \Phi(x, 0) dx = 0, \tag{4}$$

$\forall \Phi \in C_0^\infty(\mathbb{R}^3 \times [0, T)), \nabla \cdot \Phi = 0, \Phi$ *is vector function;*
(3) satisfies the energy inequality,

$$\|u(\cdot, t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\cdot, s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2, 0 \leq t \leq T, \tag{5}$$

then u is called Leray-Hopf weak solution to the Navier-Stokes equations.

In addition, we give three lemma for the proof of the theorem.

Lemma 1. *The estimate for the pressure in $L^q(\mathbb{R}^3)$:*

$$\|p\|_{L^q} \leq C \|u\|_{L^{2q}}^2, \quad 1 < q < \infty. \tag{6}$$

Proof. See [15,17]. □

Lemma 2. *Interpolation inequality:*

Assume $u \in L_t^\infty L_x^2(\mathbb{R}^3 \times I)$ and $\nabla u \in L_t^2 L_x^2(\mathbb{R}^3 \times I)$ where I is an open interval,then $u \in L_t^s L_x^r(\mathbb{R}^3 \times I)$ for $\forall\ r, s$ such that

$$\frac{2}{s} + \frac{3}{r} \leq \frac{3}{2} \tag{7}$$

and $2 \le r \le 6$. *Moreover,*

$$\|u\|_{L_t^s L_x^r} \le C\|u\|_{L_t^\infty L_x^2}^{(6-r)/2r} \|\nabla u\|_{L_t^2 L_x^2}^{(3r-6)/2r}. \tag{8}$$

Proof. See [16]. □

Lemma 3. *The estimate for the pressure in* $L_t^s L_x^r(\mathbb{R}^3 \times I)$*:*

$$\|p\|_{L_t^s L_x^r} \le C\|u_0\|_{L^2}^2, \frac{2}{s} + \frac{3}{r} = 3, 1 < r \le 3. \tag{9}$$

Proof. By Lemma 1 and Lemma 2, we get

$$\begin{aligned}
\|p\|_{L_t^s L_x^r} &\le C\|u\|_{L_t^{2s} L_x^{2r}}^2 \\
&\le C\|u\|_{L_t^\infty L_x^2}^{(6-r)/r} \|\nabla u\|_{L_t^2 L_x^2}^{(3r-6)/r} \\
&\le C(\|u\|_{L_t^\infty L_x^2}^2 + \|\nabla u\|_{L_t^2 L_x^2}^2) \\
&\le C\|u_0\|_{L^2}^2.
\end{aligned}$$

□

Proof. Proof of Theorem 1:

At first, let us multiply the equation for u_3 of (1) by $u_3^{19/5}$ and integrate over $[0,T]\times\mathbb{R}^3$, $t \in (0,T)$

$$\int_0^t \int_{\mathbb{R}^3} \frac{\partial u_3}{\partial t} \cdot u_3^{19/5} - \int_0^t \int_{\mathbb{R}^3} \triangle u_3 \cdot u_3^{19/5} = -\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u_3 \cdot u_3^{19/5} - \int_0^t \int_{\mathbb{R}^3} \partial_3 p \cdot u_3^{19/5}$$

$$I_1 + I_2 = I_3 + I_4.$$

Calculating every term,

$$\begin{aligned}
I_1 &= \frac{5}{24}\int_0^t \int_{\mathbb{R}^3} \frac{\partial u_3^{24/5}}{\partial t} = \frac{5}{24}\|u_3\|_{L^{24/5}}^{24/5}|_0^t = \frac{5}{24}\|u(\cdot,t)\|_{L^{24/5}}^{24/5} - \frac{5}{24}\|u(\cdot,0)\|_{L^{24/5}}^{24/5}, \\
I_2 &= -\sum_{i=1}^3 \int_0^t \int_{\mathbb{R}^3} \partial_{ii} u_3 u_3^{19/5} = \frac{95}{144}\sum_{i=1}^3 \int_0^t \int_{\mathbb{R}^3} (\partial_i u_3^{12/5})^2 = \frac{95}{144}\|\nabla u_3^{12/5}\|_{L_t^2 L_x^2}^2, \\
I_3 &= -\sum_{i=1}^3 \int_0^t \int_{\mathbb{R}^3} u_i \partial_i u_3 u_3^{19/5} = -\frac{5}{24}\sum_{i=1}^3 \int_0^t \int_{\mathbb{R}^3} u_i \partial_i u_3^{24/5} \\
&= \frac{5}{24}\sum_{i=1}^3 \int_0^t \int_{\mathbb{R}^3} \partial_i u_i u_3^{24/5} = 0, \\
I_4 &= \int_0^t \int_{\mathbb{R}^3} p \partial_3 u_3^{19/5} = \int_0^t \int_{\mathbb{R}^3} p \partial_3 u_3 u_3^{14/5} \\
&\le C\|p\|_{L_t^{s_1} L_x^{r_1}} \|\partial_3 u_3\|_{L_t^s L_x^r} \|u_3^{14/5}\|_{L_t^\infty L_x^{12/7}} \\
&\le C\|p\|_{L_t^{s_1} L_x^{r_1}} \|\partial_3 u_3\|_{L_t^s L_x^r} \|u_3\|_{L_t^\infty L_x^{12/7}}^{14/5} \\
&\le C\|p\|_{L_t^{s_1} L_x^{r_1}}^{12/5} \|\partial_3 u_3\|_{L_t^s L_x^r}^{12/5} + \frac{1}{24}\|u_3\|_{L_t^\infty L_x^{12/5}}^{24/5},
\end{aligned}$$

where

$$\begin{cases} \frac{1}{s_1}+\frac{1}{s_2} &= 1 \\ \frac{1}{r_1}+\frac{1}{r_2} &= \frac{5}{12} \\ \frac{2}{s}+\frac{3}{r} &= \frac{1}{4} \qquad 8 \le s \le \infty, 12 \le r \le \infty \\ \frac{2}{s_1}+\frac{3}{r_1} &= 3 \qquad 1 \le s_1 \le \frac{8}{7}, \frac{12}{5} \le r_1 \le 3. \end{cases} \tag{10}$$

By Lemma 3, we get

$$\|p\|_{L_t^{s_1}L_x^{r_1}} \le C\|u_0\|_{L^2}^2, \tag{11}$$

then

$$I_4 \le C\|u_0\|_{L^2}^{24/5}\|\partial_3 u_3\|_{L_t^sL_x^r}^{12/5} + \frac{1}{24}\|u_3\|_{L_t^\infty L_x^{12/5}}^{24/5}.$$

Summarizing the above calculating of I_1, I_2, I_3 and I_4, we get

$$\begin{aligned} &\frac{1}{6}\|u_3(\cdot,t)\|_{L^{24/5}}^{24/5} + \frac{95}{144}\|\nabla u_3^{12/5}\|_{L_t^2L_x^2}^2 \\ &\le C\|u_0\|_{L^2}^{24/5}\|\partial_3 u_3\|_{L_t^sL_x^r}^{12/5} + \frac{5}{24}\|u_3(\cdot,0)\|_{L^{24/5}}^{24/5} \\ &\le C\|u_0\|_{L^2}^{24/5}\|\partial_3 u_3\|_{L_t^sL_x^r}^{12/5} + C\|u_3(\cdot,0)\|_{L^2}^{3/5}\|u_3(\cdot,0)\|_{L^6}^{21/5} \\ &\le C\|u_0\|_{L^2}^{24/5}\|\partial_3 u_3\|_{L_t^sL_x^r}^{12/5} + C\|u_0\|_{L^2}^{3/5}\|\nabla u_0\|_{L^2}^{21/5}. \end{aligned}$$

By (3) $\partial_3 u_3 \in L_t^sL_x^r$, where, $\frac{2}{s}+\frac{3}{r} \le \frac{1}{4}, 12 \le r \le \infty$, so

$$\|u_3(\cdot,t)\|_{L_t^\infty L_x^{24/5}} + \|\nabla u_3^{12/5}\|_{L_t^2L_x^2} \le C, \tag{12}$$

moreover,

$$\|u_3(\cdot,t)\|_{L_t^{24/5}L_x^{72/5}} \le C\|\nabla u_3^{12/5}\|_{L_t^2L_x^2}^{5/12}, \tag{13}$$

then

$$u_3 \in L_t^sL_x^r, where, \frac{2}{s}+\frac{3}{r} \le \frac{5}{8}, \frac{24}{5} \le r \le \frac{72}{5}. \tag{14}$$

So the solution is regular making use of the regularity criterion in ⟨3⟩3. This implies Theorem 1 is correct. □

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