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# Variation of Curves Length Reported to Cone Metric in Lorentz Manifold

SORIN NOAGHI

ABSTRACT. On Lorentz manifold $(M, g)$ we consider a timelike, parallel and unitary vector field $Z$ with its help we define the $Z$-length of curves and we obtain their first and the second variation.

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## 1. PRELIMINARIES

In 1988 Dan I. Papuc has been started the study of differential manifold endowed with a field of tangent cones, mathematical structure on which belong the Lorentz manifold $(M, g)$ too, where we consider the cone of future directed, oriented by $Z$, timelike vector fields. So, we have in every point $p \in M$ the structure $(T_pM, K_p)$ where:

$$K_p = \{v \in T_pM \mid g(v, v) \leq 0,\ g(v, Z_p) < 0\}$$

that means a Krein space that involves the definition of ordered relation:

$$v \leq w \text{ if and only if } v - w \in K_p$$

$v, w \in T_pM$.

Moreover, this ordered relation involves the definition of a norm [2], [3] named $Z$ -norm through:

$$|v|_{Z_p} = \inf\{\lambda \geq 0 \mid -\lambda Z_p \leq v \leq \lambda Z_p\}$$

For the expression of $Z$- norm we have:

$$|v|_{Z_p} = |g(v, Z_p)| + \sqrt{g(v, v) + [g(v, Z_p)]^2}$$

and for a smooth curve $\lambda : [a, b] \to M$ we name its $Z$- length the value:

$$L_Z(\lambda) = \int_a^b |\lambda'(t)|_{Z_p} dt$$

$$= \int_a^b \left\{ \left|g\left(\lambda'(t), Z_{\lambda(t)}\right)\right| + \sqrt{g(\lambda'(t), \lambda'(t)) + g^2\left(\lambda'(t), Z_{\lambda(t)}\right)} \right\} dt$$

We say that the curve $\lambda : [a, b] \to M$ is $Z$-global in $x_0 = \lambda(t_0)$ if $\lambda'(t_0)$ and $Z_{\lambda(t)}$ are collinear. The calculus of the first variation assign same restrictive hypothesis:

A) We suppose that $Z$ is parallel that means $\nabla_X Z = 0,\ \forall X \in \mathcal{X}(M)$

B) We suppose that $\lambda : [t_p, t_q] \to M$ is not $Z$-global none of its points.

**Remark 1.** *Condition for $\lambda$ as curve that is not $Z$-global in none of its points involves that $h(\lambda'(t), \lambda'(t)) \neq 0$, where $h(X,Y) = g(X,Y) + g(X,Z)\,g(Y,Z)$. We have*

$$h(X,X) = 0 \Leftrightarrow Gram\{X.Z\} = \begin{vmatrix} g(X,X) & g(X,Z) \\ g(Z,X) & g(Z,Z) \end{vmatrix} = 0 \Leftrightarrow \{X,Z\} \textit{colinear.}$$

**Remark 2.** *If $\phi : (-\varepsilon, \varepsilon) \times [t_p, t_g] \to M$ is a piecewise smooth variation of a timelike, future directed curve $\lambda$ that is not $Z$-global, then exists $\delta > 0$ with the property that $\phi(u, \ ) : [t_p, t_g] \to M$ is timelike future directed and not $Z$-global for every $|u| < \delta$.*

Beem [1] prove the statement for a geodesic segment $\lambda$.With no difficulty we can renounce at restriction of geodesic segment, considering $\lambda$ a piecewise smooth timelike future directed curve. It remains to prove the condition that $\phi(u, \ ) : [t_p, t_g] \to M$ is not $Z$-global for $|u| < \delta$.

First the smooth differentiation of $\phi$ involve the fact that exists $\varepsilon_1 < \varepsilon$ as $\phi : [-\varepsilon_1, \varepsilon_1] \times [t_p, t_g] \to M$ is differentiable on compact, that means we can extend to an open set which contains $[-\varepsilon_1, \varepsilon_1] \times [t_p, t_g]$. Because $\lambda$ is timelike and is not $Z$-global, we have $\phi'\left(u, t_p^+\right)$ ; $\phi'\left(u, t_q^-\right)$ timelike vector not collinear with $Z_{\phi\left(u, t_p^+\right)}$ $Z_{\phi\left(u, t_q^-\right)}$ for $\forall |u| < \delta_1$ respectively. We suppose that there is no $\delta > 0$ so that $\phi(u_0, \ ) : [t_p, t_g] \to M$ not to be $Z$-global for $\forall |u_0| < \delta_1$. Then exists a sequence $u_n \to 0$ so that $\phi'(u_n, t_n)$ is collinear with $Z_{\phi(u_n, t_n)}$. Hence $(u_n, t_n) \in [-\varepsilon_1, \varepsilon_1] \times [t_p, t_q]$ which is compact it results than we have an accumulation point by form $(0, t)$. So $\phi'(0, t)$ is collinear with $Z_{\phi(0,t)}$ or $\lambda'(t)$ is collinear with $Z_{\lambda(t)}$, that involves the fact that $\lambda$ is $Z$-global in $x = \lambda(t)$, affirmation excluded from hypothesis.

Second, we consider the case when $\phi : (-\varepsilon, \varepsilon) \times [t_p, t_g] \to M$ is a piecewise smooth variation of piecewise smooth timelike future directed and not $Z$-global curve $\lambda$. It will exist a partition $t_p = t_0 < t_1 < ... < t_k = t_q$ as $\phi|_{(-\varepsilon,\varepsilon)\times[t_{i-1}, t_i]}$ is a smooth timelike not $Z$-global future directed variation of $\lambda|_{[t_{i-1}, t_i]}$, $\forall i = \overline{1,k}$. From the above affirmation, we have $\delta_i$, $i = \overline{1,k}$ so that $\phi(u, \ ) : [t_{i-1}, t_i] \to M$ is not $Z$-global and $\phi'\left(u, t_{i-1}^+\right)$; $\phi'\left(u, t_i^-\right)$ are not collinear with $Z_{\phi\left(u, t_{i-1}^+\right)}$, respectively $Z_{\phi\left(u, t_i^-\right)}$ for $|u| < \delta_i$. Considering $\delta = \min\limits_{i=\overline{1,k}} \delta_i$ we obtain that $\phi(u, \ ) : [t_p, t_g] \to M$ is not a $Z$-global curve for $|u| < \delta$.

C) We suppose that $\lambda : [t_p, t_q] \to M$ is a timelike future directed curve, $h$-unitary parameterized, that means $h(\lambda', \lambda') = 1$.

**Remark 3.** *Obviously, the $h$-unitary condition involves the not $Z$-global property of $\lambda$ from* ***Remark 1.***

Let $\Phi : \mathcal{A} \to M$ where $\mathcal{A} := (-\varepsilon, \varepsilon) \times [t_p, t_q]$ be, an proper smooth causal variation of $\lambda$, the curve $\phi(u, \ ) : [t_p, t_q] \to M$ ,$|u| < \varepsilon$ are not $Z$-global. That is:

(1) $\Phi(0, t) = \lambda(t)$, $\forall t \in [t_p, t_q]$
(2) $\Phi(u, t_p) = p$, $\Phi(u, t_q) = q$
(3) $\Phi \in \mathcal{C}^3(\mathcal{A})$
(4) $g(V,V) < 0$, $g(V,Z) < 0$, $g(V,Z)^2 + g(V,V) \neq 0$ where $V = \phi_*\left(\frac{\partial}{\partial t}\right)$

We denote the variation vector field by $X = \phi_*\left(\frac{\partial}{\partial u}\right)$ where $\left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right\}$ is a base in $T_{(u,t)}\mathcal{A}$.

We denote $L_Z(u) = L_Z(\phi(u, \ ))$. In $(A)$ and $(B)$ hypothesis we have:

**Lemma 1.** *The first variation of $Z$- length of $\lambda$ is:*

$$\frac{d}{du}L_Z(0) = -\int_{t_p}^{t_q} \frac{1}{\sqrt{h(\lambda'.\lambda')}} h\left(\lambda'', P^h_{(\lambda')^\perp} X\right)|_{(o,t)}dt$$

*where* $P^h_{(\lambda')^\perp}X = X - \frac{h(X,Y)}{h(Y,Y)}Y$ *is the* $X$ *projection, with respecting the bilinear form* $h$ *on* $(\lambda')^\perp$.

If, additionally, we suppose about $\lambda$ that is a geodesic segment, being satisfied all the $A, B, C$ hypothesis, we have:

**Lemma 2.**

$$\frac{d^2}{du^2}L_Z(0) = -\int_{t_p}^{t_q} h(R(X,\lambda')\lambda' + N'', N)|_{(0,t)}dt + \\ + \{h(\nabla_X X, \lambda') - g(\nabla_X X, Z) + h(N', N)\}|_{(0,t)}|_{t_p}^{t_q}$$

*where* $N = X - h(X,V)V$.

## 2. The first variation

We make the remark for $h : TM \times TM \to \mathbf{R}$ that is a bilinear metric, positive form, semidefined, degenerated with signature $(n-1,0,1)$ because:

$$\begin{aligned} X(h(Y_1,Y_2)) &= X[g(Y_1,Y_2) + g(Y_1,Z)g(Y_2,Z)] = \\ &= g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2) + \\ &+ [g(\nabla_X Y_1, Z) + g(Y_1, \nabla_X Z)]g(Y_2,Z) + \\ &+ g(Y_1,Z)[g(\nabla_X Y_2, Z) + g(Y_2, \nabla_X Z)] \\ &= g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2) + g(\nabla_X Y_1, Z)g(Y_2,Z) + \\ &+ g(Y_1,Z)g(\nabla_X Y_2, Z) = h(\nabla_X Y_1, Y_2) + h(Y_1, \nabla_X Y_2) \end{aligned}$$

where we use $A)$ hypothesize about $Z$, namely $\nabla_X Z = 0$.

We have for $Z$-length of curve $\phi(u, \ ) : [t_p, t_q] \to M$

$$L_Z(u) = \int_{t_p}^{t_q} \left\{-g(V,Z) + \sqrt{h(V,V)}\right\} dt$$

from where:

$$\frac{d}{du}L_Z(u) = \int_{t_p}^{t_q} \left\{-\frac{\partial}{\partial u}[g(V,Z)] + \frac{\partial}{\partial u}\sqrt{h(V,V)}\right\} dt = \\ = \int_{t_p}^{t_q} \left\{-g(\nabla_X V, Z) - g(V, \nabla_X Z) + \frac{1}{\sqrt{h(V,V)}} h(\nabla_X V, V)\right\} dt$$

Since $[X,V] = 0$, then $\nabla_X V = \nabla_V X$ and so:

$$\frac{d}{du}L_Z(u) = \int_{t_p}^{t_q} \left\{-g(\nabla_X V, Z) + \frac{h(\nabla_V X, V)}{\sqrt{h(V,V)}}\right\} dt \tag{1}$$

We calculate:

$$\frac{\partial}{\partial t}\left[g\left(X,Z\right)\right]=g\left(\nabla_V X,Z\right)+g\left(X,\nabla_V Z\right)=g\left(\nabla_V X,Z\right) \quad (2)$$

$$\frac{\partial}{\partial t}\left[\frac{h\left(X,V\right)}{\sqrt{h\left(V,V\right)}}\right]=$$

$$=\frac{\left[h\left(\nabla_V X,V\right)+h\left(X,\nabla_V V\right)\right]\sqrt{h\left(V,V\right)}-h\left(X,V\right)\frac{h\left(\nabla_V V,V\right)}{\sqrt{h\left(V,V\right)}}}{h\left(V,V\right)}$$

$$=\frac{h\left(\nabla_V X,V\right)}{\sqrt{h\left(V,V\right)}}+\frac{h\left(X,\nabla_V V\right)}{\sqrt{h\left(V,V\right)}}-\frac{h\left(X,V\right)h\left(\nabla_V V,V\right)}{\left[\sqrt{h\left(V,V\right)}\right]^3}=$$

$$=\frac{h\left(\nabla_V X,V\right)}{\sqrt{h\left(V,V\right)}}+\frac{1}{\sqrt{h\left(V,V\right)}}\left[h\left(\nabla_V V,X\right)-\frac{h\left(X,V\right)}{h\left(V,V\right)}h\left(\nabla_V V,V\right)\right]=$$

$$=\frac{h\left(\nabla_V X,V\right)}{\sqrt{h\left(V,V\right)}}+\frac{1}{\sqrt{h\left(V,V\right)}}h\left(\nabla_V V,X-\frac{h\left(X,V\right)}{h\left(V,V\right)}V\right) \quad (3)$$

Replacing (2) and (3) in (1), we have:

$$\frac{d}{du}L_Z\left(0\right)=$$

$$=\int_{t_p}^{t_q}\left\{-\tfrac{\partial}{\partial t}\left[g\left(X,Z\right)\right]+\tfrac{\partial}{\partial t}\left[\tfrac{h(X,V)}{\sqrt{h(V,V)}}\right]-\tfrac{1}{\sqrt{h(V,V)}}h\left(\nabla_V V,P^h_{V^\perp}X\right)\right\}\Big|_{(0,t)}dt$$

$$=-\int_{t_p}^{t_q}\left\{\frac{1}{\sqrt{h\left(V,V\right)}}h\left(\nabla_V V,P^h_{V^\perp}X\right)\right\}\Big|_{(0,t)}dt+$$

$$+\left\{\frac{h\left(X,V\right)}{\sqrt{h\left(V,V\right)}}-g\left(X,Z\right)\right\}\Big|_{(0,t)}\Big|_{t_p}^{t_q}$$

$$=-\int_{t_p}^{t_q}\left\{\frac{1}{\sqrt{h\left(V,V\right)}}h\left(\nabla_V V,P^h_{V^\perp}X\right)\right\}\Big|_{(0,t)}dt$$

$$=-\int_{t_p}^{t_q}\left\{\frac{1}{\sqrt{h\left(\lambda',\lambda'\right)}}h\left(\lambda'',P^h_{(\lambda')^\perp}X\right)\right\}\Big|_{(0,t)}$$

We used the hypothesis of proper variation, namely: $X\left(t_p^+\right)=X\left(t_q^-\right)=0$

**Remark 4.** *If $\lambda$ is a geodesic segment, then $\frac{d}{du}L_Z\left(0\right)=0$ because $\nabla_V V|_{(0,t)}=\frac{D\lambda'}{\partial t}=0$ namely, the geodesic segments that are not Z-global, are stationary points for the Z-length functional.*

**Remark 5.** *Let it be $\lambda:[t_p,t_q]\to M$ a piecewise smooth timelike and future directed curve corresponding to the partition $t_p=t_0<t_1<...<t_k=t_q$, $h-$unitary parametrized which is not Z- global. Denoting with $L_Z^i\left(u\right)$ the Z-length of restriction to the interval $\left[t_{i-1},t_i\right]$, the uniparametric variation of the curve $\lambda|_{[t_{i-1},t_i]}$, $i=\overline{1,k}$ we have:*

$$\frac{dL_Z^i(0)}{du} = -\int_{t_{i-1}}^{t_i} h\left(\lambda'', P^h_{(\lambda')^\perp} X\right)\Big|_{(0,t)} dt + \\ + \left\{h(\lambda', X) - g(X, Z)\right\}\big|_{(0,t)}\Big|_{t_{i-1}}^{t_i}$$

*so that*

$$\frac{dL_Z}{du}(0) = \sum_{i=1}^{k} \frac{dL_Z^i(0)}{du} = -\int_{t_p}^{t_q} h\left(\lambda'', P^h_{(\lambda')^\perp} X\right)\Big|_{(0,t)} dt - \\ - \sum_{i=1}^{k-1} h\left(X(t_i), \Delta_{t_i}(\lambda')\right)$$

*where* $\Delta_{t_i}(\lambda') = \lambda'\left(t_i^+\right) - \lambda'\left(t_i^-\right), \ \forall i = \overline{1, k-1}$ *and we have take in to account the fact that the variation is proper, namely we have:* $X(\lambda(t_p)) = X(\lambda(t_q)) = 0$.

**Remark 6.** *Let consider* $H$ *a spatial hypersurface and we assume that* $\lambda : [t_p, t_q] \to M$ *is a timelike future directed geodesics segment that is not* $Z$*-global curve with* $\lambda(t_p) = p \in H$ *and* $\lambda(t_p) = q \notin H$*. Then:*

$$0 = \frac{dL_Z(0)}{du} = g(X_p, Z_p) - g(X_p, \lambda'(t_p)) - g(X_p, Z_p)\, g(\lambda'(t_p), Z_p) = \\ = -g(X_p, \lambda'(t_p) + g(\lambda'(t_p), Z_p) Z_p - Z_p)$$

*namely* $\lambda'(t_p) + [g(\lambda'(t_p), Z_p) - 1] Z_p$ *must be* $g$ *orthogonal on* $H$.

If $\lambda : [t_p, t_q] \to M$ is a geodesics segment, we can find an affine parametrization of $\lambda$ so that $g(\lambda'(t_p), Z_p) = 1$, case in which the upper condition becomes: $\lambda'(t_p)$ is $g$ ortogonal on $H$.

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University of Petroșani
Department of Mathematics
University Street, no. 20, 332006, Petroșani, România
*E-mail address*: snoaghi@yahoo.com