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Gauss-Christoffel Type Quadrature Formulas with Fixed Nodes. Gauss-Laguerre-Radau Quadrature Formula

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ABSTRACT. We study in this paper the Gauss-Christoffel with fixed nodes quadrature formula with their principal's properties, looking up to construct a new Gauss-Laguerre-Radau quadrature formula with one fixed nodes, corresponding to the Laguerre orthogonal polynomials. We gave too an application to this formula.

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Christoffel [1] as well has given a generalised importance to this classical Gauss type formula. Christoffel's generalisation consists of: he managed to develop a formula consisting of $k+m$ nodes, where k are certain data, and m are chosen as for the respective quadrature formula to have the maximum degree of exactitude. We will consider the following formula

$$\int_a^b \rho(x) f(x)\,dx = \sum_{i=0}^{m} A_{m,i} f(a_i) + \sum_{j=1}^{k} B_{k,j} f(b_j) + R_{m+k[f]} \tag{1}$$

Where ρ is the (negative) weight function, $b_j \notin (a,b)$, $j=\overline{1,k}$ are the fix and distinctive nodes, and the nodes $a_i \in (a,b)$, $i=\overline{0,m}$ also called free nodes, will be determined so as the quadrature formula (1) will have the maximum degree of exactitude.

It is also observed that integrated function has to be defined on the exterior points of the interval (a,b) as well.

Using the theory of interpolation, multiplying by the Lagrange interpolation theory relative to the function f and the nodes $a_i, i=\overline{0,m}$ and b_j, $j=\overline{1,k}$, after integrating member by member, the quadrature formula (1) will be obtained, with a degree of exactitude of at least $m+k$. As the nodes are free, they can be determined so that the exactitude degree of the formula may be increased.

A short presentation of several important results and observations of this paragraph will be presented as follows.

Theorem 1. *The maximum degree of exactitude, $r=2m+k+1$, of the quadrature formula (1) is obtained if the nodes $a_i, i=\overline{0,m}$, are the roots of the $(m+1)$ degree orthogonal polynomial relative for the following weight:*

$$w_k(x) = \rho(x)\,|v_k(x)| = \rho(x) \prod_{j=1}^{k} |x-b_j|,\ x \in (a,b) \tag{2}$$

The following observation will be also made.

Remark 1. *If in the quadrature formula (1) with a maximum degree of exactitude* $r = 2m+k+1$*, we consider the function* $f = |v_k| \cdot g$*, where* g *is a certain integrable function, then we have:*

$$\int_a^b \rho(x) |v_k(x)| g(x)\, dx = \sum_i^m A_{m,i} |v_k(a_i)| g(a_i) + R_{m+k}[|v_k| g] \tag{3}$$

On the other hand, we may be able to write the classical Gauss type quadrature formula relative to the weight $w_k = \rho |v_k|$ and the interval (a, b) for the function g:

$$\int_a^b w_k(x) g(x)\, dx = \sum_{i=0}^m C_{m,i} g(a_i) + R_m[g] \tag{4}$$

where $C_{m,i}$, $i = \overline{0, m}$ and $R_m[g]$ are the coefficients and respectively the residual of the Gauss type quadrature formula.

By comparing the quadrature formulas (3) and (4) the following relations will result:

$$A_{m,i} = \frac{1}{|v_k(a_i)|} \cdot C_{m,i},\ \ i = \overline{0, m} \tag{5}$$

$$R_m[g] = R_{m+k}[|v_k| g]\, or R_{m+k}[f] = R_m\left[\frac{f}{|v_k|}\right] \tag{6}$$

As well as the fact that the nodes $a_i, i = \overline{0, m}$, are the roots of the $m+1$ degree orthogonal polynomial relative to the weight $w_k = \rho |v_k|$.

The following theorem gives us the explicit representation of the free nodes polynomial ω_m from formula (1), resolving the polynomial equation $\omega_m(x) = 0$. We are able to observe as well that the nodes will be situated inside the integration interval (a, b), being the roots of an orthogonal polynomial.

Theorem 2. *Be it the quadrature formula (1) with a maximum degree of exactitude and* $\left(\tilde{\mathrm{P}}_n\right) n \geq 0$ *the sequence of orthogonal polynomials relative to the weight* ρ *and the interval* (a, b) *having the coefficient of the monomial of maximum degree equal to* 1*.*

The free nodes polynomial ω_m allows the following representation:

$$\omega_m(x) = \frac{1}{v_k(x)} \cdot \frac{\begin{vmatrix} \tilde{\mathrm{P}}_{m+1}(b_1) & \dots & \tilde{\mathrm{P}}_{m+k+1}(b_1) \\ \dots & \dots & \dots \\ \tilde{\mathrm{P}}_{m+1}(b_k) & \dots & \tilde{\mathrm{P}}_{m+k+1}(b_k) \\ \tilde{\mathrm{P}}_{m+1}(x) & \dots & \tilde{\mathrm{P}}_{m+k+1}(x) \end{vmatrix}}{\begin{vmatrix} \tilde{\mathrm{P}}_{m+1}(b_1) & \dots & \tilde{\mathrm{P}}_{m+k}(b_k) \\ \dots & \dots & \dots \\ \tilde{\mathrm{P}}_{m+1}(b_k) & \dots & \tilde{\mathrm{P}}_{m+k}(b_k) \end{vmatrix}} \tag{7}$$

The following theorem refers to the coefficients of the quadrature formula (1).

Theorem 3. *The coefficients of the quadrature formula (1) with a maximum degree of exactitude are given by:*

$$A_{m,i} = \int_a^b \rho(x) \frac{\omega_m(x) \cdot v_k(x)}{(x-a_i)\,\omega_m'(a_i) \cdot v_k(a_i)} dx \ , \ i = \overline{0,m} \tag{8}$$

$$B_{k,j} = \int_a^b \rho(x) \frac{\omega_m(x) \cdot v_k(x)}{(x-b_j)\,\omega_m(b_j) \cdot v_k'(b_j)} dx \ , \ j = \overline{1,k} \tag{9}$$

where $\omega(x)$ and $v_k(x)$ are the polynomials of the free nodes respectively fix of the formula (1)

Following the same path with the one in Gauss type quadrature formulas, for the residual we will have the following results.

Theorem 4. *If we consider the formula (1) with a maximum degree of exactitude $r = 2m+k+1$, then for the term residual we will have:*

i) the representation by divided difference:

$$R_{m+k}[f] = \int_a^b \rho(x) v_k(x)\, \omega_m^2(x) \left[x, a_0, a_0, ..., a_m, a_m, b_1, ..., b_k; f\right] dx \tag{10}$$

ii) if $f \in C^{2m+k+2}[a,b]$, then

$$R_{m+k}[f] = \frac{f^{(2m+k+2)}(\xi)}{(2m+k+2)!} \int_a^b \rho(x) v_k(x)\, \omega_m^2(x)\, dx, \ \ a < \xi < b \tag{11}$$

For the demonstration of these results see [4].

The Gauss - Radau formulas, are particular cases of the Gauss - Christoffel quadrature formula, these having a single node each, being one the extremities of the interval (a,b), respectively $b_1 = a$ or $b_{1'} = b$, then the integration interval may be infinite, (a, ∞). Analogically, when $b_1 = b$ the infinite interval $(-\infty, b)$ may be considered. If $(a,b) = (-1,1)$, and the weight function is $\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}$ with $\alpha, \beta > -1$ a thorough study has been made with the result for the Gauss - Jacobi - Radau type quadrature formulas (with the fix node $b_1 = -1$ respectively $b_1 = 1$) being the polynomial free nodes, the coefficients for the free nodes, the coefficients for the fix nodes, as well as the term residual.

We will study the case where $(a,b) = (0, \infty)$, and the weight function is $\rho(x) = x^{\alpha} e^{-x}$, $\alpha > -1, x \in (0, \infty)$. We know that this weight corresponds to the Laguerre orthogonal polynomials, their expressions being obtained by the formula:

$$L_m^{[\alpha]}(x) = x^{-\alpha} e^{-x} \frac{d^m}{dxHm} \left[x^{m+\alpha} e^{-x}\right] \tag{12}$$

Considering the following type of quadrature formula:

$$\int_0^{\infty} x^{\alpha} e^{-x} f(x) dx = \sum_{i=0}^{m} A_{m,i} f(a_i) + B_{1,1} f(0) + R_{m+1}[f] \tag{13}$$

For this Gauss - Laguerre - Radau type quadrature formula, the polynomial of the fix node is $\nu_1 = x$, and the polynomial of the free nodes, according to formula (7) is:

$$\omega_m(x) = \frac{1}{x} \cdot \frac{\begin{vmatrix} \tilde{\mathrm{P}}_{m+1}(0) & \tilde{\mathrm{P}}_{m+2}(0) \\ \tilde{\mathrm{P}}_{m+1}(x) & \tilde{\mathrm{P}}_{m+2}(x) \end{vmatrix}}{\tilde{\mathrm{P}}_{m+1}(0)} \tag{14}$$

where $\tilde{\mathrm{P}}_{m+1}, \tilde{\mathrm{P}}_{m+2}$ are the $m+1$ respectively $m+2$ Laguerre orthogonal polynomials relative to the weight $\rho(x) = x^{\alpha}e^{-x}$. Therefore, the determination of free nodes $a_i, i = \overline{0,m}$ from formula (13) goes back to solving the polynomial equation $\omega_m(x) = 0$ namely

$$\tilde{\mathrm{P}}_{m+1}(0)\,\tilde{\mathrm{P}}_{m+2}(x) - \tilde{\mathrm{P}}_{m+1}(x)\,\tilde{\mathrm{P}}_{m+2}(0) = 0 \tag{15}$$

Moreover, according to the Observation 2, the nodes $a_i, i = \overline{0,m}$ are the roots for the $m+1$ Laguerre orthogonal polynomial at $x^{\alpha+1}e^{-x}$.

The $A_{m,i},\ i = \overline{0,m}$ coefficients may be determined with the formula (5), namely

$$A_{m,i} = \frac{1}{a_i} C_{m,i,\ \ i=\overline{0,m}} \tag{16}$$

Where:

$$C_{m,i} = \frac{m!\Gamma(m+\alpha+2)}{L_m^{[\alpha+1]}(a_i)\,\frac{d}{dx}\left[L_{m+1}^{[\alpha+1]}(x)\right]_{x=a_i}} \quad ,\ i = \overline{0,m} \tag{17}$$

represents the coefficients for the Gauss - Laguerre quadrature formula relative for the weight $x^{\alpha+1}e^{-x}$.

The coefficient corresponding to the fix node $b_1 = 0$ will be determined by the formula (9), namely

$$B_{1,1} = \int_0^{\infty} \rho(x)\frac{\omega_m(x)}{\omega_m(0)}dx = \int_0^{\infty} x^{\alpha}e^{-x}\frac{L_{m+1}^{[\alpha+1]}}{L_{m+1}^{[\alpha+1]}(0)}dx \tag{18}$$

For the term residual we will apply the formula (11) resulting in the expression

$$\begin{aligned} R_{m+1}[f] &= \tfrac{f^{(2m+3)}(\xi)}{(2m+3)!}\int_0^{\infty} x^{\alpha+1}e^{-x}\left[\tilde{L}_{m+1}^{[\alpha+1]}(x)\right]^2 dx = \\ &= \tfrac{f^{(2m+3)}(\xi)}{(2m+3)!}\int_0^{\infty} x^{\alpha+1}e^{-x}\left[(-1)^{m+1}L_{m+1}^{[\alpha+1]}(x)\right]^2 dx = \\ &= \tfrac{f^{(2m+3)}(\xi)}{(2m+3)!}\int_0^{\infty} x^{\alpha+1}e^{-x}\left[L_{m+1}^{[\alpha+1]}(x)\right]^2 dx \end{aligned} \tag{19}$$

If we mark by:

$$\lambda_m^{[\alpha]} = \int_0^{\infty} x^{\alpha}e^{-x}\left[L_m^{[\alpha]}\right]^2 dx = m!\Gamma(m+\alpha+1) = \left\|L_m^{[\alpha]}\right\|^2 \tag{20}$$

Where $\|\cdot\|$ represents L_2 the orthogonal polynomial norm related to the weight $x^{\alpha}e^{-x}$, the result will be:

$$\lambda_{m+1}^{[\alpha+1]} = \int_0^\infty x^{\alpha+1} e^{-x} \left[L_{m+1}^{[\alpha+1]}(x) \right]^2 dx = (m+1)!\Gamma(m+\alpha+3) \tag{21}$$

Therefore the expression of the residual will be:

$$R_{m+1}[f] = \frac{(m+1)! f^{(2m+3)}(\xi)}{(2m+3)!} \Gamma(m+\alpha+3) \tag{22}$$

Application. We will consider the case $m = 1$, namely two nodes are free a_0 and a_1. The Gauss - Laguerre - Radau type quadrature formula becomes

$$\int_0^\infty x^\alpha e^{-x} f(x) dx = A_{1,0} f(a_0) + A_{1,1} f(a_1) + B_{1,1} f(0) + R^2[f] \tag{23}$$

In order to determine the free nodes a_0 and a_1, based on the observation 1, we will determine by calculation the 2nd degree Laguerre orthogonal polynomial at weight $x^{\alpha+1}e^{-x}$ according to the definition of the Laguerre polynomial from the relation (12) we have:

$$L_{m+1}^{[\alpha+1]}(x) = x^{-(\alpha+1)} e^x \frac{d^{m+1}}{dx^{m+1}} \left[x^{m+\alpha+2} e^{-x} \right]$$

Therefore in our case

$$L_2^{[\alpha+1]}(x) = x^{-(\alpha+1)} e^x \tfrac{d^2}{dx^2} \left[x^{\alpha+3} e^{-x} \right] = x^{-(\alpha+1)} e^x \tfrac{d}{dx} \left[x^{\alpha+2} e^{-x} (\alpha+3-x) \right] = \\ = x^{-(\alpha+1) e^x x^{(\alpha+1)} e^{-x} \left[x^2 - 2x(\alpha+3) + (\alpha+2)(\alpha+3) \right]}$$

Therefore the polynomial equation:

$$L_2^{[\alpha+1]}(x) = x^2 - 2x(\alpha+3) + (\alpha+2)(\alpha+3) = \omega_1(x) = 0 \tag{24}$$

has as roots $a_0 = \alpha + 3 - \sqrt{\alpha+3}$ respectively $a_1 = \alpha + 3 + \sqrt{\alpha+3}$ the free nodes of the quadrature formula (23).

For the calculation of $A_{1,0}$ and $A_{1,1}$ coefficients we will apply the formula(16) respectively (17), namely:

$$A_{1,0} = \frac{1}{a_0} C_{1,0} \quad cu \;\; C_{1,0} = -\frac{\Gamma(\alpha+3)}{L_1^{[\alpha+1]}(a_0) \frac{d}{dx} \left[L_2^{[\alpha+1]}(x) \right]_{x=a_0}} \tag{25}$$

Respectively

$$A_{1,1} = \frac{1}{a_1} C_{1,1} \quad cu \;\; C_{1,1} = -\frac{1! \cdot \Gamma(\alpha+3)}{L_1^{[\alpha+1]}(a_1) \frac{d}{dx} \left[L_2^{[\alpha+1]}(x) \right]_{x=a_1}} \tag{26}$$

but

$$L_1^{[\alpha+1]}(x) = x^{-(\alpha+1)} e^x \frac{d}{dx} \left[x^{\alpha+2} e^{-x} \right] = \alpha + 2 - x$$

therefore

$$L_1^{[\alpha+1]}(a_0) = \alpha + 2 - a_0 = \sqrt{\alpha+3} - 1$$

and

$$L_1^{[\alpha+1]}(a_1) = \alpha + 2 - a_1 = -\left(1 + \sqrt{\alpha+3}\right)$$

We will also determine

$$\frac{d}{dx}\left[L_2^{[\alpha+1]}(x)\right] = 2x - 2(\alpha+3)$$

considering the relation (24). We will also determine the values:

$$\frac{d}{dx}\left[L_1^{[\alpha+1]}(x)\right]_{x=0} = 2\left[a_0 - (\alpha+3)\right] = -2\sqrt{\alpha+3}$$

$$\frac{d}{dx}\left[L_2^{[\alpha+1]}(x)\right]_{x=a_1} = 2\left[a_1 - (\alpha+3)\right] = -2\sqrt{\alpha+3}$$

Replacing the above values in the formulas (25) respectively in (26) we will determine the values of the free nodes coefficients:

$$A_{1,0} = \frac{\Gamma(\alpha+3)}{2} \cdot \frac{1}{\left(\alpha+3-\sqrt{\alpha+3}\right)^2} = \frac{\Gamma(\alpha+3)}{2a_0^2} \tag{27}$$

$$A_{1,1} = \frac{\Gamma(\alpha+3)}{2} \cdot \frac{1}{\left(\alpha+3+\sqrt{\alpha+3}\right)^2} = \frac{\Gamma(\alpha+3)}{2a_1^2} \tag{28}$$

For the determination of the $B_{1,1}$ coefficient we will apply the formula (18)

$$B_{1,1} = \int_0^\infty x^\alpha e^{-x} \frac{L_2^{[\alpha+1]}(x)}{L_2^{\alpha+1}(0)} dx =$$
$$= \frac{1}{(\alpha+2)(\alpha+3)} \int_0^\infty x^\alpha e^{-x}\left[x^2 - 2x(\alpha+3) + (\alpha+2)(\alpha+3)\right] dx =$$
$$\frac{1}{(\alpha+2)(\alpha+3)} \int_0^\infty x^{\alpha+2} e^{-x} dx - \frac{2}{\alpha+2} \int_0^\infty x^{\alpha+1} e^{-x} dx + \int_0^\infty x^\alpha e^{-x} dx$$

The relation (24) has also been considered.

From the definition of the special function

$$\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x} dx$$

and the functional relations:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\alpha\Gamma(\alpha)$$

$$\Gamma(\alpha+3) = (\alpha+2)\Gamma(\alpha+2) = (\alpha+2)(\alpha+1)\Gamma(\alpha+1)$$

we have:

$$\begin{array}{l} B_{1,1} = \frac{\Gamma(\alpha+3)}{(\alpha+2)(\alpha+3)} - \frac{2\Gamma(\alpha+2)}{(\alpha+2)} + \Gamma\left(\alpha+1\right) = \\ = \frac{\Gamma(\alpha+3)}{(\alpha+2)(\alpha+3)} - \frac{2\Gamma(\alpha+3)}{(\alpha+2)^2} + \frac{\Gamma(\alpha+3)}{(\alpha+1)(\alpha+2)} = \\ = \frac{2\Gamma(\alpha+3)}{(\alpha+1)(\alpha+2)^2(\alpha+3)} \end{array} \tag{29}$$

For the term residual we will have the expression according to the formula (22)

$$R_2\left[f\right] = \frac{\Gamma\left(\alpha+4\right)}{60} f^{(5)}\left(\xi\right)\,,\ f\in C^5\left(0,\infty\right)\,,\xi>0 \tag{30}$$

therefore the formula considered for this application has the 4th degree of exactitude. If the relations (26), respectively (28) and (29) as well as (30) are replaced inside the formula (23) we will obtain a Gauss - Laguerre - Radau quadrature formula with one fix node and two free ones.

$$\begin{array}{l} \int\limits_0^\infty x^\alpha e^{-x} f\left(x\right)dx = \frac{\Gamma(\alpha+3)}{2}\left[\frac{1}{a_0^2} f\left(a_0\right) + \frac{1}{a_1^2} f\left(a_1\right) + \frac{4}{(\alpha+1)(\alpha+2)(\alpha+3)} f\left(0\right)\right] + \\ + \frac{\Gamma(\alpha+4)}{60} f^{(5)}\left(\xi\right),\ \xi>0,\ \ where\ \ a_{0,1} = \alpha+3\pm\sqrt{\alpha+3} \end{array} \tag{31}$$

Where $a_{0,1} = \alpha + 3 \mp \sqrt{\alpha+3}$ represent the value of the free nodes.

Another particular case of the Gauss - Christoffel quadrature formula is the Gauss - Lobatto quadrature formula with two fix nodes, these being the extremities of the interval bordered by the integration (ab), namely $b_1 = a$ and $b_2 = b$, in the case where $(a,b) = (-1,1)$ and the weight function is $\rho\left(x\right) = \left(1-x\right)^\alpha \left(1+x\right)^\beta$, $\alpha, \beta > -1$ and the numerical integration formula will be called the Gauss - Jacobi - Lobatto quadrature formula, with the following form:

$$\begin{array}{c} \int\limits_{-1}^{1} \left(1-x\right)^\alpha \left(1+x\right)^\beta f\left(x\right) dx = \\ = \sum\limits_{i=0}^{m} A_{m,i} f\left(a_i\right) + B_{2,1} f\left(-1\right) + B_{2,2} f\left(1\right) + R_{m+2}\left[f\right] \end{array} \tag{32}$$

This formula has been studied in detail for it's different weights $\rho\left(x\right) = 1$, corresponding to the Legendre orthogonal polynomial and $\rho\left(x\right) = \left(1-x\right)^{-\frac{1}{2}}$ respectively $\rho\left(x\right) = \left(1-x\right)^{\frac{1}{2}}$ corresponding to the Cebasev type I respectively type II orthogonal polynomials.

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